Integral Closures of Ideals in Completions of Regular Local Domains

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1 Abstract

In this paper we exhibit an example of a three-dimensional regular local domain $(A, \mathbf{n})$ having a height-two prime ideal P with the property that the extension $P\widehat{A}$ of P to the $\mathbf{n}$-adic completion $\widehat{A}$ of A is not integrally closed. We use a construction we have studied in earlier papers: For $R = k[x, y, z]$, where k is a field of characteristic zero and x, y, z are indeterminates over k, the example A is an intersection of the localization of the power series ring $k[y, z][[x]]$ at the maximal ideal (x, y, z) with the field $k(x, y, z, f, g)$, where f, g are elements of $(x, y, z)k[y, z][[x]]$ that are algebraically independent over $k(x, y, z)$. The elements f, g are chosen in such a way that using results from our earlier papers A is Noetherian and it is possible to describe A as a nested union of rings associated to A that are localized polynomial rings over k in five variables.

2 Introduction and Background

We are interested in the general question: What can happen in the completion of a 'nice' Noetherian ring? We are examining this question as part of a project of constructing Noetherian and non-Noetherian integral domains using power series rings. In this paper as a continuation of that project we display an example of a three-dimensional regular local domain $(A, \mathbf{n})$ having a height-two prime ideal P with the property that the extension $P\widehat{A}$ of P to the $\mathbf{n}$-adic completion $\widehat{A}$ of A is not integrally closed. The ring A in the example is a nested union of regular local domains of dimension five.

Let I be an ideal of a commutative ring R with identity. We recall that an element $r \in R$ is *integral over* I if there exists a monic polynomial $f(x) \in R[x]$, $f(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$, where $a_i \in I^i$ for each $i, 1 \leq i \leq n$ and $f(r) = 0$. Thus $r \in R$ is integral over I if and only if $IJ^{n-1} = J^n$, where $J = (I, r)R$ and n is some positive integer. (Notice that $f(r) = 0$ implies $r^n = -\sum_{i=1}^n a_i r^{n-i} \in IJ^{n-1}$ and this implies $J^n \subseteq IJ^{n-1}$.) If $I \subseteq J$ are ideals and $IJ^{n-1} = J^n$, then I is said to be a *reduction* of J. The *integral closure* $\overline{I}$ of an ideal I is the set of elements of R integral over I. If $I = \overline{I}$, then I is said to be *integrally closed*. It is well known that $\overline{I}$ is an integrally closed ideal. An ideal is integrally closed if and only if it is not a reduction of a properly bigger ideal. A prime ideal is always integrally closed. An ideal is said to be *normal* if all the powers of the ideal are integrally closed.

We were motivated to construct the example given in this paper by a question asked by Craig Huneke as to whether there exists an analytically unramified Noetherian local ring $(A, \mathbf{n})$ having an integrally closed ideal I for which $I\widehat{A}$ is not integrally closed, where $\widehat{A}$ is the $\mathbf{n}$-adic completion of A. In Example **??**, the ring A is a 3-dimensional regular local domain and $I = P = (f, g)A$ is a prime ideal of height two. Sam Huckaba asked if the ideal of our example is a normal ideal. The answer is 'yes'. Since f, g form a regular sequence and A is Cohen-Macaulay, the powers P^n of P have no embedded associated primes and therefore are P-primary [**?**, (16.F), p. 112], [**?**, Ex. 17.4, p. 139]. Since the powers of the maximal ideal of a regular local domain are integrally closed, the powers of P are integrally closed. Thus the Rees algebra $A[Pt] = A[ft, gt]$ is a normal domain while the Rees algebra $\widehat{A}[ft, gt]$ is not integrally closed.

A problem analogous to that considered here in the sense that it also deals with the behavior of ideals under extension to completion is addressed by Loepp and Rotthaus in [**?**]. They construct nonexcellent local Noetherian domains to demonstrate that tight closure need not commute with completion.

Remark 2.1 Without the assumption that A is analytically unramified, there exist examples even in dimension one where an integrally closed ideal of A fails to extend to an integrally closed ideal in $\widehat{A}$. If A is reduced but analytically ramified, then the zero ideal of A is integrally closed, but its extension to $\widehat{A}$ is not integrally closed. An example in characteristic zero of a one-dimensional Noetherian local domain that is analytically ramified is given by Akizuki in his 1935 paper [**?**]. An example in positive characteristic is given by F.K. Schmidt [**?**, pp. 445-447]. Another example due to Nagata is given in [**?**, Example 3, pp. 205-207]. (See also [**?**, (32.2), p. 114].)

Remark 2.2 Let R be a commutative ring and let R' be an R-algebra. We list cases where extensions to R' of integrally closed ideals of R are again integrally closed. The R-algebra R' is said to be *quasi-normal* if R' is flat over R and the following condition $(N_{R,R'})$ holds: If C is any R-algebra and D is a C-algebra in which C is integrally closed, then also $C \otimes_R R'$ is integrally closed in $D \otimes_R R'$.

1. By [**?**, Lemma 2.4], if R' is an R-algebra satisfying $(N_{R,R'})$ and I is an integrally closed ideal of R, then IR' is integrally closed in R'.

2. Let $(A, \mathbf{n})$ be a Noetherian local ring and let $\widehat{A}$ be the $\mathbf{n}$-adic completion of A. Since $A/\mathbf{q} \cong \widehat{A}/\mathbf{q}\widehat{A}$ for every $\mathbf{n}$-primary ideal $\mathbf{q}$ of A, the $\mathbf{n}$-primary ideals of A are in one-to-one inclusion preserving correspondence with the $\widehat{\mathbf{n}}$-primary ideals of $\widehat{A}$. It follows that an $\mathbf{n}$-primary ideal I of A is a reduction of a properly larger ideal of A if and only if $I\widehat{A}$ is a reduction of a properly larger ideal of $\widehat{A}$. Therefore an $\mathbf{n}$-primary ideal I of A is integrally closed if and only if $I\widehat{A}$ is integrally closed.

3. If A is excellent, then the map $A \to \widehat{A}$ is quasi-normal by [**?**, (7.4.6) and (6.14.5)], and in this case every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$.

4. If $(A, \mathbf{n})$ is a local domain and A^h is the Henselization of A, then every integrally closed ideal of A extends to an integrally closed ideal of A^h. This follows because A^h is a filtered direct limit of étale A-algebras [**?**, (iii), (i), (vii) and (ix), pp. 800- 801].

5. In general, integral closedness of ideals is a local condition. Suppose R' is an R-algebra that is *locally normal* in the sense that for every prime ideal P' of R', the local ring $R'_{P'}$ is an integrally closed domain. Since principal ideals of an integrally closed domain are integrally closed, the extension to R' of every principal ideal of R is integrally closed. In particular, if $(A, \mathbf{n})$ is an analytically normal Noetherian local domain, then every principal ideal of A extends to an integrally closed ideal of $\widehat{A}$.

6. If R is an integrally closed domain, then for every ideal I and element x of R we have $\overline{xI} = x\overline{I}$. If $(A, \mathbf{n})$ is analytically normal and also a

UFD, then every height-one prime ideal of A extends to an integrally closed ideal of $\widehat{A}$. In particular if A is a regular local domain, then $P\widehat{A}$ is integrally closed for every height-one prime P of A. If $(A, \mathbf{n})$ is a 2-dimensional regular local domain, then every nonprincipal integrally closed ideal of A has the form xI, where I is an $\mathbf{n}$-primary integrally closed ideal and $x \in A$. In view of item 2, every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ in the case where A is a 2-dimensional regular local domain.

7. Suppose R and R' are Noetherian rings and assume that R' is a flat R-algebra. Let I be an integrally closed ideal of R. The flatness of R' over R implies every $P' \in \text{Ass}\,(R'/IR')$ contracts in R to some $P \in \text{Ass}\,(R/I)$ [**?**, Theorem 23.2]. Since a regular map is quasi-normal, if the map $R \to R'_{P'}$ is regular for each $P' \in \text{Ass}\,(R'/IR')$, then IR' is integrally closed.

3 A non-integrally closed extension

In the construction of the following example we make use of results from [**?**]-[**?**].

Construction of Example 3.1 Let k be a field of characteristic zero and let x, y and z be indeterminates over k. Let $R := k[x, y, z]_{(x,y,z)}$ and let R^* be the (xR)-adic completion of R. Thus $R^* = k[y, z]_{(y,z)}[[x]]$, the formal power series ring in x over $k[y, z]_{(y,z)}$.

Let α and β be elements of $xk[[x]]$ which are algebraically independent over $k(x)$. Set

$$f = (y - \alpha)^2, \quad g = (z - \beta)^2, \quad \text{and } A = k(x, y, z, f, g) \cap R^*.$$

Then the (xA)-adic completion A^* of A is equal to R^* [**?**, Lemma 2.3.2, Prop. 2.4.4].

In order to better understand the structure of A, we recall some of the details of the construction of a nested union B of localized polynomial rings over k in 5 variables associated to A. (More details may be found in [**?**].)

Approximation Technique 3.2 With k, x, y, z, f, g, R and R^* as in (3.1), Write

$$f = y^2 + \sum_{j=1}^{\infty} b_j x^j, \qquad g = z^2 + \sum_{j=1}^{\infty} c_j x^j,$$

for some $b_j, \in k[y]$ and $c_j \in k[z]$. There are natural sequences $\{f_r\}_{r=1}^{\infty}$, $\{g_r\}_{r=1}^{\infty}$ of elements in R^*, called the r^{th} *endpiece*s for f and g respectively which "approximate" f and g. These are defined for each $r \geq 1$ by:

$$f_r := \sum_{j=r}^{\infty}(b_j x^j)/x^r, \qquad g_r := \sum_{j=r}^{\infty}(c_j x^j)/x^r.$$

For each $r \geq 1$, define B_r to be $k[x, y, z, f_r, g_r]$ localized at the maximal ideal generated by $(x, y, z, f_r - b_r, g_r - c_r)$. Then define $B = \bigcup_{r=1}^{\infty} B_r$. The endpieces defined here are slightly different from the notation used in [**?**]. Also we are using here a localized polynomial ring for the base ring R. With minor adjustments, however, [**?**, Theorem 2.2] applies to our setup.

THEOREM 3.3 *Let A be the ring constructed in (3.1) and let $P = (f, g)A$, where f and g are as in (3.1) and (3.2). Then*

1. *$A = B$ is a three-dimensional regular local domain that is a nested union of five-dimensional regular local domains.*

2. *P is a height-two prime ideal of A.*

3. *If A^* denotes the (xA)-adic completion of A, then $A^* = k[y, z]_{(y,z)}[[x]]$ and PA^* is not integrally closed.*

4. *If $\widehat{A}$ denotes the completion of A with respect to the powers of the maximal ideal of A, then $\widehat{A} = k[[x, y, z]]$ and $P\widehat{A}$ is not integrally closed.*

Proof: Notice that the polynomial ring $k[x, y, z, \alpha, \beta] = k[x, y, z, y-\alpha, z-\beta]$ is a free module of rank 4 over the polynomial subring $k[x, y, z, f, g]$ since $f = (y - \alpha)^2$ and $g = (z - \beta)^2$. Hence the extension

$$k[x, y, z, f, g] \to k[x, y, z, \alpha, \beta][1/x]$$

is flat. Thus item (1) follows from [**?**, Theorem 2.2].

For item (2), it suffices to observe that P has height two and that, for each positive integer r, $P_r := (f, g)B_r$ is a prime ideal of B_r. We have $f = xf_1 + y^2$ and $g = xg_1 + z^2$. It is clear that $(f, g)k[x, y, z, f, g]$ is a height-two prime ideal. Since B_1 is a localized polynomial ring over k in the variables $x, y, z, f_1 - b_1, g_1 - c_1$, we see that

$$P_1 B_1[1/x] = (xf_1 + y^2, xg_1 + z^2)B_1[1/x]$$

is a height-two prime ideal of $B_1[1/x]$. Indeed, setting $f = g = 0$ is equivalent to setting $f_1 = -y^2/x$ and $g_1 = -z^2/x$. Therefore the residue class ring $(B_1/P_1)[1/x]$ is isomorphic to a localization of the integral domain

$k[x,y,z][1/x]$. Since B_1 is Cohen-Macaulay and f, g form a regular sequence, and since $(x,f,g)B_1 = (x, y^2, z^2)B_1$ is an ideal of height three, we see that x is in no associated prime of $(f,g)B_1$ (see, for example [**?**, Theorem 17.6]). Therefore $P_1 = (f,g)B_1$ is a height-two prime ideal.

For $r > 1$, there exist elements $u_r \in k[x,y]$ and $v_r \in k[x,z]$ such that $f = x^r f_r + u_r x + y^2$ and $g = x^r g_r + v_r x + z^2$. An argument similar to that given above shows that $P_r = (f,g)B_r$ is a height-two prime of B_r. Therefore $(f,g)B$ is a height-two prime of $B = A$.

For items 3 and 4, $R^* = B^* = A^*$ by Construction **??** and it follows that $\widehat{A} = k[[x,y,z]]$. To see that $PA^* = (f,g)A^*$ and $P\widehat{A} = (f,g)\widehat{A}$ are not integrally closed, observe that $\xi := (y-\alpha)(z-\beta)$ is integral over PA^* and $P\widehat{A}$ since $\xi^2 = fg \in P^2$. On the other hand, $y - \alpha$ and $z - \beta$ are nonassociate prime elements in the local unique factorization domains A^* and $\widehat{A}$. An easy computation shows that $\xi \notin P\widehat{A}$. Since $PA^* \subseteq P\widehat{A}$, this completes the proof. □

REMARK 3.4 In a similar manner it is possible to construct for each integer $d \geq 3$ an example of a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height $h := d - 1$ such that $P\widehat{A}$ is not integrally closed. Indeed, let k be a field of characteristic zero and let $x, y_1, \ldots, y_h$ be indeterminates over k. Let $\alpha_1, \ldots, \alpha_h \in xk[[x]]$ be algebraically independent over $k(x)$. For each i with $1 \leq i \leq h$, define $f_i = (y_i - \alpha_i)^h$. Proceeding in a manner similar to what is done in (3.1) we obtain a d-dimensional regular local domain A and a prime ideal $P = (f_1, \ldots, f_h)A$ of height h such that the $y_i - \alpha_i \in \widehat{A}$. Let $\xi = \prod_{i=1}^{h}(y_i - \alpha_i)$. Then $\xi^h = f_1 \cdots f_h \in P^h$ implies ξ is integral over $P\widehat{A}$, but using that $y_1 - \alpha_1, \ldots, y_h - \alpha_h$ is a regular sequence in $\widehat{A}$, we see that $\xi \notin P\widehat{A}$.

4 Comments and Questions

In connection with Theorem **??** it is natural to ask the following question.

QUESTION 4.1 For P and A as in Theorem **??**, is P the only prime of A that does not extend to an integrally closed ideal of $\widehat{A}$?

COMMENTS 4.2 In relation to the example given in Theorem **??** and to Question **??**, we have the following commutative diagram, where all the maps shown are the natural inclusions:

$$
\begin{array}{ccccc}
B = A & \xrightarrow{\gamma_1} & A' := k(x,y,z,\alpha,\beta) \cap R^* & \xrightarrow{\gamma_2} & R^* = A^* \\
\delta_1 \Big\uparrow & & \delta_2 \Big\uparrow & & \psi \Big\uparrow \\
S := k[x,y,z,f,g] & \xrightarrow{\varphi} & T := k[x,y,z,\alpha,\beta] & = & T
\end{array}
\tag{1}
$$

Let $\gamma = \gamma_2 \cdot \gamma_1$. Referring to the diagram above, we observe the following:

1. The discussion in [**?**, bottom p. 668 to top p. 669] implies that [**?**, Thm. 3.2] applies to the setting of Theorem 3.3. By [**?**, Prop. 4.1 and Thm. 3.2], $A'[1/x]$ is a localization of T. By Theorem 3.3 and [**?**, Thm 3.2], $A[1/x]$ is a localization of S. Furthermore, by [**?**, Prop. 4.1] A' is excellent. (Notice, however, that A is not excellent since there exists a prime ideal P of A such that $P\widehat{A}$ is not integrally closed.) The excellence of A' implies that if $Q^* \in$ Spec A^* and $x \notin Q^*$, then the map $\psi_{Q^*} : T \to A^*_{Q^*}$ is regular [**?**, (7.8.3 v)].

2. Let $Q^* \in$ Spec A^* be such that $x \notin Q^*$ and let $\mathbf{q}' = Q^* \cap T$. By [**?**, Theorem 32.1] and Item 1 above, if $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ is regular, then $\gamma_{Q^*} : A \to A^*_{Q^*}$ is regular.

3. Let I be an ideal of A. Since A' and A^* are excellent and both have completion $\widehat{A}$, Remark **??**.3 shows that the ideals IA', IA^* and $I\widehat{A}$ are either all integrally closed or all fail to be integrally closed.

4. The Jacobian ideal of the extension $\varphi : S = k[x,y,z,f,g] \to T = k[x,y,z,\alpha,\beta]$ is the ideal of T generated by the determinant of the matrix
$$
\mathcal{J} := \begin{pmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} \\ \frac{\partial f}{\partial \beta} & \frac{\partial g}{\partial \beta} \end{pmatrix}.
$$
Since the characteristic of the field k is zero, this ideal is $(y-\alpha)(z-\beta)T$.

In Proposition **??**, we relate the behavior of integrally closed ideals in the extension $\varphi : S \to T$ to the behavior of integrally closed ideals in the extension $\gamma : A \to A^*$.

PROPOSITION 4.3 *With the setting of Theorem* **??** *and Comment* **??***.2, let I be an integrally closed ideal of A such that $x \notin Q$ for each $Q \in Ass\ (A/I)$. Let $J = I \cap S$. If JT is integrally closed (resp. a radical ideal) then IA^* is integrally closed (resp. a radical ideal).*

Proof: Since the map $A \to A^*$ is flat, x is not in any associated prime of IA^*. Therefore IA^* is contracted from $A^*[1/x]$ and it suffices to show $IA^*[1/x]$ is integrally closed (resp. a radical ideal). Our hypothesis implies

$I = IA[1/x] \cap A$. By Comment **??**.1, $A[1/x]$ is a localization of S. Thus every ideal of $A[1/x]$ is the extension of its contraction to S. It follows that $IA[1/x] = JA[1/x]$. Thus $IA^*[1/x] = JA^*[1/x]$.

Also by Comment **??**.1, the map $T \to A^*[1/x]$ is regular. If JT is integrally closed, then Remark **??**.7 implies that $JA^*[1/x]$ is integrally closed. If JT is a radical ideal, then the regularity of the map $T \to A^*[1/x]$ implies the $JA^*[1/x]$ is a radical ideal. $\square$

PROPOSITION 4.4 *With the setting of Theorem* **??** *and Comment* **??**, *let* $Q \in \mathit{Spec}\ A$ *be such that* $Q\widehat{A}$ *(or equivalently* QA^**) is not integrally closed. Then*

1. Q *has height two and* $x \notin Q$.
2. *There exists a minimal prime* Q^* *of* QA^* *such that with* $\mathbf{q}' = Q^* \cap T$, *the map* $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ *is not regular.*
3. Q *contains* $f = (y - \alpha)^2$ *or* $g = (z - \beta)^2$.
4. Q *contains no element that is a regular parameter of* A.

Proof: By Remark **??**.6, the height of Q is two. Since $A^*/xA^* = A/xA = R/xR$, we see that $x \notin Q$. This proves item 1.

By Remark **??**.7, there exists a minimal prime Q^* of QA^* such that $\gamma_{Q^*} : A \to A^*_{Q^*}$ is not regular. Thus item 2 follows from Comment **??**.2.

For item 3, let Q^* and $\mathbf{q}'$ be as in item 2. Since γ_{Q^*} is not regular it is not essentially smooth [**?**, 6.8.1]. By [**?**, (2.7)], $(y - \alpha)(z - \beta) \in \mathbf{q}'$. Hence $f = (y - \alpha)^2$ or $g = (z - \beta)^2$ is in $\mathbf{q}'$ and thus in Q. This proves item 3.

Suppose $w \in Q$ is a regular parameter for A. Then A/wA and A^*/wA^* are two-dimensional regular local domains. By Remark **??**.6, QA^*/wA^* is integrally closed, but this implies that QA^* is integrally closed, which contradicts our hypothesis that QA^* is not integrally closed. This proves item 4. $\square$

QUESTION 4.5 In the setting of Theorem **??** and Comment **??**, let $Q \in$ Spec A with $x \notin Q$ and let $\mathbf{q} = Q \cap S$. If QA^* is integrally closed, does it follow that $\mathbf{q}T$ is integrally closed?

QUESTION 4.6 In the setting of Theorem **??** and Comment **??**, if a prime ideal Q of A contains f or g, but not both, and does not contain a regular parameter of A, does it follow that QA^* is integrally closed ?

In Example **??**, the three-dimensional regular local domain A contains height-one prime ideals P such that $\widehat{A}/P\widehat{A}$ is not reduced. This motivates us to ask:

Question 4.7 Let $(A, \mathbf{n})$ be a three-dimensional regular local domain and let $\widehat{A}$ denote the $\mathbf{n}$-adic completion of A. If for each height-one prime P of A, the extension $P\widehat{A}$ is a radical ideal, i.e., the ring $\widehat{A}/P\widehat{A}$ is reduced, does it follow that $P\widehat{A}$ is integrally closed for each $P \in \operatorname{Spec} A$?

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