Homework and Pre-Class reading for Math 152H-1 August 30

Homework:

1. Review section 1.1, pgs 3-7 on absolute values, and solving inequalities. You should be able to work problems 9, 17, 27, and 39 in section 1.1. The answers are in the back of the book.

2. Prove that $\sqrt{6}$ is irrational (follow the proof for $\sqrt{2}$). Once you've done that try showing that $\sqrt{2}+\sqrt{3}$ is irrational (this is a little trickier, but start by trying to mimic the same proof).

3. Use the trick from class to prove that
$$1+p+p^2+\ldots p^n = \frac{1-p^{n+1}}{1-p}$$
If $|p|<1$ find what happens as $n\to\infty$. This should verify the formula from class.

4. Consider the following argument. Let $a_n = 1+2+2^2+\ldots+2^n$. Let L be the limit of a_n as $n\to\infty$. Then we have
$$L = 1+2+2^2+2^3+\ldots = 1+2(1+2+2^2+\ldots) = 1+2L$$
Solving we get $L=-1$. This must be incorrect, since the sum of positive numbers is positive. What have we done that's wrong?

5. Let $b_n = \frac{1}{2}\big(b_{n-1}+\frac{1}{b_{n-1}}\big)$ and $b_1 = 2$. Here are the first few terms to get you started: $b_1 = 2$, $b_2 = \frac{1}{2}\big(2+\frac{1}{2}\big) = \frac{5}{4}$, $b_3 = \frac{1}{2}\big(\frac{5}{4}+\frac{1}{\frac{5}{4}}\big)$, ... Convince yourself that this is a sequence of rational numbers. I assure you that the sequence is convergent (see below). To find the limit, ask yourself what happens to each side of the equation as $n\to\infty$. You should get an equation for L, which you can then solve.

Reading:

First let me answer the questions we didn't get to:

1. Which of the following sequences is non-decreasing? non-increasing? bounded?

   (a) $a_n = 2+\dfrac{(-1)^n}{n}$

   The sequence is bounded between 1 and 3 since we are always adding or subtracting a number less than 1. However, the terms alternate between being bigger than 2 and less than 2, so they can't be just getting bigger or smaller. Thus the sequence is neither increasing or decreasing.

   (b) $a_n = a_{n-1}+\sqrt{n}$ and $a_1 = 1$ (calculate a few terms first, each term is found from its predecessor)

   Calculate the first few terms: $a_1 = 1$, $a_2 = a_1+\sqrt{2} = 1+\sqrt{2}$, $a_3 = a_2+\sqrt{3} = 1+\sqrt{2}+\sqrt{3}$. Following this pattern $a_n = 1+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}$. This sequence is bounded below, but not above since we get to the next term by adding something bigger than 1. It is non-decreasing.

2. Is a bounded sequence of real numbers necessarily convergent? No, $a_n = (-1)^n$ is divergent but bounded. What if it is non-decreasing? (see below)

3. Is every real number the limit of a sequence of distinct rational numbers? (distinct means no term in the sequence is equal to any other. To do this think about what you know about decimal expansions of real numbers, but be careful about terminating decimals!) We've seen how to get every real number as a sequence of rationals: $\pi = 3.1415\ldots$ so we take the sequence 3, 3.1, 3.14, ..., tacking on a single additional place in the decimal each time. Are these all distinct? Consider 1.1. If we used this procedure we'd get 1, 1.1, 1.10, ..., but 1.1 is the same number as 1.10 and 1.100. All the additional terms will be equal to 1.1. To get distinct terms recall that $1.1000\overline{0}\ldots$ is the same number as $1.099999\overline{9}$. To see this consider the following
$$1.099\overline{9} = 1+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+\ldots = 1+\frac{9}{100}\big(1+\frac{1}{10}+\frac{1}{10^2}+\ldots\big)$$
This last can be summed using $1+p+p^2+\ldots = \frac{1}{1-p}$ when $|p|<1$, using $p=\frac{1}{10}$. When we do that (you should check me) we get 1.1

The key property that distinguishes the reals from the rationals can be summarized in the following fact:

1. An increasing sequence of real numbers that is also bounded above converges to a real number. Likewise a decreasing sequence of real numbers that is bounded below converges to a real number. To see that this is not true of the rationals, i.e. that there are bounded increasing sequences of rationals which do not converge to a *rational* number, just look at the sequence $1, 1.4, 1.41, 1.414, \ldots$ found from the decimal expansion of $\sqrt{2}$. Try to convince yourself that it is true of the reals. (it's hard to prove)

This is very useful as it gives us a way to prove that a sequence converges, rather than merely guessing. The next few examples will show some appplications of this. We start with the sequence

$$u_n = \frac{2n-7}{3n+2}$$

This is bounded above by 2, since $\frac{2n-7}{3n+2} < 2$ when $2n - 7 < 6n + 4$, i.e. $-11 < 4n$, which will be true when $n > 0$. Now we show that it is increasing. To do that, we check when $u_{n+1} > u_n$. This requires us to find when

$$\frac{2(n+1)-7}{3(n+1)+2} > \frac{2n-7}{3n+2} \qquad \Rightarrow \qquad \frac{2n-5}{3n+5} > \frac{2n-7}{3n+2}$$

We multiply by the denominators (which are positive) to get $(2n-5)(3n+2) > (2n-7)(3n+5)$. This reduces to $6n^2 - 11n - 10 > 6n^2 - 11n - 35$, which is the same as $-11 > -35$ and thus is always true. Hence the sequence has a limit (which we already know how to find). However, this property can also be used to show that $f_{n+1} = \sqrt{3f_n}$, $f_1 = 1$ has a limit and converges to 3 (which is not obvious to me). We'll do this in class.

The main use of this property is to guarantee that a sequence converges. But what do we do about sequences which are not increasing or decreasing? For example, how do we *prove* that $2 + \frac{(-1)^n}{n}$ converges to 2 or that $n - \frac{1}{n}$ diverges to infinity instead of just bouncing around. In general, we first have to guess what happens and guess the limit if we think there is one. We then check our guess using the **definition** of convergence. Ok, we've talked about convergence already and we have an intuitive idea of what it means: as $n \to \infty$, the values of a_n should "get closer, and closer" to some specific value L. Why do we need a definition? Because "closer and closer" is too vague; when we have complicated sequences we need something better. In particular, when we want to understand the properties of limits, we need the definition. So here it is:

Definition: We say that a sequence $\{a_n\}$ converges to a limit L when for each $\epsilon > 0$, there is a number N_ϵ such that when $n \geq N_\epsilon$ we have $|a_n - L| < \epsilon$.

You might say that this doesn't look any better, but it is. In particular, it gives specific mathematical properties to be checked. Furthermore, we have ways to manipulate inequalities, so checking convergence gets turned into algebraic calculation rather than guessing. Note, however, that you need to have an L in mind to be able to use this. What the definition really does is say, how close do you want to be to L? within ϵ? then you need only wait until $n > N_\epsilon$ and *all* the subsequent terms will be within ϵ of L, i.e. if we choose ϵ very small, then they will be very close to L. Now by choosing ϵ even smaller, and waiting until the new N_ϵ, we can force the terms to be even closer.

Here's an example: Lets shown that $\frac{1}{n}$ converges to 0. Given ϵ, we choose $N_\epsilon = \frac{1}{\epsilon}$. Then when $n > N$, $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| < \frac{1}{N} = \epsilon$. So for each ϵ we have found an N_ϵ and verified the inequality. Therefore 0 is the limit.

I mentioned on the first day that we could talk about an infinite list by giving a rule for calculating each term (rather than wasting away our lives actually writing down the list). This definition builds upon that. Instead of having to know the entire sequence to determine whether we are getting "closer, and closer", we need only verify that after some point all the terms are squeezed near the limit by the inequality. Here's another example:

Let $b_n = \frac{3n+1}{2n-1}$. We will show that $b_n \to \frac{3}{2}$. To do this, we need to know how to choose N_ϵ for a given ϵ. For that we first compute how far away the terms are from the limit:

$$\left|\frac{3n+1}{2n-1} - \frac{3}{2}\right| = \frac{5}{2n-1}$$

Now when $\frac{5}{2n-1} < \epsilon$ we will be okay. This occurs when $2n - 1 > \frac{5}{\epsilon}$ or $n > \frac{1}{2} + \frac{5}{2\epsilon}$. We thus take N_ϵ to be $\frac{1}{2} + \frac{5}{2\epsilon}$.