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COMPUTER-AIDED ANALYSIS OF MONOTONIC SEQUENCE GAMES

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ABSTRAC

We propose a game based on a theorem of Erdös and Szekeres about monotonic sequences. The outcomes of small-sized games are generated by computer and then proved mathematically. The efficiency of the algorithm used is also discussed.

1. *Introduction.*

A well-known theorem of Erös and Szekeres [2] extends at once to the following statement:

Theorem ES. Let N and M be non-negative integers. Any sequence of at least $NM+1$ distinct integers must contain an increasing subsequence of length $N+1$ or a decreasing subsequence of length $M+1$. □

For short proofs of this result see Blackwell [1] and Seidenberg [8]. The lower bound $NM+1$ is sharp since there exist sequences of length NM whose longest increasing subsequence is of length N, and whose longest decreasing subsequence is of length M, e.g.,

$$M,\ M-1,\dots,1,\ 2M,\ 2M-1,\dots,M+1,\dots,NM,\dots,N(M-1)+1.$$

Recently one of us has been investigating achievement and avoidance games based on theorems [3, 4]. To turn.

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Theorem ES into a game, imagine two players, A and B, who take turns picking distinct numbers from a finite set, Σ, of integers and use them to build a sequence (or permutation) $P(1)$, $P(2)$, $P(3)$,... . Player A always starts the game and P(1) is his first move, then player B chooses $P(2)$, and so on.

In the achievement version of the game, the first player to create an increasing subsequence of length $N+1$ or a decreasing subsequence of length $M+1$ is the winner. In the avoidance game, the first player to do so loses. Note that in view of Theorem ES we are guaranteed a winner when $|\Sigma| \geq NM+1$. Clearly only the cardianality S of Σ matters, not the actual numbers in the set, so we will always take $\Sigma = \{1, 2, ..., S\}$.

The following notation will be useful in the sequel. For the achievement game we set

$$w(N, M, S) \begin{cases} A & \text{if } A \text{ can force a win} \\ B & \text{if } B \text{ can} \\ 0 & \text{if neither can.} \end{cases}$$

We assume that both players play rationally, i.e., both A and B play to maximize their chances of winning. The abbreviation $w(N, M)$ is used when $S = NM+1$. The *achievement number* is

$$a(N, M) = \min\{S \mid w(N, M, S) \neq 0\}.$$

We have already observed that $a(N, M) \leq NM+1$. If $w(N, M, S) \neq 0$ we define the *move number* $m(N, M, S)$ as the smallest number of moves in which the winner can force the game to a close. In particular let $m(N, M)$ be the move number for the game with $S = NM+1$. The corresponding concepts for the avoidance game will be denoted $\bar{w}(N, M, S)$, $\bar{a}(N, M)$, etc..

We do not have a general strategy for either game, however with the aid of a computer we have determined the values of the six functions w, a, m, $\bar{w}$, $\bar{a}$, $\bar{m}$, for small N and M. Using standard backtrackking techniques (see, e.g., [7]) we were able to obtain complete results through $NM+1 = 10$, even though the game tree has $O((NM+1)$ leaves.

However, employment of tree pruning and a suitable heuristic improved our run times enough to push the bound up to $NM+1=15$. Although these devices reduce the exponent of growth, the tree size remains exponential in $NM+1$. Hence it is not expected that much larger games will be tractable.

In Section 2 we provide a rigorous analysis of the small achievement games. Section 3 is concerned with the statistics for the avoidance game. Finally in Section 4 e provide a comparison between the various tree search techniques used and conclude with some open questions.

2. *The achievement game.*

Table 1 lists the values of $w(N, M)$, $a(N, M)$ and $m(N, M)$ that we have obtained; three dots indicate the continuation of a general pattern. First some trivial observations

(1) $$w(N, M, S)=w(M, N\ S).$$

For all S we have

(2) $$w(0, M, S)=A,$$

and,

(3) $$a(0, M)=m(0, M, S)=1.$$

Table 1 — ACHIEVEMENT GAMES.

$w(N, M)$:

N \ M	0	1	2	3	4	
0	*A*	*A*	*A*	*A*	*A*	. . .
1	*A*	*B*	*A*	*B*	*A*	. . .
2	*A*	*A*	*A*	*A*	*A*	. . .
3	*A*	*B*	*A*	*A*	*A*	
4	*A*	*A*	*A*	*A*	?	
	⋮	⋮	⋮			

$a(N, M)+m(N, M)$:

N \ M	0	1	2	3	4	
0	1	1	1	1	1	. . .
1	1	2	3	4	5	. . .
2	1	3	5	5	7	. . .
3	1	4	5	9	11	
4	1	5	7	11	?	
	⋮	⋮	⋮			

We will now derive some simple results that will verify the rest of the entires in Table 1.

$$(4)\qquad w(1, M)=\begin{cases} A & \text{if } M \text{ is even} \\ B & \text{if } M \text{ is odd.} \end{cases}$$

$$(5)\qquad m(1, M)=M+1.$$

$$(6)\qquad a(1, M)=M+1.$$

For $S \geq a(1, M)$ we have

$$(7)\qquad \begin{cases} w(1, M, S)=w(1, M) \\ m(1, M, S)=m(1, M). \end{cases}$$

Proof of (4)-(7). Clearly A must first play the largest number, $M+1$, otherwise he will lose on the next move. For the same reasons B must now play M, and so forth until a decreasing sequence is completed giving us (4) and (5). Since this is the only possible strategy, equations (6) and (7) follow as well. □

$$(8)\qquad \text{If } w(N, M, S)=B \text{ then } w(N+1, M, S+1)= =w(N, M+1, S+1)=A.$$

Proof. For the game with parameters $(N+1, M, S+1)$ player A can begin by playing the smallest integer, 1. This will result in a game of type (N, M ,S) with the roles of A and B reversed. The other case is similar. □

Equation (8) will helps us to determine the winner for the case $N=2$.

$$(9)\qquad w(2, M)=A \text{ for all } M.$$

$$(10)\qquad a(2, M)=\begin{cases} M+2 & \text{for all odd } M \geq 3, \\ M+3 & \text{for all even } N \geq 2, \end{cases}$$

For $S \geq a(2, M)$ we have

$$\begin{cases} w(2, M, S) = A, \\ m(2, M, S) = a(2, M). \end{cases} \tag{11}$$

Proof of (9)-(11). First consider equation (9). When M is odd this follows from (4) and (8).

When M is even, A begins by playing 2. If B does not play 1 or $2M+1$ then A will win on his next move. But if B plays 1 then A can play $2M+1$ (and vice-versa) which reduces the game to one of type $(1, M-1)$ with the roles of A and B interchanged. Hence A must win by (4) and the fact that $M-1$ is odd.

To verify (10), note that the strategy above will guarantee a win for A after at most $M+2$ moves for odd M or at most $M+3$ moves for even M. Now $a(2, M) \geq a(1, M) = M+1$ and if A can win then $a(2, M)$ must be odd. Hence if M is odd then $a(2, M) = M+2$. If, on the other hand, M is even we have

$$a(2, M) = M+1 \quad \text{or} \quad M+3.$$

But B can prevent A from winning in $M+1$ moves by playing 1 at some point. So in this case $a(2, M) = M+3$. It also follows that the move number is equal to the achievement number whenever $S \geq a(2, M)$. □

We next derive results for $N = M = 3$.

$$w(3, 3) = 4, \tag{12}$$

$$a(3, 3) = 9 \tag{13}$$

For $S \geq a(3, 3)$ we have

$$m(3, 3, S) = 9, \tag{14}$$

$$w(3, 3, S) = A, \tag{15}$$

Proof of (12)-(15). A beings by playing 5. If B responds with 6 then A follows with 4 and vice versa. This creates both an increasing and a decreasing sequence of length two which constitutes a double threat: if B now plays anything except the largest or smallest number then he will create an increasing or decreasing sequence of length three which can be completed by A on the very next move. Hence both players must now adopt the strategy of playing the largest or smallest of the remaining numbers which will force a win for A after six more moves. If B plays $P(2) \neq 4$ or 6 then A can create a similar double threat by choosing any $P(3)$ which is between 5 and $P(2)$. This proves (12) and the fact that $a(3, 3) \leq 9$.

To finish the proof of (13) we need to show that $a(3, 3) \neq 8$. This is a tedious exercise that we leave to the reader. Clearly the same strategy will work for any $S \geq 9$ (A picks $P(1) = [S/2]$ and continues as before). Hence (14) and (15) follow. □

As the entries for $N=3$, $M=4$ in Table 1 are verified by techniques similar to those used above for the case $N=M=3$, we omit the details.

3. *The avoidance game*

Table 2 contains computer generated statistics about the avoidance game. It is much more difficult to analyze than the achievement games. We only have total information for two trivial cases:

$$(16) \qquad \begin{cases} \bar{w}(0, M, S) = B, \\ \bar{a}(0, M) = \overline{m}(0, M, S) = 1, \end{cases}$$

and for $S \geq 2 = \bar{a}(1, M)$,

$$(17) \qquad \begin{cases} \bar{w}(1, M, S) = A, \\ \overline{m}(1, M, S) = 2. \end{cases} \qquad \square$$

Table 2 — AVOIDANCE GAMES.

$\overline{w}(N, M)$:

N \ M	0	1	2	3	4	5	6	7	
0	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	...
1	*B*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	...
2	*B*	*A*	*A*	*A*	*B*	*A*	*A*	*A*	...
3	*B*	*A*	*A*	*A*	*A*				
4	*B*	*A*	*B*	*A*					
5	*B*	*A*	*A*						
6	*B*	*A*	*A*						
7	*B*	*A*	*A*						
	⋮	⋮	⋮						

$\overline{a}(N, M) = \overline{m}(N, M)$:

N \ M	0	1	2	3	4	5	6	7	
0	1	1	1	1	1	1	1	1	...
1	1	2	2	2	2	2	2	2	...
2	1	2	4	6	7	8	10	12	...
3	1	2	6	8	8				
4	1	2	7	8					
5	1	2	8						
6	1	2	10						
7	1	2	12						
	⋮	⋮	⋮						

For $N=2$ we can only offer the conjectures

$$\bar{w}(2, M)=\begin{cases} A & \text{if } M \not\equiv 0 \pmod 4; \\ B & \text{if } M \equiv 0 \pmod 4; \end{cases} \tag{18}$$

$$\bar{a}(2, M)=\bar{m}(2, M)=\begin{cases} 2M-2\left[\frac{M}{4}\right] & \text{if } M \not\equiv 0 \pmod 4; \\ 1+\frac{3M}{2} & \text{if } M \equiv 0 \pmod 4. \end{cases} \tag{19}$$

For N, $M \geq 2$ the general strategy seems to be the use of a double threat. The winner constructs both an increasing subsequence

Table 3 — OPENING MOVES OR THE AVOIDANCE GAME.

A WINNING OPENING FOR A.

N \ M	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	3	2	5	*	6	12	13
3	4	3	2	7			
4	5	*	5				
5	6	6					
6	7	2					
7	8	3					

of length N ending, say, in n and a decreasing subsequence of length M ending, say, in m such that either $n \leq m$ or $n > m$ and all numbers between n and m have been used. One can also get away with constructing one subsequence if it happens to be $1, 2, \ldots, N$ or $S, S-1, \ldots, S-M+1$. Unfortunately it is hard to convert the above observations into a precise proof.

It is unusual that A wins in almost all the known cases where $N, M \geq 1$. In Table 3 we have listed a winning opening for A in each of these games. For the case $N=2$, $M=4$ when B wins, we have listed a winning response to each of A's openings in Table 3 (*). Perhaps these will be useful in future attempts to understand this game.

4. *The algorithm and unsolved problems*

The computer program consisted of three almost independent modules: the tree transverser, which was oblivious to the detailed semantics of the tree nodes; the problem definition module, which maintained the data structures implementing the semantics, and which made available to the tree transverser such operations as Enumerate Successors; and the driver, which is not of interest here. The tree transverser initially performed a depth-first search of the entire tree, and was then modified to perform forward pruning. The enumeration of successor nodes was initially done in arbitrary order and then modified to use the heuristic that, where appropriate, one should try to force an early conclusion by extending the subsequence that is nearest to causing termination of the game.

The program was written in IBM Fottran *H* and ran on an Amdahl 470v/8, which has an effective memory access time of 5E-8 sec. In Table 4 we give the number of games played, number of branches pruned, and the *CPU* time used for each of the three programs in the achievement version. Note that even though the algorithm with the heuristic sometimes plays more games than the one without, it is still faster since the games it plays are shorter (being forced to an early conclusion). The average tree branching factor for the full program hovers around 1.85 for the large games. However, even with this version, the case $N=3$, $M=5$, was still incomplete after two full minutes of *CPU* time.

(*) A winnng response for B in the game $N=2$, $M=4$.

a_0	1	2	3	4	5	6	7	8	9
a_1	2	4	4	7	7	7	6	6	6

Table 4 — PROGRAM STATISTICS FOR THE ACHIEVEMENT GAME.

	N	M	Games played	Branches pruned	CPU time (seconds)
Backtracking:	1	1	2	0	1.11E—04
	2	1	6	0	2.73E—04
	3	1	18	0	7.97E—04
	4	1	50	0	2.23E—03
	2	2	100	0	5.26E—03
	5	1	130	0	5.89E—03
	6	1	322	0	1.49E—02
	3	2	2170	0	1.31E—01
	7	1	770	0	3.64E—02
	8	1	1794	0	8.72E—02
	4	2	50022	0	3.38E+00
	9	1	4098	0	2.06E—01
	3	3	598668	0	4.46E+01
With pruning:	1	1	2	0	1.17E—04
	2	1	2	1	1.27E—04
	3	1	5	4	3.16E—04
	4	1	5	5	3.96E—04
	2	2	15	13	1.10E—03
	5	1	10	10	7.39E—04
	6	1	10	11	8.81E—04
	3	2	110	116	1.00E—02
	7	1	17	18	1.40E—03
	8	1	17	19	1.72E—03
	4	2	668	691	6.80E—02
	9	1	26	28	2.36E—03
	3	3	1945	2023	2.19E—01
With pruning and heuristic:	1	1	2	0	1.03E—04
	2	1	2	1	1.24E—04
	3	1	5	4	2.90E—04
	4	1	5	5	3.29E—04
	2	2	20	17	1.10E—03
	5	1	10	10	6.06E—04
	6	1	10	11	6.55E—04
	3	2	184	168	1.02E—02
	7	1	17	18	1.07E—03
	8	1	17	19	1.10E—03
	4	2	798	821	4.60E—02
	9	1	26	28	1.58E—03
	3	3	2150	2225	1.24E—01

One other optimization consideration is the method of keeping track of the subsequences. For brevity, we discuss only increasing subsequences; decreasing ones are handled analogously. We maintain an auxiliary vector UPLEN (I) containing the length of a longest increasing subsequence terminating at position I. When a new term $P(J)$ is added to the sequence, UPLEN (J) may be calculated by comparing P (J) with each P (K), K<J. If implemented in a straightforward manner, the number of comparisons required, $J-1$, has an upper bound NM. However, one can do much better by observing that certain of the comparisons render others superfluous. If we proceed in decreasing order of UPLEN (K) then once we find that $P(J)>P(K)$ we need not compare $P(J)$ to any $P(K')$ such that $K'<K$ and UPLEN $(K')=$UPLEN (K).

A more complete application of this principle requires two more auxiliary vectors, but the number of comparisons now has an upper bound of N. Even for a game as small as $N=M=3$, this technique saves a factor of 2 in *CPU* time over the straightforward method (not recorded in Table 4) despite the need to maintain more complex data structures. In fact this amounts to an efficient implementation of an algorithm of Schensted [6] where only the first row of Schensted's array is kept.

In addition to the obvious problem of giving a complete solution to either the achievement or avoidance game, there are several other interesting questions that arise from the cases we have already studied.

(1) The size of the set, S, appears to be immaterial, i.e., for all $S \geq a(N, M)$ we have;

$$w(N, M, S)=w(N, M) \quad \text{and} \quad m(N, M, S)=m(N, M).$$

Is this true in general? Note that this is not always the case for other games, e.g., the winner in graphical achievement and avoidance games depends heavily on the board size [5].

(2) It is striking that we always seem to have $a(N, M)= =m(N, M, a(N, M))$. Although this is not a suprising conjecture, as one expects the first non-draw game to use all the elements in S, a proof is far from obvious.

(3) Failing a precise determination of $a(N, M)$ and $m(N, M, S)$, what bounds can be constructed? Of course we already have

$$\min(N+1, M+1) \leq a(N, M), \quad m(N, M, S) \leq NM+1.$$

(4) The above questions also have obvious counterparts for the avoidance game.

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