MATH 152H SECTION 1
FALL 2006
MIDTERM # 2

Problem 1: (10 pts) Compute

(a) (5 pts) $\displaystyle\lim_{x\to 8}\frac{6\sqrt{1+x}-10-x}{x^2-16\,x+64}\overset{\frac{0}{0}}{=}\lim_{x\to 8}\frac{\frac{3}{\sqrt{1+x}}-1}{2\,x-16}\overset{\frac{0}{0}}{=}\lim_{x\to 8}\frac{-\frac{3}{2}(1+x)^{-\frac{3}{2}}}{2}=-\frac{3}{108}$

(b) (5 pts) $\displaystyle\int\frac{sin\,2x}{\sqrt{3+cos\,2x}}+4\,sin\,x-1\,dx=\int-\frac{1}{2\sqrt{u}}\,du-4\,cos\,x-x+C=-\sqrt{u}-4\,cox\,x-x+C$

$$=-\sqrt{3+cos\,2x}-2\,cos\,x-x+C$$

Problem 2: (10 pts) Let $f(x)=x^3-3x+5$.

(a) (3 pts) Where is $f(x)$ increasing? decreasing? Where are its critical points?

$f'(x)=3x^2-3=3(x-1)(x+1)$. So $f(x)$ has critical points at $x=\pm 1$. It is increasing on $(-\infty,-1)\cup(1,\infty)$ and decreasing on $(-1,1)$.

(b) (3 pts) Where is $f(x)$ concave up? concave down? where are its inflection points?

$f''(x)=6x$. $f''>0$ on $(0,\infty)$ and <0 on $(-\infty,0)$. There is a point of inflection at $x=0$ and the function is concave up for $x>0$ and concave down for $x<0$.

(c) (4 pts) Find the absolute maximum and minimum of $f(x)$ on the interval $[-1,4]$.

On $[-1,4]$ there are two critical points, $x=\pm 1$. As one is already an endpoint, we need only consider $x=-1,1,4$. The absolute minimum is $f(1)=1-3+5=3$; the absolute maximum is $f(4)=64-12+5=57$.

Problem 3: (10 pts) You wish to make a box with a square base to store some books. The largest book requires that one side of the box be at least 8" long. You estimate that you need $2000\,in^3$ of volume in the box. Due to the weight you also need the base of the box to be three times as strong as the sides or the top, and thus three times as costly to buy. What are the dimensions of the box that will minimize the cost *and* store all your books?

Call the side length of the square base x, and the height y. We need to minimixe $C(x,y)=3x^2+4xy+x^2=4x^2+4xy$ subject to $x^2y=2000$ and $x\geq 8$. We have $y=\frac{2000}{x^2}$, so $C(x)=4x^2+\frac{8000}{x}$. Thus $C'(x)=8x-\frac{8000}{x^2}$. This is 0 at $x=10$ (and $y=20$). So a $10\times 10\times 20$ box will suffice. Note that since $C''(10)>0$ this is a local minimum. Checking $C'(x)$ we see that the function is decreasing for $x<10$ and increasing for $x>10$, so this is the absolute minimum as well.

Problem 4: (10 pts) You are on a pier overlooking a lake. You are holding a piece of string which runs over a pulley and out to a float bobbing on the perfectly calm water. The pulley is 6 ft above the *s*urface of the lake, directly above the end of the pier. You wind in the string, shortening its length at a constant rate of 4 ft/min. Assuming the float remains on the surface of the water, how fast is the float moving across the lake when it is 8 ft from the point in the water directly below pulley?

Call the length of the string from the pulley to the float l. Let x be the distance from the float to the point directly below the pulley. From the diagram we have $x^2+36=l^2$ or $2x\frac{dx}{dt}=2l\frac{dl}{dt}$. When $x=8$, $l=10$ and $\frac{dl}{dt}=-4$. Thus $\frac{dx}{dt}=\frac{10}{8}\cdot(-4)=-5\ ft/min$.

Problem 5: (10 pts) (**5 pts**) State the Mean Value theorem for a function $f(x)$ defined on $[a, b]$. Be sure to include all necessary assumptions.

Look in the book!

(**5 pts**) Consider a function $f(x)$ defined on $(-1, \infty)$ with the following property:

$$f'(x) = \sqrt{16 + x^2}, \qquad f(0) = 1$$

Explain why the Mean Value Theorem applies to $f(x)$ on $[0, 3]$, then use the Mean Value Theorem on $[0, 3]$ to show that $13 \leq f(3) \leq 16$.

Since the function has a derivative on $[0, 3]$ (given in the espression above) it is continuous on $[0, 3]$. Hence the mean value theorem applies. There is a point $c \in (0, 3)$ such that

$$\frac{f(3) - f(0)}{3 - 0} = f'(c) = \sqrt{16 + c^2}$$

In other words, $f(3) = 1 + 3\sqrt{16 + c^2}$ for some $c \in (0, 3)$. But $4 \leq \sqrt{16 + c^2} \leq 5$ for $c \in (0, 3)$. Hence $13 \leq f(3) \leq 16$.

Problem 6:(10 pts) You are given a function which satisfies

$$y' = 3x^2 y - \frac{6x}{1 + \sqrt{y}} \qquad y(1) = 1$$

Compute $y''(1)$ and find the quadratic approximation at $x = 1$.

We know $y(1) = 1$. Substituting above give $y'(1) = 0$. Use implicit differentiation to get:

$$y'' = 6xy + 3x^2 y' - \frac{6(1 + \sqrt{y}) - \frac{3x}{\sqrt{y}} \cdot y'}{(1 + \sqrt{y})^2}$$

Plugging in $x = 1, y = 1$ and $y' = 0$ gives $y'' = 6 - \frac{6 \cdot 2 - 0}{4} = 3$. Thus $Q(x) = 1 + \frac{3}{2}(x - 1)^2$.

Which of the following could be a graph of $y(x)$ near $x = 1$? Circle it, and explain why the others cannot.

The graph needs to have a horizontal tangent and be concave up. This is the third in the top row. None of the others have these properties.