Homework for Math 152H-1 September 18

Reading: Read section 2.7

Homework:

(1) First, let's give a solution to the homework problem from last time about whether we can extend

$$f(x) = \frac{sin(x^2 - x - 2)}{x^2 - 5x + 3}$$

to be continuous anywhere it is not currently defined. The domain of $f(x)$ is everywhere but the solutions of $x^2 - 5x + 3 = 0$. Solving we find that the function is defined everywhere but $x = \frac{5 \pm \sqrt{13}}{2}$. It is also continuous except at those points. Can we extend the function to these points and make the result continuous? No, we cannot. Since $x^2 - x - 2$ is not an integer multiple of π for $x = \frac{5 \pm \sqrt{13}}{2}$, the numerator does not go to 0 as the denominator goes to zero. Thus there are vertical asymptotes at $\frac{5 \pm \sqrt{13}}{2}$. Since the limit of $f(x)$ as $x \to \frac{5 \pm \sqrt{13}}{2}$ does not exist, there is no way to extend and get a continuous function. Now try the same problem, but with a different function (one without a typo):

Find the domain of

$$g(x) = \frac{sin(x^2 - 1)}{x^2 - x}$$

Where is this function continuous? Can we extend the function, by defining it at points currently not in the domain, so that it is continuous at any of these points?

(2) Let $f(x) = x^3 - x$. Use the Intermediate Value Theorem to answer "how many solutions are there to $f(x) = \frac{1}{4}$"? (There are at most three, but is the actual number 1,2, or 3? Try plugging in values that are integers or integers divided by 2. Don't forget to state why *all* the assumptions of the theorem are satisfied).

(3) How many solutions are there to $sin\, x = \frac{x}{5\pi}$ with $x \geq 0$? (Draw a graph!!)

(4) Let R be a connected region in the plane such that every point in R has $y > 0$. Assuming that R has a smooth boundary and an area of 1, let $A(v)$ be the area of that part of R consisting of points with x-coordinate less than v. Explain why $A(v)$ is a continuous function of v, and show that there is a point with $A(v) = \frac{1}{2}$. How many such points are there? If we change the assumptions on R, can we get more than one such point?

(5) You are given a region in the plane with area 1 (but it is otherwise complicated). It sits on one side of a line while on the other side there is a circle of area 1. Is there a line which divides both the region and the circle in half? (think diameters)

(6) A rubber string is laid out from 0 to 1 on the number line. A friend pulls at both ends, stretching the ends beyond their original positions left and right but with different amounts of force. Is it always the case that some point on the string will wind up at the same number as where it started? (Let $f(x)$ be the new value on the number line for the point starting at x, and consider the difference between x and $f(x)$).

(7) We have used the Intermediate Value Theorem to show that there are solutions to $f(x) = C$ under certain circumstances. In fact, we can be more specific about the location of a solution, as well. Let's assume $f(a) < C < f(b)$ for $f(x)$ continuous on $[a, b]$. Now consider the mid-point $m = \frac{a+b}{2}$.

(a) Either $f(m) = C$ or $f(m) < C$ or $f(m) > C$. For each of these possibilities, in what closed interval should we look for a solution to $f(x) = C$?

(b) Now you have a new closed interval. It too has a mid-point, what are the possibilities for $f(x)$ at this new mid-point?

(c) Repeat (a) on the smaller interval. If we keep repeating this process without finding a point where $f(x) = C$, we get smaller and smaller closed intervals, each one half the size and included in the previous one. There is only one point in all the intervals. What is true of this point?

(8) You have a wire tetrahedron (the edges of a pyramid with a triangular base). Show that there are at least four points, on different edges, where the temperatures are the same. (**Hint:** Label the 4 vertices by A, B, C, and D. There are six edges. For edge AB, let $f_{AB}(t)$ be the temperature at the point a fraction t along the way from A to B (A is $t = 0$, B is $t = 1$ and $t = \frac{1}{2}$ corresponds to the mid-point). Now graph this function on $[0, 1]$. Do this for each of the six edges, keeping track of the vertices, and use the intermediate value theorem).