Solutions to the Review for Math 152H-1 Test #2

1.

$$\lim_{x\to 3}\frac{4\sqrt{1+x}-5-x}{(x-3)^2}\overset{\frac{0}{0}}{=}\lim_{x\to 3}\frac{\frac{2}{\sqrt{1+x}}-1}{2(x-3)}\overset{\frac{0}{0}}{=}\lim_{x\to 3}\frac{-\frac{1}{(1+x)^{\frac{3}{2}}}}{2}=\frac{-\frac{1}{8}}{2}=-\frac{1}{16}$$

$$\lim_{x\to\frac{\pi}{2}+}(x-\frac{\pi}{2})\,tan\,x=\lim_{x\to\frac{\pi}{2}+}\frac{x-\frac{\pi}{2}}{cot\,x}\overset{\frac{0}{0}}{=}\lim_{x\to\frac{\pi}{2}+}\frac{1}{-csc^2\,x}=-1$$

2. Calculate the following anti-derivatives:

$$\int 4\,x^{\frac{3}{2}}+5-\frac{2}{x^3}\,dx=\frac{8}{5}\,x^{\frac{5}{2}}+5\,x+x^{-2}+C$$

$$\int\frac{1}{(3+x)^2}+x^2\,sin^2(x^3)\,dx=-(x+3)^{-1}+\int sin^2(u)\frac{1}{3}\,du=-(x+3)^{-1}+\frac{1}{3}\int\frac{1-cos\,2u}{2}\,du=$$
$$-(x+3)^{-1}+\frac{1}{6}(u-\frac{1}{2}\,sin\,2u)+C=-(x+3)^{-1}+\frac{1}{6}(x^3-\frac{1}{2}sin\,2x^3)+C$$

$$\int\frac{sec^2\big((3x+4)^{\frac{3}{4}}\big)}{(3x+4)^{\frac{1}{4}}}\,dx\overset{u=(3x+4)^{\frac{3}{4}}}{=}\int sec^2(u)\frac{4}{9}\,du=\frac{4}{9}\,tan\,u+C=\frac{4}{9}\,tan\big((3x+4)^{\frac{3}{4}}\big)+C$$

$$\int\frac{x^2-1}{x^{\frac{3}{2}}-\sqrt{x}}\,dx=\int\frac{(x-1)(x+1)}{\sqrt{x}(x-1)}\,dx=\int\frac{x+1}{\sqrt{x}}\,dx=\int x^{\frac{1}{2}}+x^{-\frac{1}{2}}\,dx=\frac{2}{3}x^{\frac{3}{2}}+2\,x^{\frac{1}{2}}+C$$

$$\int x^3\sqrt{1+x}\,dx\overset{u=x+1}{=}\int(u-1)^3\sqrt{u}\,du=\int u^{\frac{7}{2}}-3\,u^{\frac{5}{2}}+3\,u^{\frac{3}{2}}-u^{\frac{1}{2}}\,du=$$
$$\frac{2}{9}(1+x)^{\frac{9}{2}}-\frac{6}{7}(1+x)^{\frac{7}{2}}+\frac{6}{5}(1+x)^{\frac{5}{2}}-\frac{2}{3}(1+x)^{\frac{3}{2}}+C$$

3. Find y' at a point (x,y) satisfying $x^2-3x\sqrt{1+y}+y^2=1$.

   $2x-3\sqrt{1+y}-3x\big(\frac{1}{2\sqrt{1+y}}\cdot y'\big)+2yy'=0$. Solve for y'.

4. Find y' when $x^2+3xy-y^2=3$. Where are the horizontal tangents to the curve? Where are the vertical tangents? Are there any points where there is no unique tangent line? (Check if these are on the curve!!)

   $2x+3y+3xy'-2yy'=0$ gives $y'=-\frac{2x+3y}{3x-2y}$. There are horizontal tangents where $y'=0$, i.e. $y=-\frac{2}{3}x$ and $x^2+3xy-y^2=3$. Substituting gives $x^2-2x^2-\frac{4}{9}x^2=3$. This has no solutions as the left side is negative after simplification. Thus there are no horizontal tangents. There are also no points wher y' is undefined like $\frac{0}{0}$. For vertical tangents we need to solve $3x-2y=0$ or $y=\frac{3}{2}x$. Again we substitute back into the equation to get $x^2+\frac{9}{2}x^2-\frac{9}{4}x^2=3$. This is the same as $\frac{13}{4}x^2=3$ or $x=\pm\sqrt{\frac{12}{13}}$.

5. Find the linearization and quadratic approximation to $f(x)=\sqrt{x}-x$ at $x=4$. Is the graph increasing or decreasing near this point? Is it concave up or concave down?

   $f(4)=2-4=-2$, $f'(4)=\frac{1}{2\cdot 2}-1=-\frac{3}{4}$, $f''(4)=-\frac{1}{4}\cdot 4^{-\frac{3}{2}}=-\frac{1}{32}$. Hence the linearization at $x=4$ is

$L(x) = -2 - \frac{3}{4}(x-4)$ and the quadratic approximation is $Q(x) = -2 - \frac{3}{4}(x-4) - \frac{1}{64}(x-4)^2$ (don't forget to divide $f''(4)$ by 2!). The function is decreasing and concave down near $x = 4$. Note that since the second derivative of $y = x$ is 0, the concavity is determined by $\sqrt{x}$, and this is always concave down on $(0, \infty)$.

6. Find the linearization and quadratic approximation to a function satisfying $f'(x) = \frac{1}{1+f(x)}, f(3) = -2$ at $x = 3$. If you could graph $f(x)$ would the function be increasing or decreasing at $x = 3$?

   We are given $f(3) = -2$. From the expression $f'(3) = \frac{1}{1+-2} = -1$. Taking another derivative gives $f''(x) = -(1+f(x))^{-2}f'(x)$ so $f''(3) = -(1+-2)^{-2}(-1) = 1$. Thus $L(x) = -2-(x-3) = 1-x$ and $Q(x) = 1-x+\frac{1}{2}(x-3)^2$. The graph would be concave up and decreasing near $x = 3$. Note that at this point, knowing only the information about the derivatives, we can compute a reasonable approximation for $f(3.01)$. We can use $L(3.01) = -2.01$.

7. Where is $g(x) = \frac{x}{1-x^2}$ increasing/decreasing? Where is it concave up/concave down? What are its inflection points, if any? Graph this function as completely as possible.

   $g'(x) = \frac{1+x^2}{(1-x^2)^2}$. There are vertical asymptotes at ± 1, but otherwise the function is increasing. It only crosses the x-axis when $x = 0$. Concavity requires more work: $g''(x) = \frac{2x(1-x^2)^2-(1+x^2)(2(1-x^2)(-2x)}{(1-x^2)^4} = \frac{2x(1-x^2)\left((1-x^2)+2(1+x^2)\right)}{(1-x^2)^4}$ $= \frac{2x(3+x^2)}{(1-x^2)^3}$. The $3+x^2$ term is always positive. The other possible changes in concavity are at $x = 0, \pm 1$. Checking signs gives: concave up on $(-\infty,-1) \cup (0,1)$ and concave down on $(-1,0) \cup (1,\infty)$. The only point of inflection is at $x = 0$ ($x = \pm 1$ are not in the domain!).

8. Draw a graph of the solution of

   $$\frac{d}{dx}y = \sqrt{x}(4-2x^{\frac{3}{2}}),\ y(1) = \sqrt{2}$$

   Find the solution.

   The solution is $y = \frac{8}{3}x^{\frac{3}{2}} - \frac{2}{3}x^3 + (\sqrt{2}-2)$. The domain of the function is $[0,\infty)$ since that is where the derivative is defined. On this interval $\sqrt{x} \geq 0$. To determine increasing and decreasing regions we look at when $4-2x^{\frac{3}{2}} > 0$ or not. This is equivalent to $2 > x^{\frac{3}{2}}$ or $4^{\frac{1}{3}} > x$. Thus the function increases on $(0, 4^{\frac{1}{3}})$ and decreases on $(4^{\frac{1}{3}}, \infty)$. We have critical points at 0 and $x = 4^{\frac{1}{3}}$. The former is a minimum (on the boundary of the domain) and the latter is a local maximum. Since we know the solution we have $y(0) = \sqrt{2}-2 < 0$. However, $y(4^{\frac{1}{3}}) = \frac{16}{3} - \frac{8}{3} + \sqrt{2} - 2 = \frac{2}{3} + \sqrt{2} > 0$. We also have $y'' = 2x^{-\frac{1}{2}} - 4x = 2x^{-\frac{1}{2}}(1-2x^{\frac{3}{2}})$. This implies that the function is concave up when $x < (\frac{1}{4})^{\frac{1}{3}}$. After this point it remains concave down.

9. Find and classify as local max/local min all the critical points of $f(x) = x^4 - 6x^3 + 5$. Where are the absolute max/min for $f(x)$ on the interval $[-1, 5]$.

   $f'(x) = 4x^3 - 18x^2$ and $f''(x) = 12x^2 - 36x$. The critical points are when $f'(x) = 0$ or $0, \frac{9}{2}$. $f''(\frac{9}{2}) = 12 \cdot \frac{81}{4} - 36 \cdot \frac{9}{2} = 243 - 162 > 0$. Thus $x = \frac{9}{2}$ is a local minimum. However, $f''(0) = 0$ so the test fails, and we need to go back and look at the sign pattern of the first derivative in order to determine what is going on. Check that $f' < 0$ when $x < \frac{9}{2}, x \neq 0$ (since $x^2 > 0$ except when $x = 0$). Thus, the sign of f' does not change so $x = 0$ is neither a local max nor a local min. To locate the absolute max/min we add $x = -1$ and $x = 5$ to $x = \frac{9}{2}$ as points to check. We know already that $f(x)$ decreases from $x = -1$ to $x = \frac{9}{2}$. It then increases on $(\frac{9}{2}, \infty)$. Thus $x = \frac{9}{2}$ is the absolute minimum, and since $f(-1) = 12$ while $f(5) = 625 - 750 + 5 < 0$, the absolute maximum occurs at $x = -1$.

10. A farmer wants to enclose a plot of land by a straight river. This plot should have one side on the river and be in the shape of a rectangle. He will then subdivide the plot into two equal pieces, each touching the river. Due to a slope it costs twice as much to build a fence perpendicular to the river then it does to build parallel to the river. He wants to have at least 294 ft^2 of fenced pasture when he is done. What are the dimensions of the plot that will minimize his

cost? He wishes to put sheep in one part of the pasture, and they need to be able to graze up to 10 ft away from the river. What dimensions will minimize his cost subject to this additional restriction?

Let x be the length of the pasture in the direction perpendicular to the river. Let y be the length parallel. Then the total amount of fence he needs, using the river as one side, is thre sections of length x and one of length y. Thus the cost will be $a(6x + y)$ where a is the cost per foot of fence parallel to the river. On the other hand $xy = 294$, so we wish to minimixe $C(x) = 6x + \frac{294}{x}$. This occurs when $x = 7$, $y = 42$. Notice that such a pasture won't work for his sheep. The restriction comes down to specifying that x is in $[10, \infty)$. The derivative $C'(x)$ is greater than 0 on this interval, thus the minimum occurs where $x = 10$, $y = 29.4$.

11. What is the smallest volume of a right circular cone which encloses a sphere of radius 2?
Ask if you want to know how this is done.

12. Hot water (100 degrees) flows into a tub at a rate of $0.4\, gal/min$. In the tub already are 10 gallons of 40 degree water. Assume that the temperature of the resulting mix is given by $\frac{1}{V}\left(100 \cdot v_h + 40 \cdot v_c\right)$ where $V = v_c + v_h$ and $v_c = 10$ is the volume of cold water mixed with v_h gallons of hot water. How fast is the temperature changing 5 minutes after we begin adding the hot water?

The temperature is given by $\frac{100(0.4t)+400}{10+0.4t}$. Take the derivative to get $\frac{40(10+0.4t)-(40t+400)(0.4)}{(10+0.4t)^2} = \frac{240}{(10+0.4t)^2}$. When $t = 5$ this yields $\frac{240}{144} = \frac{5}{3}\, deg/min$.

13. You are standing $100ft$ from a building. A ball is dropped from the top of the building and falls along the side of the building. At time t it will have fallen $5\,t^2$ feet. You watch the ball fall and record that the angle you see it at is decreasing at a rate of $\frac{1}{\sqrt{2}}$ rad/s when the angle is $\frac{\pi}{4}$. How tall is the building? (Assume that your height is negligible).

Let θ be the angle you see the ball at. If x is the height of the ball then $tan\,\theta = \frac{x}{100}$. Thus $sec^2\theta \frac{d}{dt}\,\theta = \frac{1}{100}\frac{d}{dt}\,x$. However, $x + 5t^2 = h$ where h is the height of the building. Thus $\frac{d}{dt}\,x = -10t$. Putting these together give $t = -10sec^2\theta\frac{d}{dt}\,\theta$. When $\theta = \pi/4$, $sec^2\theta = 2$ and $\frac{d}{dt}\,\theta = -\frac{1}{\sqrt{2}}$. Thus $t = 10\sqrt{2}$. This is how long the ball has been falling. Thus it has fallen $1000\,ft$. On the other hand it still has $100\,ft$ to go (since the angle is $\pi/4$). So the height of the building is $1100\,ft$.

14. Explain why $x = \frac{1}{1+\sqrt{x}}$ has one solution of the interval $[0, \infty)$.

First, let $f(x) = x - \frac{1}{1+\sqrt{x}}$. This is a continuous function on $[0, \infty)$. $f(0) = -1$ while $f(4) = 4 - \frac{1}{3} > 0$. By the intermediate value theorem there is at least one solution. To show that there is at most one solution we compute $f'(x) = 1 + \frac{1}{2\sqrt{x}(1+\sqrt{x})^2}$. Thus $f'(x) > 0$ on $(0, \infty)$. Thus the function is increasing there, and so can only cross $y = 0$ once. If it crossed more than once then Rolle's theorem implies that there must be a c where $f'(c) = 0$, and this does not happen.

15. Use the Mean Value Theorem to explain why a polynomial $P(x) = a_0\,x^3 + a_1\,x^2 + a_2\,x^1 + a_3$ has at most 3 roots. Use this to explain why a fourth degree polynomial $a_0x^4 + a_1x^3 + \ldots + a_4$ has at most four roots. If you keep doing this, how do you show that $a_0\,x^n + a_1\,x^{n-1} + \ldots a_{n-1}\,x + a_n$ has at most n roots?

Polynomials are continuous and differentiable on every interval. They thus satisfy the Mean Value theorem. Choosing a and b to be consecutive roots, the Mean Value theorem guarantees a c where $P'(x) = 0$ between them. If there are n roots to $P(x)$, then $P'(x)$ must equal 0 in at least $n - 1$ places. However, $P'(x)$ is a quadratic polynomial and thus can equal 0 at most twice. If you start with a fourth degree polynomial, it's derivative has at most three roots. Hence the fourth degree can have at most four. Clearly, this can be continued for fifth degree, sixth degree, etc. Thus, by induction, an n^{th} degree polynomial has at most n roots.