Homework for Math 152H-1 September 13

Squeeze Theorem: If $f(x) \leq h(x) \leq g(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$ then $\lim_{x\to a} h(x) = L$. Also if $f(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \infty$ then $\lim_{x\to a} h(x) = \infty$.

This also works for right and left hand limits and for limits at infinity. Here are some problems for using the squeeze theorem.

1. Suppose $\sqrt{2-x^2} \leq f(x) \leq \sqrt{5-4x^2}$ for all x in $[-\sqrt{2}, \sqrt{2}]$. What is $\lim_{x\to 1} f(x)$?

2. $\lim_{x\to\infty} \dfrac{cos\ 3\,x}{\sqrt{x}}$

3. If all you know is that the values of $f(x)$ will lie between x^2+x and x^3+x^2, are there any values for a where you can compute $\lim_{x\to a} f(x)$? you might want to draw a graph!

These problems have to do with asymptotics and rates of growth.

5. You may remember that a polynomial $P(x) = A\,x^3 + B\,x^2 + C\,x + D$ can cross the x-axis once, twice, or three times, but never 0 times and never more than three times. To show that it never crosses 0 times we will need two steps. We take the first today. Calculate

$$\lim_{x\to\infty} \frac{P(x)}{x^3} \qquad\qquad \lim_{x\to-\infty} \frac{P(x)}{x^3}$$

What do these limits tell you about the graph of $P(x)$ compared to x^3? What else do we need to know about the graph of $P(x)$ to conclude that it must cross the x axis at least once. (Later we will see why there cannot be more than three roots, but if you know how to find maxima/minima you can try doing it now).

6. Physicists (often) do things like: our angle θ is at most 0.02, so we'll just remove that $tan\,\theta$ which makes the equations so difficult and replace it with θ. Why is this (almost) justifiable? What is it that they're not checking too carefully?

Here is the $\epsilon\delta$-definition of a limit (see also section 2.3):

Definition: $\lim_{x\to a} f(x) = L$ if for each $\epsilon > 0$ there is a $\delta > 0$ so that $\left|f(x) - L\right| < \epsilon$ when $x \in (a-\delta, a) \cup (a+\delta, a)$.

7. You have been asked to manufacture flat steel squares with an area 9. Recognizing that machines don't align correctly, the purchaser will accept anything within 0.01 of this value. What accuracy should you ensure in cutting the sides to comply with the order? Why is this problem in the definition of the limit section?

8. For a line $f(x) = mx + b$ you can always choose $\delta = \frac{\epsilon}{m}$ regardless of a. Verify this, and use a graph to explain why the slope appears in this way.

9. For $f(x) = \frac{x}{2x-2}$, find δ for a given $\epsilon > 0$ so that the definition above guarantess $\lim_{x\to 2} f(x) = 1$ (Assume that x is between $\frac{3}{2}$ and $\frac{5}{2}$).

10. Write a quantitative definition for $\lim_{x\to\infty} f(x) = L$ (think about the definition for sequences where $n \to \infty$ is now $x \to \infty$). Write a quantitative definition for $\lim_{x\to a} f(x) = \infty$ (again try modelling this off what it means for a sequence to diverge to ∞)