Reading for Math 152H-1 September 15

We have said that $\lim_{x\to a} f(x) = L$ means that "near" a, $f(x)$ should be arbitrarily well approximated by L (at least if we are willing to get closer to a). Now consider the function:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \text{ is irrational} \\ \frac{1}{q} & \text{if } 0 \leq x = \frac{p}{q} \text{ is a reduced rational number} \end{cases}$$

We are going to show that $\lim_{x\to a} f(x) = 0$ for every $a > 0$. What we need to see is that for any sequence of rational numbers, distinct from a, but converging to a, the denominators necesarily go to infinity. Why is this true? Well consider the rationals in $[0, 1]$ with denominators $1, 2, 3, 4$ and 5, for example. There are only finitely many of them, and the reduced ones are thus a minimal distance apart. Obviously, this is still true if the denominators are restricted to $1, 2, 3, \ldots, M$, although the minimal distance is much smaller. Nevertheless, to converge to any number without equalling it, we cannot use a finite set of numbers. Thus, at some point in the sequence the denominators of the rational terms must get bigger than M. After that point, the value of the function is 0 if the term in the sequence is irrational and $< \frac{1}{M}$ if the term is rational. But this implies that $\lim_{x\to a} f(x) = 0$ for all a.

This takes a little time to get one's mind around. Nevertheless, this says that near any point $f(x)$ is well-approximated by 0, even near a rational point. How can this be? Remember we are not saying anything about what happens at the rational, only in an interval $(a - \delta, a) \cup (a, a + \delta)$ near it. There it can be shown that the irrationals are "much denser" than the rationals, and those rationals which do appear have very large denominators. Nevertheless, this isn't quite what we would like. What we would ideally like is to find limits by plugging into the function. Failing that, we will allow nice failures: jumps, infinite asymptotes, and missing points. Nevertheless, we really, really would like to be able to plug into the function.

The property that makes this possible is called **continuity**. Intuitively it means that when we draw a graph, we don't lift our pencil. More specifically, we don't lift our pencil on the domain of the function. Mathematically, this becomes:

Definition: A function $f(x)$ is continuous at a if

1. a is in the domain of $f(x)$ (we don't allow this point to be missing or the graph to have an asymptote)
2. $\lim_{x\to a} f(x)$ exists (no jumps or $sin\,\frac{1}{x}$ misbehaviour)
3. $\lim_{x\to a} f(x) = f(a)$ (we can evaluate the limit by plugging a into the function)

A function is continuous if it is continuous at each point in its domain. For example, $f(x) = \frac{1}{x}$ is continuous because 0 is not in its domain. It is not continuous *at* 0, though. We will require the domains of our functions to be unions of intervals (otherwise we can get some odd things). When we want to broaden our class of functions, we'll generally ask that they be piecewise continous, i.e. continuous except at a finite number of jumps. Finally, if $f(x)$ is defined on $[a, b]$ we can talk about right-hand continuity at a by requiring a to be in the domain of f and $\lim_{x\to a^+} f(x) = f(a)$.

Furthermore, it is continuity that allows us to compute $\lim_{x\to 1} \sqrt{x+2} = \sqrt{3}$. If $g(x)$ is continuous at L (as $\sqrt{x}$ will be near 3) and $\lim_{x\to a} f(x) = L$ (no continuity required!) we have $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)) = g(L)$. In other words, we can take limits inside continuous functions (but not unless the function is continuous).

For evaluating limits this is all quite nice, but it raises two questions:

1. How can we tell when a function is continuous?
2. Why does the mathematical definition correspond to the intuitive notion of continuity?

The answer to the former question is to use the definition of the limit. Fortunately, there are faster ways for most functions we will use. These are contained in the following facts. We assume that both $f(x)$ and $g(x)$ are continuous at a, then the follwoing are also continuous at a

1. $f(x) \pm g(x)$
2. $f(x)g(x)$
3. $\frac{f(x)}{g(x)}$ when $g(a) \neq 0$
4. $C \cdot f(x)$
5. if $h(x)$ is continuous at $f(a)$ then $h(f(x))$ is continuous at a.

The first four follow from the properties of limits. The last one is new. Now we combine this with the fact that $|\cdot|$, x^l, $sin\ x$, $cos\ x$ are all continuous to see that many more complicated functions are continuous.

For example, where is $\sqrt{\frac{(x-1)(x-2)}{x-3}}$ continuous? First, it can only be continuous on its domain: we need $x \neq 3$ and $\frac{(x-1)(x-2)}{x-3} \geq 0$. The domain is $[1, 2] \cup (3, \infty)$. However at each of these points each of $x - 1$, $x - 2$, and $x - 3$ is continuous using the rules above. Thus the fraction $\frac{(x-1)(x-2)}{x-3}$ is continuous. And since $\sqrt{x} = x^{\frac{1}{2}}$ is continuous wherever it is defined so will be the composition $\sqrt{\frac{(x-1)(x-2)}{x-3}}$.

A partial answer to the second question is contained in the following two facts:

1. If $f(x)$ is continuous on $[a, b]$ then $f(x)$ attains a maximum and a minimum on this set.
2. If $f(x)$ is continuous on $[a, b]$ and $f(a) < C < f(b)$ then there is a point c where $f(c) = C$

See section 2.6 for continuity