Homework for Math 152H-1 September 18

Reading: Read section 2.7

Homework:

(1) Find the domain of
$$g(x) = \frac{sin(x^2-1)}{x^2-x}$$
Where is this function continuous? Can we extend the function, by defining it at points currently not in the domain, so that it is continuous at any of these points?

The function has domain $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$. Since $sin\,x$ is continuous everywhere, and $x^2 - x$ and $x^2 - 1$ are polynomials (hence continuous everywhere), the function is continuous except where the denominator is zero. This follows from the composition and quotient of continuous functions being continuous except where the denominator is zero (section 2.6). So the two points where we might try to define $g(x)$ to extend it continuously are at 0 and 1. To do this we consider $\lim_{x\to 0} g(x)$ and $\lim_{x\to 1} g(x)$. As $x \to 0$, $x^2 - 1 \to -1$, so the numerator tends to $sin(-1)$ while the denominator tends to 0. Without further work, we don't know if the limit is infinite or undefined, but we do know that it's not a finite number. Thus we cannot define $g(x)$ at 0 in a way which will make the resulting function continuous. On the other hand, as $x \to 1$ both the numerator and denominator go to zero and we can use one of our standard tricks:
$$\lim_{x\to 1} \frac{sin(x^2-1)}{x^2-x} = \lim_{x\to 1} \frac{sin(x^2-1)}{x^2-1} \cdot \frac{x^2-1}{x^2-x} = 1 \cdot \lim_{x\to 1} \frac{x+1}{x} = 2$$
Thus,
$$h(x) = \begin{cases} \frac{sin(x^2-1)}{x^2-x} & x \neq 1, 0 \\ 2 & x = 0 \end{cases}$$
now has domain $(-\infty, 0) \cup (0, \infty)$, and is continuous at every point in this domain. It equals $g(x)$ where both are defined.

(2) Let $f(x) = x^3 - x$. Use the Intermediate Value Theorem to answer "how many solutions are there to $f(x) = \frac{1}{4}$"? (There are at most three, but is the actual number 1,2, or 3? Try plugging in values that are integers or integers divided by 2. Don't forget to state why *all* the assumptions of the theorem are satisfied).

(3) How many solutions are there to $sin\,x = \frac{x}{5\pi}$ with $x \geq 0$? (Draw a graph!!)

(4) Let R be a connected region in the plane such that every point in R has $y > 0$. Assuming that R has a smooth boundary and an area of 1, let $A(v)$ be the area of that part of R consisting of points with x-coordinate less than v. Explain why $A(v)$ is a continuous function of v, and show that there is a point with $A(v) = \frac{1}{2}$. How many such points are there? If we change the assumptions on R, can we get more than one such point?

Once we now that $A(v)$ is continuous, then we will be okay. For v very negative $A(v) = 0$ and for v very positive $A(v) = 1$ (more precisiely, if your region is unbounded you should use limits as $v \to -\infty$ and $v \to \infty$, but we'll assume the region is bounded). Then there must be a point in between where $A(v) = \frac{1}{2}$. If the region is allowed to be disconnected, we could have half the region in one part of the plane and half the region in another part, and every v in between would have this property. For a connected region the question is more difficult. For "nice" regions there should only be one, but we can contrive situations where this isn't true. (It depends on what you mean by "region"). So why is $A(v)$ continuous? Consider $A(v + \delta) - A(v)$ where $|\delta|$ is small. This is the area of R in a thin vertical strip. Surely as the strip narrows, this area goes to 0, especially as the total area of R is assumed to be 1. (you need the smmoth boundary to be sure).

(5) You are given a region in the plane with area 1 (but it is otherwise complicated). It sits on one side of a line while on the other side there is a circle of area 1. Is there a line which divides both the region and the circle in half? (think diameters)

> Look at the diameters of the circle. All of these divide the circle in half. There is one for every angle between 0 and 2π, at which point we catch up with ourselves. The condition for the line dividing the circle from the region tells you that the region is intersected by the diameter only for a smaller interval of angles. (It doesn't enclose the circle!). Now take $A(\theta)$ to be the area of the region lying to the right of the diameter with angle θ, when we are looking out from the center of the circle. For some θ this is 0. As θ increases, we arrive at some θ where the entire region lies on the right. For the same reason as in the previous problem, the function is continuous. Hence there must be some place in between where exactly half the area of the region is on one side of the diameter, and the other half is on the other side.

(6) A rubber string is laid out from 0 to 1 on the number line. A friend pulls at both ends, stretching the ends beyond their original positions left and right but with different amounts of force. Is it always the case that some point on the string will wind up at the same number as where it started? (Let $f(x)$ be the new value on the number line for the point starting at x, and consider the difference between x and $f(x)$).

> We think of the string as being an interval on the x-axis. To start it is $[0,1]$. To end it is $[a,b]$ where $a < 0 < 1 < b$, since it has been stretched at both ends. The point at x is taken to $f(x)$ by the stretching. $f(x)$ is continuous since points near each other will still be near each other after stretching (no cutting – corresponding to a jump – can occur). Now $f(x) - x = 0$ precisely when the point winds up at the same location as it started. Let $g(x) = f(x) - x$. Then $g(0) = f(0) - 0 = a < 0$ and $g(1) = f(1) - 1 = b - 1 > 0$, but since $f(x)$ and x are both continuous so is $f(x) - x$. Thus there is some x value where $g(x) = 0$ and this point winds up where it started.

(7) We have used the Intermediate Value Theorem to show that there are solutions to $f(x) = C$ under certain circumstances. In fact, we can be more specific about the location of a solution, as well. Let's assume $f(a) < C < f(b)$ for $f(x)$ continuous on $[a,b]$. Now consider the mid-point $m = \frac{a+b}{2}$.

> (a) Either $f(m) = C$ or $f(m) < C$ or $f(m) > C$. For each of these possibilities, in what closed interval should we look for a solution to $f(x) = C$?
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> (b) Now you have a new closed interval. It too has a mid-point, what are the possibilities for $f(x)$ at this new mid-point?
>
> (c) Repeat (a) on the smaller interval. If we keep repeating this process without finding a point where $f(x) = C$, we get smaller and smaller closed intervals, each one half the size and included in the previous one. There is only one point in all the intervals. What is true of this point?

(8) You have a wire tetrahedron (the edges of a pyramid with a triangular base). Show that there are at least four points, on different edges, where the temperatures are the same. (**Hint:** Label the 4 vertices by A, B, C, and D. There are six edges. For edge AB, let $f_{AB}(t)$ be the temperature at the point a fraction t along the way from A to B (A is $t = 0$, B is $t = 1$ and $t = \frac{1}{2}$ corresponds to the mid-point). Now graph this function on $[0,1]$. Do this for each of the six edges, keeping track of the vertices, and use the intermediate value theorem).

> To illustrate the point, assume that each of the vertices has a different temperature. Let's say A is the hottest, and D is the coldest, with the others in order alphabetically. Now consider the edges between $\{A, B\}$ and $\{C, D\}$. There are four edges and thus four functions f_{CA}, etc. We take these functions so that t measures from the colder to the hotter. Each of these function is continuous. If we choose a temperature between that of B and C, then each of the four functions must cross this temperature as their endpoints at 0 have temperature below that amount, and the endpoints for $t = 1$ have temperature above. Thus there are at least four points where the temperatures are the same.