Solutions for September 13 Homework

1. Suppose $\sqrt{2-x^2} \le f(x) \le \sqrt{5-4x^2}$ for all x in $[-\sqrt{2}, \sqrt{2}]$. What is $\lim_{x\to 1} f(x)$?

The far right and left sides of the inequality go to $\sqrt{1}$ as $x \to 1$. Thus $\lim_{x\to 1} f(x) = 1$.

2. $\lim_{x\to\infty} \frac{cos\ 3\,x}{\sqrt{x}}$

As $x \to \infty$, we still have $-1 \le cos\ 3\,x \le 1$, hence $\frac{-1}{\sqrt{x}} \le \frac{cos\ 3\,x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$. As $x \to \infty$, the ends of this inequality go to 0. Thus $\lim_{x\to\infty} \frac{cos\ 3\,x}{\sqrt{x}} = 0$

3. If all you know is that the values of $f(x)$ will lie between $x^2 + x$ and $x^3 + x^2$, are there any values for a where you can compute $\lim_{x\to a} f(x)$? you might want to draw a graph!

$x^2 + x$ and $x^3 + x^2$ are equal when $x^2 + x = x^3 + x^2$, or $x^3 - x = 0$. Solving gives $x = \pm 1, 0$. When $a = \pm 1, 0$ we must have that $\lim_{x\to a} f(x)$ equals the value of either (hence both) of these functions (namely, 0 for -1, 0 for 0, and 2 for $+1$).

5. You may remember that a polynomial $P(x) = A\,x^3 + B\,x^2 + C\,x + D$ can cross the x-axis once, twice, or three times, but never 0 times and never more than three times. To show that it never crosses 0 times we will need two steps. We take the first today. Calculate

$$\lim_{x\to\infty} \frac{P(x)}{x^3} \qquad\qquad \lim_{x\to-\infty} \frac{P(x)}{x^3}$$

What do these limits tell you about the graph of $P(x)$ compared to x^3? What else do we need to know about the graph of $P(x)$ to conclude that it must cross the x axis at least once. (Later we will see why there cannot be more than three roots, but if you know how to find maxima/minima you can try doing it now).

The point is that both limits will give A. Assuming $A \neq 0$ this tells us that far from 0, $P(x)$ is shaped like Ax^3. However, that means that far away from 0 $P(x)$ is on one side of the x-axis for $x \to \infty$ and the other side for $x \to -\infty$ (because x^3 is like this). Now if we knew that $P(x)$ had no jumps or asymptotes or other problems, then the only way for this to happen is if the graph of $P(x)$ crosses the x-axis. In short, if we knew that $P(x)$ is continuous. Note that $x^3 + x^2$ is shaped like x^3 far from 0, but the x^2 term will still be huge."Shaped like" is a *qualitative* statement, not a *quantitative* one!!

6. Physicists (often) do things like: our angle θ is at most 0.02, so we'll just remove that $tan\,\theta$ which makes the equations so difficult and replace it with θ. Why is this (almost) justifiable? What is it that they're not checking too carefully?

$\lim_{\theta\to 0} \frac{tan\,\theta}{\theta} = 0$ and since θ limits to a finite value, this tells us that near enough to 0, $tan\,\theta$ and θ are very, very close. What they have not checked is how close to 0 we need to choose θ to ensure that $tan\,\theta$ is close enough to θ. To be fair to physicists, they're usually just trying to get an idea of whats going on, and these sorts of approximations are well understood. If they really had to be accurate they would check these details.

7. You have been asked to manufacture flat steel squares with an area 9. Recognizing that machines don't align correctly, the purchaser will accept anything within 0.01 of this value. What accuracy should you ensure in cutting the sides to comply with the order? Why is this problem in the definition of the limit section?

Let's write the inequalities, we want to know how when $|l - 3| < \delta$, which says that the side length l is within $\pm\delta$ of 3, implies that $|l^2 - 9| < 0.01$. In other words, we want to know the δ for $\epsilon = 0.01$ for the function x^2. That such a δ exists is guaranteed by $\lim_{x\to 3} x^2 = 9$. Here's how to do this

$$|x - 3| < \delta \Rightarrow |x^2 - 9| < \delta|x + 3|$$

. If we make $\delta|x + 3| < 0.01$ then we will be okay. This tells us that $x - 3 < -6 + \frac{0.01}{\delta}$. Now here's the tricky part: we choose $0 < \delta < -6 + \frac{0.01}{\delta}$, i.e. $\delta^2 + 6\delta - 0.01 < 0$. This works for $0 < \delta < 0.001662$.

When you have a calculator, of course, you can just calculate $\sqrt{9.01} = 3.001666204...$ and $\sqrt{8.99} = 2.99833287..$ and see that as long as you are within 0.00166, you will be okay. The point of the inequalities approach is to show how it is related to limits.

8. For a line $f(x) = mx + b$ you can always choose $\delta = \frac{\epsilon}{|m|}$ regardless of a. Verify this, and use a graph to explain why the slope appears in this way.

I'll just explain the point. Recall that the slope is the ratio of a change in y to a change in x. So the function scales the distance between two x-values by a factor of $|m|$, when you look at the distance between the corresponding y-values. If you want $f(x)$ to be within ϵ of $f(a)$ you need to choose x-values within $\epsilon/|m|$ to account for the expansion or contraction. Alternately, you can solve:

$$\big|(mx + b) - (ma + b)\big| < \epsilon \Leftrightarrow |m|\big|x - a\big| < \epsilon \Leftrightarrow \big|x - a\big| < \frac{\epsilon}{|m|}$$

9. For $f(x) = \frac{x}{2x-2}$, find δ for a given $\epsilon > 0$ so that the definition above guarantess $\lim_{x \to 2} f(x) = 1$ (Assume that x is between $\frac{3}{2}$ and $\frac{5}{2}$).

Consider $\big|\frac{x}{2x-2} - 1\big| < \epsilon$. This is the same as $\big|2 - x\big| < 2\epsilon\big|x - 1\big|$ after some algebra. But $\big|x - 1\big|$ is at least $\frac{1}{2}$ for values of x near 2. So if we choose $\delta = 2\epsilon \cdot \frac{1}{2} = \epsilon$, then $\delta < 2\epsilon\big|x - 1\big|$ and we've seen that this implies that $\big|\frac{x}{2x-2} - 1\big| < \epsilon$.

10. Write a quantitative definition for $\lim_{x \to \infty} f(x) = L$ (think about the definition for sequences where $n \to \infty$ is now $x \to \infty$). Write a quantitative definition for $\lim_{x \to a} f(x) = \infty$ (again try modelling this off what it means for a sequence to diverge to ∞)

See sections 2.4 and 2.5 in the book.