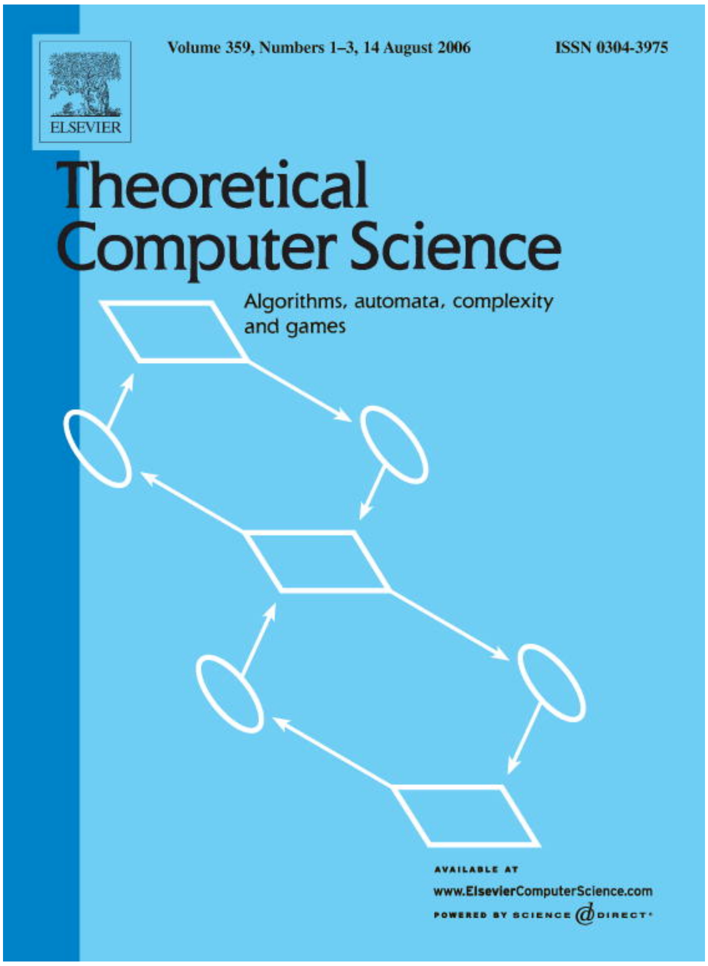
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Rationality of the Möbius function of a composition poset

Anders Björner[a], Bruce E. Sagan[b,*,1]

[a]Department of Mathematics, Kungliga Tekniska Högskolan, S-100 44 Stockholm, Sweden
[b]Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

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Abstract

We consider the zeta and Möbius functions of a partial order on integer compositions first studied by Bergeron, Bousquet-Mélou, and Dulucq. The Möbius function of this poset was determined by Sagan and Vatter. We prove rationality of various formal power series in noncommuting variables whose coefficients are evaluations of the zeta function, ζ, and the Möbius function, μ. The proofs are either directly from the definitions or by constructing finite-state automata.

We also obtain explicit expressions for generating functions obtained by specializing the variables to commutative ones. We reprove Sagan and Vatter's formula for μ using this machinery. These results are closely related to those of Björner and Reutenauer about subword order, and we discuss a common generalization.

Keywords: Automaton; Composition; Generating function; Hypergeometric series; Monoid; Rationality; Subword order

1. Introduction

Let A be an arbitrary set and consider the *free monoid*, A^*, of all words over A:

$$A^* = \{w = w(1)w(2)\dots w(n) \mid n \geqslant 0 \text{ and } w(i) \in A \text{ for all } i\}.$$

We let $\ell(w)$ denote the length (number of elements) of w.

If $\mathbb{P}$ is the positive integers, then $\mathbb{P}^*$ is just the set of integer compositions (ordered partitions). We put a partial order on $\mathbb{P}^*$ by saying that $u \leqslant w$ if and only if w contains a subword $w(i_1)w(i_2)\dots w(i_l)$ where $l = \ell(u)$ and

$$u(j) \leqslant w(i_j) \quad \text{for } 1 \leqslant j \leqslant l.$$

To illustrate, $\mathbf{334} \leqslant \mathbf{34261}$ as can be seen by considering the subword $\mathbf{346}$. Note that integers will be typeset in boldface when considered as elements of $\mathbb{P}^*$. Bergeron et al. [2] initiated the study of $\mathbb{P}^*$ by counting its saturated lower chains.

* Corresponding author. Tel.: +1 517 355 8329.
E-mail addresses: bjorner@math.kth.se (A. Björner), sagan@math.msu.edu (B.E. Sagan).
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This work was carried on by Snellman [12,13] who also considered saturated chains in two other partial orders on $\mathbb{P}^*$. One of these posets was originally defined by Björner and Stanley [6] who showed that it has analogues of many of the properties of Young's lattice. Sagan and Vatter [11] determined the Möbius function of the poset we are considering. Here we will use generating functions over monoids to give more information about the Möbius and zeta functions of $\mathbb{P}^*$ as well as rederiving the theorem of Sagan and Vatter using this machinery.

There is a strong connection between this order on $\mathbb{P}^*$ and subword order. Considering A to be arbitrary, we define *subword order* on A^* by letting $u \leqslant w$ if and only if there is a subword $w(i_1)w(i_2)\ldots w(i_l)$ of length $l = \ell(u)$ with

$$u(j) = w(i_j) \quad \text{for } 1 \leqslant j \leqslant l.$$

For example, $abba \leqslant ababbbaa$ since $w(1)w(4)w(6)w(8) = abba$. Context will make it clear whether "$\leqslant$" refers to subword order or composition order. Björner [4] was the first to give a complete characterization of the Möbius function for subword order. See [11] for a history of this problem. In particular, Björner and Reutenauer [5] showed that the Möbius and zeta functions have rational generating functions and were able to reprove the formula for μ using these ideas.

The rest of this paper is structured as follows. In the next section, we provide the necessary definitions to state Björner's formula for μ in A^* as well as Sagan and Vatter's result in $\mathbb{P}^*$, see Theorems 2.1 and 2.2, respectively. In Section 3, we prove the rationality of monoid generating functions for μ and ζ on the subposet $\{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}^*$ of $\mathbb{P}^*$. Our demonstrations are either based directly on the definitions or use finite-state automata. By specializing the variables, we obtain explicit formulas for related generating functions in Section 4. Surprisingly, results about hypergeometric series are needed to do some of the computations. The next section is devoted to another proof of the formula for μ in $\mathbb{P}^*$ using the generating function approach. Sagan and Vatter showed that both Theorems 2.1 and 2.2 are special cases of a more general result about certain partial orders which they called generalized subword orders (and which have been studied in the context of well-quasi-ordering, see Kruskal [9]). In Section 6, we indicate which of our results can be proved in this level of generality. We end with a section of comments and open problems.

2. Subword and composition order

We will first present the formula for the Möbius function of A^* in a way that will help motivate our definitions when we get to $\mathbb{P}^*$. We will not define the Möbius function itself, but that background can be found in the text of Stanley [16, Sections 3.6 and 3.7].

We begin by giving an equivalent formulation for subword order which will be useful when we get to μ. Suppose we have a special symbol 0 with $0 \notin A$. Then the *support* of a word $\eta = \eta(1)\eta(2)\ldots\eta(n) \in (A \cup 0)^*$ is

$$\operatorname{Supp} \eta = \{i \mid \eta(i) \neq 0\}.$$

An *expansion* of $u \in A^*$ is a word $\eta_u \in (A \cup 0)^*$ such that the restriction η_u to its support is u. Taking $u = abba$ as before, then one possible expansion is $\eta_u = a00b0b0a$. An *embedding* of u into w is an expansion η_u of u having length $\ell(w)$ and satisfying

$$\eta_u(i) = w(i) \quad \text{for all } i \in \operatorname{Supp} \eta_u.$$

Clearly $u \leqslant w$ in subword order if and only if there is an embedding of u into w. In fact, the example η_u above is the embedding which corresponds to the subword of $w = ababbbaa$ given in the previous section.

The Möbius function of subword order counts a particular type of embedding. Suppose $a \in A$. A *run* of a's in w is a maximal interval of indices $[r, t]$ such that

$$w(r) = w(r + 1) = \cdots = w(t) = a.$$

Continuing with our example, $w = ababbbaa$ has runs [1, 1], [2, 2], [3, 3], [4, 6], and [7, 8]. An embedding η_u into w is *normal* if, for every $a \in A$ and every run $[r, t]$ of a's, we have

$$(r, t] \subseteq \operatorname{Supp} \eta_u$$

for the half-open interval $(r, t]$. In our running example, this means that the b's in positions 5 and 6 as well as the a in position 8 must be in any normal embedding. (Runs of one element impose no restriction since if $r = t$ then $(r, t] = \emptyset$.)

So in this case there are exactly two normal embeddings η_u into w, namely

$$\eta_u = a000bb0a \quad \text{and} \quad 00a0bb0a.$$

Let $\binom{w}{u}_n$ denote the number of normal embeddings of u into w.

Theorem 2.1 (*Björner [4]*). *If* $u, w \in A^*$ *then*

$$\mu(u, w) = (-1)^{|w|-|u|} \binom{w}{u}_n.$$

Putting everything together in our example, we obtain

$$\mu(abba, ababbbaa) = (-1)^{8-4} \cdot 2 = 2.$$

In $\mathbb{P}^*$, the definitions of support and expansion are the same as in A^*. However, the definition of embedding must be changed to reflect the different partial order. In this case, define an *embedding* of u into w as an expansion η_u such that $\ell(\eta_u) = \ell(w)$ and

$$\eta_u(i) \leqslant w(i) \quad \text{for } 1 \leqslant i \leqslant \ell(w).$$

As before, $u \leqslant w$ in $\mathbb{P}^*$ if and only if there exists an embedding of u into w.

Of particular interest to us will be the rightmost embedding. Suppose $u \leqslant w$. The *rightmost* embedding ρ_u into w is the one such that for any other embedding η_u into w we have $\text{Supp}(\rho_u) \geqslant \text{Supp}(\eta_u)$. (If $S = \{i_1 < \cdots < i_m\}$ and $S' = \{i'_1 < \cdots < i'_m\}$ then $S \geqslant S'$ means $i_j \geqslant i'_j$ for $1 \leqslant j \leqslant m$.)

The definition of a run is again the same in $\mathbb{P}^*$ as it was in A^*. So we call an embedding η_u into w *normal* if it satisfies the following two criteria:

(1) For $1 \leqslant i \leqslant \ell(w)$, we have $\eta_u(i) = w(i)$, $w(i) - 1$, or 0.
(2) For all $k \geqslant 1$ and every run $[r, t]$ of $\boldsymbol{k}$'s in w, we have
   (a) $(r, t] \subseteq \text{Supp}\, \eta_u$ if $k = 1$,
   (b) $r \in \text{Supp}\, \eta_u$ if $k \geqslant 2$.

Note that in $\mathbb{P}^*$ a normal embedding can have three possible values at each position instead of the two permitted in A^*. Also note that the run condition for ones is the same as in A^*, while that condition for integers greater than one is complementary. For example, if $u = \mathbf{21113}$ and $w = \mathbf{2211133}$, then there are two normal embeddings, namely $\eta_u = \mathbf{2101130}$ and $\mathbf{2011130}$. Also, $\mathbf{2001113}$ and $\mathbf{0211130}$ are not normal since they violate conditions (1) and (2), respectively.

Another difference between A^* and $\mathbb{P}^*$ is that, in the former, the sign of an embedding only depends on the length difference, while in the latter, it depends on the embedding itself. If η_u into w is normal then define its *defect* to be

$$d(\eta_u) = \#\{i \mid \eta_u(i) = w(i) - 1\}.$$

The formula for the Möbius function of $\mathbb{P}^*$ is as follows:

Theorem 2.2 (*Sagan and Vatter [11]*). *If* $u, w \in \mathbb{P}^*$ *then*

$$\mu(u, w) = \sum_{\eta_u} (-1)^{d(\eta_u)}$$

where the sum is over all normal embeddings η_u *into* w.

Finishing off the example above

$$\mu(\mathbf{21113}, \mathbf{2211133}) = (-1)^2 + (-1)^0 = 2.$$

Although this example does not show it, it is possible to have cancellation among the terms in the sum for μ.

3. Rationality

Let ε denote the empty word in A^*. For this section and the next one we will assume that A is a finite set. Let $\mathbb{Z}\langle\langle A\rangle\rangle$ be the algebra of formal power series in the noncommuting variables A with integer coefficients. So every $f \in \mathbb{Z}\langle\langle A\rangle\rangle$ has the form

$$f = \sum_w c_w w,$$

where $w \in A^*$ and $c_w \in \mathbb{Z}$. If f has no constant term, i.e., $c_\varepsilon = 0$, then define

$$f^* = \varepsilon + f + f^2 + f^3 + \cdots = (\varepsilon - f)^{-1}. \tag{1}$$

(One needs the restriction on f to make sure that the sum is well-defined as a formal power series.) We say f is *rational* if it can be constructed from a finite set of monomials using a finite number of applications of the algebra operations and the star operation. For more information about rational series, see the books of Eilenberg [8] or Berstel and Reutenauer [3]. We will show in this section that various series related to the Möbius and zeta functions are rational.

It will be convenient to define $[n] = [1, n]$. We will also use such interval notations with elements of $\mathbb{P}^*$ in the obvious way. So, for example,

$$[\mathbf{k}, \mathbf{n}] = \{\mathbf{k}, \mathbf{k+1}, \ldots, \mathbf{n}\}.$$

Consider $[\mathbf{n}]^*$ as a subposet of $\mathbb{P}^*$. Given $u \in [\mathbf{n}]^*$, we have the associated formal series

$$Z(u) = \sum_{w \geqslant u} w = \sum_w \zeta(u, w) w, \tag{2}$$

where ζ is the zeta function of $[\mathbf{n}]^*$. We also wish to consider

$$M(u) = \sum_{w \geqslant u} \left(\sum_{\eta_u} (-1)^{d(\eta_u)} \right) w, \tag{3}$$

where the inner sum is over all normal embeddings η_u into w. Note that if we assume Theorem 2.2 then $M(u) = \sum_w \mu(u, w) w$, but we will not need this fact to do our computations. Indeed, in Section 5 we will use the displayed definitions of $Z(u)$ and $M(u)$ above to reprove Theorem 2.2.

The crucial observation underlying our method is that $Z(u)$ and $M(u)$ can be expressed in terms of simpler series. To define these series, it will help to have a bit more notation. If $S \subseteq [\mathbf{n}]^*$ then we will also let S stand for the generating function $\sum_{w\in S} w$. Context will make it clear which interpretation is meant. If S is empty then the corresponding generating function is the zero series. If f is a series without constant term then we let

$$f^+ = f + f^2 + f^3 + \cdots = f^* - \varepsilon.$$

Note that f^+ is rational if f is. Finally, a function $F : [\mathbf{n}]^* \to \mathbb{Z}\langle\langle [\mathbf{n}]\rangle\rangle$ is called *multiplicative* if for any $u \in [\mathbf{n}]^*$ we have

$$F(u) = F(u(1))F(u(2)) \cdots F(u(l)),$$

where $l = \ell(u)$.

Now define two multiplicative functions from $[\mathbf{n}]^*$ to $\mathbb{Z}\langle\langle [\mathbf{n}]\rangle\rangle$ by setting, for all $\mathbf{k} \in [\mathbf{n}]$,

$$z(\mathbf{k}) = [\mathbf{k}, \mathbf{n}] \cdot [\mathbf{k-1}]^*$$

and

$$m(\mathbf{k}) = \begin{cases} \mathbf{1} - \mathbf{2}^+(\varepsilon - \mathbf{1}) & \text{if } \mathbf{k} = \mathbf{1}, \\ (\mathbf{k}^+ - (\mathbf{k+1})^+)(\varepsilon - \mathbf{1}) & \text{if } \mathbf{k} \geqslant \mathbf{2}. \end{cases}$$

(Note that by convention, $[\mathbf{k-1}] = \emptyset$ when $\mathbf{k} = \mathbf{1}$ and $\mathbf{k+1} = \emptyset$ when $\mathbf{k} = \mathbf{n}$.) These are the building blocks for $Z(u)$ and $M(u)$.

Lemma 3.1. *For any* $u \in [\boldsymbol{n}]^*$ *we have*

$$Z(u) = [\boldsymbol{n}]^* z(u)$$

and

$$M(u) = (\varepsilon - \mathbf{1}) m(u).$$

Proof. To prove the first equation, it suffices to show that the product on the right-hand side produces each $w \geqslant u$ according to the rightmost embedding ρ_u of u into w. So such w will occur exactly once since the rightmost embedding is unique. Suppose $\boldsymbol{k}$ is the last element of u. Then $z(\boldsymbol{k})$ is the last factor of the product. The term $\boldsymbol{l}$ chosen from $[\boldsymbol{k}, \boldsymbol{n}]$ corresponds to the element of w greater than $\boldsymbol{k}$ in the rightmost embedding, while the product $[\boldsymbol{k-1}]^*$ contains all possible subwords which could appear after $\boldsymbol{l}$ in w while keeping $\boldsymbol{k}$ in its rightmost position. Similar considerations apply to the other factors in $z(u)$. Finally, the initial $[\boldsymbol{n}]^*$ accounts for everything to the left of the element of w corresponding to the first element of u.

The proof of the second equation is similar except that we must have an unique term for every normal embedding η_u into w and each term must have sign $(-1)^{d(\eta_u)}$. Again, consider the last element $\boldsymbol{k}$ of u. If $\boldsymbol{k} = \mathbf{1}$ then by the first normality condition, the corresponding element of w must be $\boldsymbol{l} = \mathbf{1}$ or $\mathbf{2}$. If $\boldsymbol{l} = \mathbf{1}$ then the second normality condition ensures that there is no element to the right of $\boldsymbol{l}$ in w and there is no contribution to the defect in this case. This corresponds to the initial $\mathbf{1}$ in the expression for $m(\mathbf{1})$. If $\boldsymbol{l} = \mathbf{2}$ then (by normality again) the subword of w to the right of $\boldsymbol{l}$ must consist only of $\mathbf{2}$'s, possibly with a final $\mathbf{1}$. The factor $-\mathbf{2}^+$ accounts for the string of $\mathbf{2}$'s with the appropriate sign and the final factor of $\varepsilon - \mathbf{1}$ takes care of the possibilities at the right end of w. The only difference between the formula for $m(\boldsymbol{k})$ for $\boldsymbol{k} = \mathbf{1}$ and $\boldsymbol{k} \geqslant \mathbf{2}$ is that the summand $\mathbf{1}$ has been replaced by $\boldsymbol{k}^+(\varepsilon - \mathbf{1})$. This is because if $\boldsymbol{k} \geqslant \mathbf{2}$, then the second normality condition permits w to end with a string of $\boldsymbol{k}$'s possibly with a $\mathbf{1}$ at the end. Also note that the initial factor of $\varepsilon - \mathbf{1}$ in $M(u)$ represents the fact that w may or may not begin with a $\mathbf{1}$ not accounted for by the other $m(\boldsymbol{k})$ factors. This completes the proof of the lemma. □

Note that directly from their definition, $z(\boldsymbol{k})$ and $m(\boldsymbol{k})$ are rational series. So, by the previous lemma, we have the following result.

Theorem 3.2. *For any* $u \in [\boldsymbol{n}]^*$, $Z(u)$ *and* $M(u)$ *are rational series.*

We will now prove analogous results for the generating functions of ζ and μ using the alphabet of ordered pairs $[\boldsymbol{n}] \times [\boldsymbol{n}] = [\boldsymbol{n}]^2$. We could do so by modifying the arguments which led to the previous theorem. But for variety's sake, we will use finite-state automata. We write the elements of $\mathbb{Z}\langle\langle [\boldsymbol{n}]^2 \rangle\rangle$ as

$$f = \sum_{u,w} c_{u,w}\, u \otimes w.$$

Given an alphabet A, a *finite-state automaton* is a digraph D with the following properties. The vertex set V and directed edge (arc) set E are both finite with loops and multiarcs permitted. There is a distinguished initial vertex and a distinguished final vertex denoted α and ω, respectively. Each $e \in E$ is assigned a monomial label $f(e) \in \mathbb{Z}\langle\langle A \rangle\rangle$.

Now given a finite walk W with arcs $e_1, \ldots, e_l$, we assign it the monomial

$$f(W) = \prod_{i=1}^{l} f(e_i).$$

The formal power series *accepted* by D is

$$f(D) = \sum_{W} f(W),$$

where the sum is over all finite walks from α to ω. Note that if $e_1, \ldots, e_j$ are all arcs from a vertex β to a vertex γ, then replacing these arcs by a single arc $e = \overrightarrow{\beta\gamma}$ and setting

$$f(e) = \sum_{i=1}^{j} f(e_i)$$

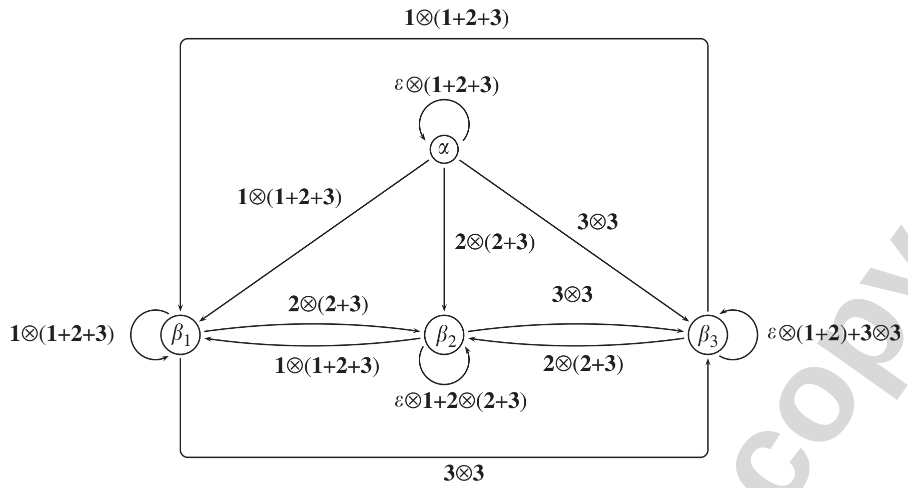

Fig. 1. The automaton for $Z_\otimes$ when $n = 3$ (with vertex ω omitted).

does not change the series accepted by D. So we will do this when constructing automata without further comment. We will also use algebraic operations to simplify the sum for $f(e)$ if possible.

The crucial fact which we will need is the well-known result that a series is rational if and only if it is accepted by some finite-state automaton D, see e.g. [3].

Theorem 3.3. *In* $\mathbb{Z}\langle\langle[\boldsymbol{n}]^2\rangle\rangle$ *the series*

$$Z_\otimes = \sum_{u,w} \zeta(u, w) u \otimes w$$

and

$$M_\otimes = \sum_{u,w} \mu(u, w) u \otimes w$$

are rational.

Proof. For both series, we will build finite-state automata accepting them.

The automaton D for $Z_\otimes$ has vertices $\{\alpha, \omega, \beta_1, \ldots, \beta_n\}$. A picture of the digraph when $n = 3$ is given in Fig. 1. The vertex ω is not shown since it simply has an incoming arc, labeled $\varepsilon \otimes \varepsilon$, from every other vertex. To describe the arc set, we will consider each of the other vertices in turn and describe all its incoming arcs.

If the vertex is α, then the only incoming arc is a loop labeled $\varepsilon \otimes [\boldsymbol{n}]$. If it is ω, then we have already described the arcs into it. If the vertex is β_k for some k then there is an incoming arc from every vertex except ω, as well as a loop, which are labeled

$$f(\overrightarrow{\beta\beta_k}) = \begin{cases} \varepsilon \otimes [\boldsymbol{k-1}] + \boldsymbol{k} \otimes [\boldsymbol{k}, \boldsymbol{n}] & \text{if } \beta = \beta_k, \\ \boldsymbol{k} \otimes [\boldsymbol{k}, \boldsymbol{n}] & \text{else.} \end{cases}$$

To show D accepts $Z_\otimes$, we need to prove that for every pair $u \otimes w$ with $u \leqslant w$ there is an unique way to obtain $u \otimes w$ as a monomial along some walk from α to ω, and that these are the only monomials in $f(D)$. We will indicate how one can find the walk W given $u \otimes w$, since then the reader should be able to fill in the details of the rest of the proof. In fact, we will show that W constructs w and u in its rightmost embedding ρ_u into w in the following sense. If e_i is the ith arc of W then $f(e_i)$ contains the term $\boldsymbol{a} \otimes \boldsymbol{b}$ where $\boldsymbol{b} = w(i)$ and $\boldsymbol{a} = \rho_u(i)$ or ε depending on whether $\rho_u(i) \in [\boldsymbol{n}]$ or $\rho_u(i) = \boldsymbol{0}$, respectively.

To begin, W loops $i - 1$ times at α, where i is the smallest index with $\rho_u(i) \neq \boldsymbol{0}$. (If $u = \varepsilon$ then let $i = \ell(w) + 1$.) The walk W finishes at ω if $i = \ell(w) + 1$, while if $i \leqslant \ell(w)$ it goes to β_k where $\rho_u(i) = \boldsymbol{k}$. Now W loops at β_k through arc e_{j-1}, where $j > i$ is the next index with $\rho_u(j) \neq \boldsymbol{0}$. The $\varepsilon \otimes [\boldsymbol{k-1}]$ summand on the arc contains the necessary monomial. Then e_j goes from β_k to β_l where $\rho_u(j) = \boldsymbol{l}$. Note that we could have $\boldsymbol{k} = \boldsymbol{l}$ so that this would also be a

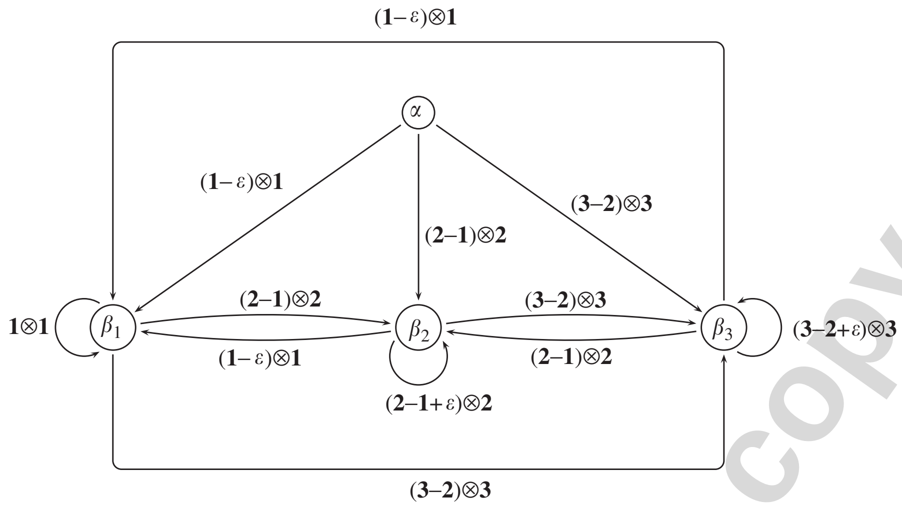

Fig. 2. The automaton for $M_\otimes$ when $n = 3$ (with the vertex ω omitted).

loop, in which case the $\boldsymbol{k} \otimes [\boldsymbol{k}, \boldsymbol{n}]$ summand contains the desired monomial. One continues in this manner until W has gone through $\ell(w)$ arcs, after which it takes an arc to ω.

The automaton for $M_\otimes$ has the same vertex set as the one for $Z_\otimes$. See Fig. 2 for the picture when $n = 3$. Again, ω only has incoming arcs from the other vertices and they are all labeled $\varepsilon \otimes \varepsilon$, so it is not shown. Since the construction of this automaton and the proof that it does accept $M_\otimes$ is parallel to what we did for $Z_\otimes$, we will content ourselves with a description of its arc set. Note that the interpretation of $\mu(u, w)$ built into the automaton relies on Theorem 2.2.

For α there are no incoming arcs and we have already described what happens for ω. If the vertex is β_1, then there are incoming arcs from every vertex except ω and they are labeled

$$f(\overrightarrow{\beta\beta_1}) = \begin{cases} \mathbf{1} \otimes \mathbf{1} & \text{if } \beta = \beta_1, \\ (\mathbf{1} - \varepsilon) \otimes \mathbf{1} & \text{else.} \end{cases}$$

If the vertex is β_k for $k \geqslant 2$ then we have the same set of incoming arcs with labels

$$f(\overrightarrow{\beta\beta_k}) = \begin{cases} (\boldsymbol{k} - (\boldsymbol{k}\mathbf{-1}) + \varepsilon) \otimes \boldsymbol{k} & \text{if } \beta = \beta_k, \\ (\boldsymbol{k} - (\boldsymbol{k}\mathbf{-1})) \otimes \boldsymbol{k} & \text{else.} \end{cases}$$

This completes the description of the automaton for $M_\otimes$. □

4. Generating functions in commuting variables

By specialization of variables, we can get generating functions for ζ and μ in terms of the length function $\ell(w)$ or in terms of the sum of the parts, or *norm*, of the composition, which will be denoted $|w|$. We will also need to keep track of the *type of* w, $t(w) = (l_1, l_2, \ldots, l_n)$, where l_k is the number of $\boldsymbol{k}$'s in w. So $\sum_k l_k = \ell(w)$ and $\sum_k l_k k = |w|$.

Suppose x is a variable and we substitute x^k for $\boldsymbol{k}$ in $Z(u)$. Then the generating function becomes

$$Z(u; x) = \sum_{w \geqslant u} x^{|w|}.$$

Doing the same thing with $z(\boldsymbol{k})$ and summing the resulting geometric series gives

$$z(\boldsymbol{k}; x) = \frac{x^k + x^{k+1} + \cdots + x^n}{1 - (x + x^2 + \cdots + x^{k-1})} = \frac{x^k - x^{n+1}}{1 - 2x + x^k}.$$

If $t(u) = (l_1, \ldots, l_n)$, then appealing to Lemma 3.1 yields a norm generating function in $[\boldsymbol{n}]^*$ of

$$Z(u; x) = \frac{1 - x}{1 - 2x + x^{n+1}} \prod_{k=1}^{n} \left(\frac{x^k - x^{n+1}}{1 - 2x + x^k} \right)^{l_k}.$$

Note that this generating function depends only on the type of u and not on u itself. Note also that one can take $n \to \infty$ in this series (reflecting the fact that there are only finitely many compositions with given norm) to obtain the norm generating function in $\mathbb{P}^*$

$$Z_{\mathbb{P}}(u;x) = \frac{1-x}{1-2x}\prod_{k\geqslant 1}\left(\frac{x^k}{1-2x+x^k}\right)^{l_k}.$$

When $u=\varepsilon$, this shows that the rank generating function for $\mathbb{P}^*$ (which is graded by norm) is $(1-x)/(1-2x)$. This can also be seen from the fact that there are 2^{N-1} compositions of N for $N \geqslant 1$. This same procedure can be applied to the generating function $M(u)$.

If one wants the generating function by length, then one substitutes the same variable, say t, for each $\boldsymbol{k}$. Under this substitution $m(\boldsymbol{k};t)=0$ for $1 \leqslant k \leqslant n$ and so $M(u;t)=0$ unless $u=\varepsilon$. Also, in this case one needs to remain in $[\boldsymbol{n}]^*$ since there are infinitely many compositions in $\mathbb{P}^*$ of a given nonzero length. The details of these computations are routine, so we will merely state the results.

Theorem 4.1. *Let $t(u)=(l_1,\ldots,l_n)$ where $u \in [\boldsymbol{n}]^*$. Then we have the norm generating functions*

$$Z(u;x) = \frac{1-x}{1-2x+x^{n+1}}\prod_{k=1}^{n}\left(\frac{x^k-x^{n+1}}{1-2x+x^k}\right)^{l_k}$$

and

$$M(u;x) = \frac{x^{|u|}(1-x)^{2\ell(u)+1}}{(1-x)^{l_1+l_n}}\prod_{k=2}^{n}\frac{1}{(1-x^k)^{l_{k-1}+l_k}}.$$

We also have the length generating functions

$$Z(u;t) = \frac{1}{1-nt}\prod_{k=1}^{n}\left(\frac{(n-k+1)t}{1-(k-1)t}\right)^{l_k}$$

and

$$M(u;t) = \begin{cases} 1-t & \text{if } u=\varepsilon,\\ 0 & \text{else}.\end{cases}$$

In $\mathbb{P}^*$ we have norm generating functions

$$Z_{\mathbb{P}}(u;x) = \frac{1-x}{1-2x}\prod_{k\geqslant 1}\left(\frac{x^k}{1-2x+x^k}\right)^{l_k}$$

and

$$M_{\mathbb{P}}(u;x) = \frac{x^{|u|}(1-x)^{2\ell(u)+1}}{(1-x)^{l_1}}\prod_{k\geqslant 2}\frac{1}{(1-x^k)^{l_{k-1}+l_k}}.$$

We would now like to calculate the generating function for ζ^m. This is of interest because $\zeta^m(u,w)$ counts the number of multichains of length m from u to w. (As mentioned in the introduction, the original motivation of Bergeron et al. in studying $\mathbb{P}^*$ was to count saturated chains in $[\varepsilon, w]$.) To do this, we will have to exploit a connection between the incidence algebra $I([\boldsymbol{n}]^*)$ and the algebra $\operatorname{End}\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ of continuous linear endomorphisms of $\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ (for the meaning of "continuity" here, see e.g. [3, p. 55]). This relationship will also be important in the next section where we will reprove the formula for μ.

Note that (2) already defines a map $Z : [\boldsymbol{n}]^* \to \mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$. We can extend this to an element of $\operatorname{End}\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ as follows. Take any $\phi \in I([\boldsymbol{n}]^*)$ and define a corresponding map $F_\phi : [\boldsymbol{n}]^* \to \mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ by

$$F_\phi(u) = \sum_w \phi(u,w)w,$$

where the sum is over all $w \in [\boldsymbol{n}]^*$, or equivalently over all $w \geqslant u$ since $\phi(u, w) = 0$ otherwise. By continuity and linearity, we can extend F_ϕ to a function in End $\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ by letting

$$F_\phi\left(\sum_u c_u u\right) = \sum_u c_u F_\phi(u).$$

Note that the right-hand side converges since any $v \in [\boldsymbol{n}]^*$ occurs with nonzero coefficient in only finitely many of the summands $F_\phi(u)$. Lifting elements of $I([\boldsymbol{n}]^*)$ to End $\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ in this way is well-behaved.

Theorem 4.2. *The map $\phi \mapsto F_\phi$ is an algebra anti-isomorphism of $I([\boldsymbol{n}]^*)$ with a subalgebra of* End $\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$.

Proof. Checking the various needed properties of the map are easy, so we will just indicate why multiplication is anti-preserved to illustrate. Recall that the product of $\phi, \psi \in I([\boldsymbol{n}]^*)$ is their convolution $\phi * \psi$ while the product in End $\mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle$ is composition of functions. To show that the two multiplications correspond, it suffices to check that they do so on elements $u \in [\boldsymbol{n}]^*$. So we compute

$$\begin{aligned} F_\psi \circ F_\phi(u) &= F_\psi\left(\sum_v \phi(u, v)v\right) \\ &= \sum_{v,w} \phi(u, v)\psi(v, w)w \\ &= \sum_w (\phi * \psi)(u, w)w \\ &= F_{\phi*\psi}(u) \end{aligned}$$

as desired. □

Now we can factor the generating function for ζ^m as follows. Let $\mathbb{Z}[[X]]$ be the formal power series ring over the integers in the set $X = \{x_1, x_2, \ldots, x_n\}$ of commuting variables. Consider the projection map $\rho : \mathbb{Z}\langle\langle[\boldsymbol{n}]\rangle\rangle \to \mathbb{Z}[[X]]$ which sends $\boldsymbol{k}$ to x_k. Then we have

$$\rho \circ z(\boldsymbol{k}) = \frac{x_k + \cdots + x_n}{1 - x_1 - \cdots - x_{k-1}}.$$

Define a multiplicative function $f : \mathbb{Z}[[X]] \to \mathbb{Z}[[X]]$ by

$$f(x_k) = \frac{x_k + \cdots + x_n}{1 - x_1 - \cdots - x_{k-1}}. \tag{4}$$

Clearly f is constructed so that

$$\rho \circ z = f \circ \rho.$$

We now apply the same idea to the function Z. If $u \in [\boldsymbol{n}]^*$ then we let $X^u = \prod_k x_k^{l_k}$, where $t(u) = (l_1, \ldots, l_n)$. So $\rho(u) = X^u$. Define a continuous, linear map $F : \mathbb{Z}[[X]] \to \mathbb{Z}[[X]]$ by

$$F(X^u) = \frac{1}{1 - x_1 - \cdots - x_n} f(X^u).$$

It follows that

$$\rho \circ Z = F \circ \rho.$$

From Theorem 4.2 we have that

$$\sum_w \zeta^m(u, w)w = Z^m(u).$$

So letting $t(u) = (l_1, \ldots, l_n)$ and applying ρ to both sides, we see that the generating function for ζ^m in $\mathbb{Z}[[X]]$ is

$$\begin{aligned}\sum_w \zeta^m(u,w)X^w &= \rho \circ Z^m(u)\\ &= F^m \circ \rho(u)\\ &= F^m(X^u)\\ &= \prod_{i=0}^{m-1} \frac{1}{1 - f^i(x_1) - \cdots - f^i(x_n)} \prod_{k=1}^{n} \left(f^m(x_k)\right)^{l_k},\end{aligned}$$

where the last equality follows from an easy induction on m.

Thus to find ζ^m for all m, it suffices to find $f^m(x_k)$ for all m and k. Since this turns out to be surprisingly hard to do, we will just consider what happens when $n = 2$. This case is of independent interest because then the poset has rank numbers given by the Fibonacci sequence. However, this is different from the Fibonacci posets defined by Stanley [14,15].

For simplicity when $n = 2$, let $x = x_1$ and $y = x_2$. In this case (4) becomes

$$f(x) = x + y \quad \text{and} \quad f(y) = \frac{y}{1-x}.$$

To simplify notation again, let

$$a_m = f^m(x) \quad \text{and} \quad b_m = f^m(y).$$

Now we have, for $m \geqslant 1$

$$f^m(x) = f^{m-1}(f(x)) = f^{m-1}(x+y) = f^{m-1}(x) + f^{m-1}(y)$$

or

$$a_m = a_{m-1} + b_{m-1}. \tag{5}$$

Similarly, one obtains

$$b_m = \frac{b_{m-1}}{1 - a_{m-1}} \tag{6}$$

for $m \geqslant 1$, and it is easy to see that

$$a_0 = x \quad \text{and} \quad b_0 = y. \tag{7}$$

Hence, we have to solve two recurrence relations in two unknowns.

Let us first make the norm substitution $y = x^2$. In this case we will denote a_m and b_m by $a_m(x)$ and $b_m(x)$. To state our result, we will need the round-down function $\lfloor\cdot\rfloor$ and round-up function $\lceil\cdot\rceil$. We will also use the conventions that the binomial coefficient $\binom{n}{k}$ equals 0 for $k < 0$ or $k > n$ and equals 1 for $k = 0$ and any n.

Theorem 4.3. *Suppose $u \in [\mathbf{2}]^*$ has type $t(u) = (l_1, l_2)$. Then*

$$\sum_w \zeta^m(u,w)x^{|w|} = a_m(x)^{l_1} b_m(x)^{l_2} \prod_{i=0}^{m-1} \frac{1}{1 - a_i(x) - b_i(x)}.$$

Furthermore, for all $m \geqslant 0$ we have

$$a_m(x) = \frac{x\overline{a}_m(x)}{d_m(x)} \quad \textit{and} \quad b_m(x) = \frac{x^2}{d_m(x)d_{m+1}(x)}, \tag{8}$$

where

$$\overline{a}_m(x) = \sum_i (-1)^{\lfloor i/2 \rfloor} \binom{\lfloor \frac{m+i}{2} \rfloor}{i} x^i \quad \textit{and} \quad d_m(x) = \sum_i (-1)^{\lceil i/2 \rceil} \binom{\lfloor \frac{m+i-1}{2} \rfloor}{i} x^i. \tag{9}$$

Proof. It suffices to show that the equations for $a_m(x)$ and $b_m(x)$ given in the statement of the theorem satisfy (5)–(7). Checking the boundary conditions is easy.

To prove that (5) holds, substitute (8) into the recursion, multiply by $d_m(x)d_{m-1}(x)/x$, substitute (9), and take the coefficient of x^k on both sides. Thus, we need to prove

$$\sum_i (-1)^{\lfloor i/2\rfloor+\lceil (k-i)/2\rceil}\binom{\lfloor\frac{m+i}{2}\rfloor}{i}\binom{\lfloor\frac{m+k-i-2}{2}\rfloor}{k-i} = \sum_i (-1)^{\lfloor i/2\rfloor+\lceil (k-i)/2\rceil}\binom{\lfloor\frac{m+i-1}{2}\rfloor}{i}\binom{\lfloor\frac{m+k-i-1}{2}\rfloor}{k-i} \tag{10}$$

for $k \neq 1$. (When $k = 1$ we need to add a 1 onto the right-hand side corresponding to the x obtained from $b_m(x)$ after doing the multiplication. But this identity is easy to verify.) The proof now breaks down into four cases depending on the parities of m and k. We will only discuss what happens when m is even and k odd, as the other demonstrations are similar.

So suppose $m = 2l$ and $k = 2j + 1$ for integers l, j. Then the terms in (10) corresponding to even i cancel. Rewriting the odd i terms using rising factorials yields, after some cancellation, the equivalent hypergeometric series identity

$$(l-1)_{j+1}(2-l)_j\, {}_4F_3\left[\begin{matrix} l+1, & -l, & -j, & -j-\frac{1}{2}; \; 1 \\ & l+j-1, & 1-l-j & \frac{1}{2} \end{matrix}\right]$$
$$= (l)_{j+1}(1-l)_j\, {}_4F_3\left[\begin{matrix} l, & 1-l, & -j, & -j-\frac{1}{2}; \; 1 \\ & l-j, & -l-j & \frac{1}{2} \end{matrix}\right].$$

Using the implementation of Zeilberger's algorithm [18,19] due to Paule and Schorn [10], one can verify that both sides of this equation satisfy the same three-term recurrence relation in l. Also, $j \neq 0$ since $k \neq 1$. For positive j both sides of the equation are clearly zero for $l = 0, 1$. So since both sides also satisfy the same boundary conditions, they must be equal. Also, Dennis Stanton has pointed out that one can give a more traditional proof of this identity (and, in fact, prove a generalization of it) using Tchebyshev polynomials and trigonometric identities.

Verifying (6) turns out to be much simpler. Substituting (8), clearing denominators, and dividing by x^2, leads to the equivalent identity

$$x\overline{a}_{m-1}(x) + d_{m+1}(x) - d_{m-1}(x) = 0.$$

This follows easily from (9) and the binomial recursion. □

To get the corresponding length generating functions, we need only change the boundary conditions to $x = y = t$. In this case, we write $a_m(t)$ and $b_m(t)$ for a_m and b_m. Since the computations are similar, we will simply state the result.

Theorem 4.4. *Suppose $u \in [\mathbf{2}]^*$ has type $t(u) = (l_1, l_2)$. Then*

$$\sum_w \zeta^m(u,w)t^{\ell(w)} = a_m(t)^{l_1} b_m(t)^{l_2} \prod_{i=0}^{m-1} \frac{1}{1-a_i(t)-b_i(t)}.$$

Furthermore, for all $m \geqslant 0$ we have

$$a_m(t) = \frac{t\overline{a}_m(t)}{d_m(t)} \quad \text{and} \quad b_m(t) = \frac{t}{d_m(t)d_{m+1}(t)},$$

where $\overline{a}_m(t) = \sum_i (-1)^i \alpha_{m,i} t^i$ and $d_m(t) = \sum_i (-1)^i \delta_{m,i} t^i$ with the coefficients $\alpha_{m,i}$ and $\delta_{m,i}$ being given by

$$\alpha_{m,i} = \begin{cases} \dfrac{(m+1)2^i}{2i+1}\dbinom{\frac{m+2i}{2}}{\frac{m-2i}{2}} & \text{if } m \text{ is even,} \\[2ex] 2^{i+1}\dbinom{\frac{m+2i+1}{2}}{\frac{m-2i-1}{2}} & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\delta_{m,i} = \begin{cases} \dfrac{m2^i}{m+2i}\dbinom{\frac{m+2i}{2}}{\frac{m-2i}{2}} & \text{if } m \text{ is even,} \\[2ex] 2^i\dbinom{\frac{m+2i-1}{2}}{\frac{m-2i-1}{2}} & \text{if } m \text{ is odd.} \end{cases}$$

5. Reproving the formula for μ in $\mathbb{P}^*$

We will now reprove the formula for μ in Theorem 2.2. Our principal tools will be the descriptions of Z and M in Lemma 3.1 and the anti-isomorphism in Theorem 4.2. Although we only stated the latter result for $[\boldsymbol{n}]^*$, it clearly holds also for $\mathbb{P}^*$. The lemma must be modified slightly by letting n tend to ∞. So the formulas for z and Z become

$$z(\boldsymbol{k}) = [\boldsymbol{k}, \infty)[\boldsymbol{k-1}]^*$$

and

$$Z(u) = \mathbb{P}^* z(u).$$

Proof (of Theorem 2.2). We wish to show that $\zeta * \mu$ is the identity element of the incidence algebra. So by Theorem 4.2, it suffices to show that $M \circ Z$ is the identity endomorphism. For any $u \in \mathbb{P}^*$ we have, using the multiplicativity of m,

$$M \circ Z(u) = M(\mathbb{P}^* z(u)) = (\varepsilon - \mathbf{1}) m(\mathbb{P})^* m(z(u)).$$

So it will be enough to show

$$(\varepsilon - \mathbf{1}) m(\mathbb{P})^* = \varepsilon \quad \text{and} \quad m(z(\boldsymbol{k})) = \boldsymbol{k}$$

for all $\boldsymbol{k} \in \mathbb{P}$.

For the first equation, note that

$$\begin{aligned} m(\mathbb{P}) &= m(\mathbf{1}) + m(\mathbf{2}) + m(\mathbf{3}) + \cdots \\ &= (\mathbf{1} - \mathbf{2}^+(\varepsilon - \mathbf{1})) + (\mathbf{2}^+(\varepsilon - \mathbf{1}) - \mathbf{3}^+(\varepsilon - \mathbf{1})) + (\mathbf{3}^+(\varepsilon - \mathbf{1}) - \mathbf{4}^+(\varepsilon - \mathbf{1})) + \cdots \\ &= \mathbf{1}. \end{aligned}$$

Now (1) gives

$$(\varepsilon - \mathbf{1}) m(\mathbb{P})^* = (\varepsilon - \mathbf{1}) \mathbf{1}^* = (\varepsilon - \mathbf{1})(\varepsilon - \mathbf{1})^{-1} = \varepsilon.$$

For the second equation, we note that the case $\boldsymbol{k} = \mathbf{1}$ has already been done in the previous paragraph, since

$$m(z(\mathbf{1})) = m(\mathbb{P}) = \mathbf{1}.$$

Note that for $k \geqslant 2$ the same telescoping phenomenon gives

$$m([\boldsymbol{k}, \infty)) = \boldsymbol{k}^+(\varepsilon - \mathbf{1}) \quad \text{and} \quad m([\boldsymbol{k-1}]) = \mathbf{1} - \boldsymbol{k}^+(\varepsilon - \mathbf{1}).$$

Combining this with (1), we obtain

$$\begin{aligned} m(z(\boldsymbol{k})) &= m([\boldsymbol{k}, \infty)) m([\boldsymbol{k-1}])^* \\ &= \boldsymbol{k}^+(\varepsilon - \mathbf{1})(\mathbf{1} - \boldsymbol{k}^+(\varepsilon - \mathbf{1}))^* \\ &= \boldsymbol{k}^+(\varepsilon - \mathbf{1})(\varepsilon - \mathbf{1} + \boldsymbol{k}^+(\varepsilon - \mathbf{1}))^{-1} \\ &= \boldsymbol{k}^+(\varepsilon - \mathbf{1})((\varepsilon + \boldsymbol{k}^+)(\varepsilon - \mathbf{1}))^{-1} \\ &= \boldsymbol{k}^+(\varepsilon - \mathbf{1})(\varepsilon - \mathbf{1})^{-1}(\varepsilon + \boldsymbol{k}^+)^{-1} \\ &= \boldsymbol{k}\boldsymbol{k}^*(\boldsymbol{k}^*)^{-1} \\ &= \boldsymbol{k}. \end{aligned}$$

This finishes the proof of Theorem 2.2. □

6. Generalized subword order

We now present a rubric due to Sagan and Vatter [11] under which the theorems about rationality of the Möbius and zeta functions for A^* and $\mathbb{P}^*$ both become special cases. Let P be any poset. Turn P^* into a poset by letting $u \leqslant_{P^*} w$ if there is a subword $w(i_1) \ldots w(i_l)$ of w having length $l = \ell(u)$ such that

$$u(i) \leqslant_P w(i_l) \quad \text{for } 1 \leqslant i \leqslant l.$$

We call this the *generalized subword order* on P^*. Note that we recover A^* or $\mathbb{P}^*$ if we take P to be an anti-chain or a well-ordered countably infinite chain, respectively. Note also that we will leave off the subscripts on inequalities if it is clear from context which poset is meant.

Many of our results about ζ for $\mathbb{P}^*$ from Sections 3 and 4, as well as the corresponding ones for A^* of Björner and Reutenauer [3], generalize easily to P^*. Given an element $a \in P$ we consider the *upper order ideal generated by a* and its set-theoretic complement

$$I_a = \{c \in P \mid c \geqslant_P a\} \quad \text{and} \quad J_a = P - I_a,$$

respectively. We define $Z(u)$ in P^* by (2), as before, and also define a multiplicative map from P^* to $\mathbb{Z}\langle\langle P\rangle\rangle$ by

$$z(a) = I_a J_a^*.$$

The proofs we have already seen contain all the ideas needed to demonstrate the next result, so we suppress the details. We will also use the same notation as in the earlier results, as we did with $Z(u)$.

Theorem 6.1. *Let P be any finite poset. Then for any $u \in P^*$ we have*

$$Z(u) = P^* z(u)$$

and so $Z(u)$ is rational. Similarly, in $\mathbb{Z}\langle\langle P^2\rangle\rangle$ the series

$$Z_\otimes = \sum_{u,w} \zeta(u, w) u \otimes w$$

is rational. Finally, if u has l_a occurrences of a for each $a \in P$, then we have the length generating function

$$Z(u; t) = \frac{1}{1 - |P|t} \prod_{a \in P} \left(\frac{|I_a|t}{1 - |J_a|t} \right).$$

Generalizing our results about μ is more delicate. Indeed, there is no known formula for the Möbius function in P^* for arbitrary P. However, there is a class of posets for which μ has been found. To characterize the Möbius function in these posets, we need the appropriate definition of a normal embedding. Suppose $\hat{0}$ is a new element not in P and form a poset $\hat{P}$ on $P \cup \hat{0}$ by adding the relations $\hat{0} <_{\hat{P}} a$ for all $a \in P$. One defines support and expansion exactly as before, just replacing 0 with $\hat{0}$. Then for $u, w \in P^*$, an *embedding* of u into w is an expansion $\eta_u \in \hat{P}^*$ of length $\ell(w)$ such that

$$\eta_u(i) \leqslant_{\hat{P}} w(i) \quad \text{for } 1 \leqslant i \leqslant \ell(w).$$

Clearly, $u \leqslant_{P^*} w$ if and only if there is an embedding of u into w.

To define normality, call P a *rooted tree* if its Hasse diagram is a tree having an unique minimal element. More generally, call P a *rooted forest* if the connected components of its Hasse diagram are rooted trees. Note that in this case $\hat{P}$ is a rooted tree. So given $a \in P$, we can define a^- to be the element adjacent to a on the unique path from a to $\hat{0}$ in $\hat{P}$. If P is a rooted forest, define an embedding η_u of u into w to be *normal* if it satisfies the following pair of conditions:

(1) For $1 \leqslant i \leqslant \ell(w)$ we have $\eta_u(i) = w(i), w(i)^-$, or $\hat{0}$.
(2) For all $a \in P$ and every run $[r, t]$ of a's in w, we have
 (a) $(r, t] \subseteq \operatorname{Supp} \eta_u$ if a is minimal in P,
 (b) $r \in \operatorname{Supp} \eta_u$ otherwise.

In this situation, the definition of the *defect* of a normal embedding η_u into w should come as no surprise:

$$d(\eta_u) = \#\{i \mid \eta_u(i) = w(i)^-\}.$$

The following theorem generalizes both Theorems 2.1 and 2.2.

Theorem 6.2 (*Sagan and Vatter [11]*). *Let P be a rooted forest. Then the Möbius function of P^* is given by*

$$\mu(u,w) = \sum_{\eta_u} (-1)^{\eta_u},$$

where the sum is over all normal embeddings η_u of u into w.

With this result in hand, generalizing the results for μ follows the same lines as for ζ. If P is any poset then let O_P be the set of minimal elements of P. (So if $P = \mathbb{P}$ then $O_P = \{1\}$.) Also, if $a \in P$ then the set of elements *covering a* is

$$C_a = \{c \in P \mid c > a \text{ and there is no } b \text{ with } c > b > a\}.$$

Now let P be a rooted forest and define $M(u)$ for $u \in P^*$ by Eq. (3). The corresponding multiplicative function is

$$m(a) = \begin{cases} a - \left(\sum_{c \in C_a} c^+\right)(\varepsilon - O_P) & \text{if } a \in O_P, \\ \left(a^+ - \sum_{c \in C_a} c^+\right)(\varepsilon - O_P) & \text{else.} \end{cases}$$

Again, there is nothing really new in considering an arbitrary rooted forest instead of $\mathbb{P}$, so we will merely state the results.

Theorem 6.3. *Let P be a finite rooted forest. Then for any $u \in P^*$ we have*

$$M(u) = (\varepsilon - O_P)m(u)$$

and so $M(u)$ is rational. Similarly, in $\mathbb{Z}\langle\langle P^2 \rangle\rangle$ the series

$$M_\otimes = \sum_{u,w} \mu(u,w) u \otimes w$$

is rational. Finally, if u has l_a occurrences of a for each $a \in P$, then we have the length generating function

$$M(u;t) = \left(\frac{t}{1-t}\right)^{l(u)} \prod_{a \in O_P} [1 - t - |C_a|(1 - |O_P|t)]^{l_a} \prod_{b \notin O_P} (1 - |C_b|)^{l_b} (1 - |O_P|t)^{l_b}.$$

In particular, if u contains any element which is covered by exactly one other element then $M(u;t) = 0$.

As a final remark, one can give a proof of Theorem 6.2 in the same way as was done for Theorem 2.2 in the previous section.

7. Comments and open problems

We end with some comments and open problems.

7.1. Generating functions for ζ^m

It would be interesting to compute the generating function for ζ^m in $[\boldsymbol{n}]^*$ for arbitrary n. It appears that one can say something, at least for $n = 3$. Let $a_m(t), b_m(t), c_m(t)$ stand for $f^m(x_1)$, $f^m(x_2)$, $f^m(x_3)$, respectively, when using the length generating function. Then numerical evidence suggests that there is a polynomial $d_m(t)$ such that the denominators of our three rational functions factor as $d_1 d_2 \cdots d_{2m-2}$, $d_{2m-3} d_{2m-2} d_{2m-1}$, and $d_{2m-2} d_{2m}$, respectively. Note that the denominator of $a_m(t)$ behaves differently from the $n = 2$ case in that the number of factors increases with m. Table 1 gives the values of $a_m(t), b_m(t), c_m(t)$ when $0 \leqslant m \leqslant 2$.

Table 1
The values of $a_m(t)$, $b_m(t)$, $c_m(t)$ when $0 \leqslant m \leqslant 2$

m	$a_m(t)$	$b_m(t)$	$c_m(t)$
0	t	t	t
1	$3t$	$\frac{2t}{1-t}$	$\frac{t}{1-2t}$
2	$\frac{2t(3-7t+3t^2)}{(1-t)(1-2t)}$	$\frac{t(3-5t)}{(1-t)(1-2t)(1-3t)}$	$\frac{t(1-t)}{(1-2t)(1-6t+3t^2)}$

Fig. 3. The Hasse diagram for the poset Λ.

7.2. The poset Λ

Can anything be said about the Möbius function of P^* if P is not a rooted forest? Again, computer evidence suggests that the answer is "yes". Consider the poset Λ in Fig. 3 which is the smallest one to which Theorem 6.2 does not apply. Let $T_n(x)$ denote the *nth Tchebyshev polynomial of the first kind,* which can be defined as the unique polynomial such that

$$T_n(\cos\theta) = \cos(n\theta).$$

Conjecture 7.1 (*Sagan and Vatter [11]*). *For all $i \leqslant j$, $\mu(a^i, c^j)$ is the coefficient of x^{j-1} in $T_{i+j}(x)$.*

Finding a proof of this conjecture by using generating functions or any other means would be most welcome.

7.3. The pattern poset

The original motivation of Sagan and Vatter [11] for studying this poset of compositions came from the theory of patterns. Let $\pi = a_1a_2\ldots a_n$ be a sequence of distinct integers and let $\sigma = b_1b_2\ldots b_n$ be another such sequence. Saying that π and σ are *order isomorphic* means that $a_i < a_j$ if and only if $b_i < b_j$ for all distinct pairs $i, j \in [n]$. Let $\mathfrak{S}_n$ denote that symmetric group on $[n]$ and let $\mathfrak{S} = \cup_{n\geqslant 0}\mathfrak{S}_n$. If $\pi = a_1a_2\ldots a_k \in \mathfrak{S}_k$ and $\sigma = b_1b_2\ldots b_n \in \mathfrak{S}_n$ then we say that σ *contains* π *as a pattern*, and write $\pi \leqslant \sigma$, if there is a subsequence $b_{i_1}b_{i_2}\ldots b_{i_k}$ of σ which is order isomorphic to π. Note that this relation turns $\mathfrak{S}$ into a poset. If σ does not contain π then σ *avoids* π. If Π is a set of permutations, then we write $\mathfrak{S}_n(\Pi)$ for the set of permutations in $\mathfrak{S}_n$ which avoid every $\pi \in \Pi$ and similarly for $\mathfrak{S}(\Pi)$. For example, $\sigma = 42513$ contains $\pi = 132$ because of the subsequence 253, but $\sigma \in \mathfrak{S}_5(123)$. We say that two sets of permutations Π and Σ are *Wilf equivalent* if $|\mathfrak{S}_n(\Pi)| = |\mathfrak{S}_n(\Sigma)|$ for all $n \geqslant 0$. The field of pattern containment and avoidance is currently very active and the reader is referred to Bóna's book [7] for more details.

One much-studied subset of $\mathfrak{S}$ is the lower order ideal $\mathfrak{L}$ of *layered permutations* which are those having the form

$$\pi = k, k-1, \ldots, 1, k+l, k+l-1, \ldots, k+1, k+l+m, k+l+m-1, \ldots, k+l+1, \ldots$$

for certain integers $k, l, m, \ldots$ called the *layer lengths*. Note that the layered permutations can also be defined by avoidance as $\mathfrak{L} = \mathfrak{S}(231, 312)$. The map $g : \mathfrak{L} \to \mathbb{P}^*$ gotten by reading off the layer lengths

$$g(\pi) = klm\ldots$$

is easily seen to be an order isomorphism between the two posets. Theorem 2.2 was discovered in an attempt to answer the following question of Wilf, which is still open in general.

Question 7.2 (*Wilf [17]*). *What can be said about the Möbius function of permutations under the pattern-containment ordering?*

To any subset $S \subseteq \mathfrak{S}$ one can associate a generating function

$$f_S = f_S(x) = \sum_{\pi \in S} x^{|\pi|}$$

and there has been much activity centered around determining for which S we have f_S algebraic or even rational. Note that if $S \subseteq \mathfrak{L}$ then f_S becomes a norm generating function in the sense of Section 4. Of particular interest to us here is the work of Albert and Atkinson [1] using simple permutations to compute these generating functions.

A *block* of a permutation π is an interval $[i,j]$ whose image under the map π is also an interval. Any $\pi \in \mathfrak{S}_n$ has *trivial blocks* consisting of singletons and the interval $[n]$. A permutation having no nontrivial blocks is *simple*. Albert and Atkinson showed that any lower order ideal $S \subset \mathfrak{S}$ containing only finitely many simple permutations has an algebraic generating function. In fact, their proof is constructive, giving a way to actually find f_S (or at least a polynomial which it satisfies). In particular cases, their construction will show that f_S is actually rational.

Proposition 7.3 (*Albert and Atkinson [1]*). *Every lower order ideal properly contained in* $\mathfrak{S}(231)$ *has a rational generating function*.

(Albert and Atkinson actually proved this for $\pi = 132$, but there is an isomorphism between the posets $\mathfrak{S}(132)$ and $\mathfrak{S}(231)$.) This proposition implies that the generating function f_S of any lower order ideal S in $\mathfrak{L} = \mathbb{P}^*$ is rational. Thus the same is also true of any upper order ideal in $\mathbb{P}^*$ since its complement is a lower order ideal. So in one sense, this is much more general than our results about the norm generating function of ζ. On the other hand, using their algorithm to compute $Z(u; x)$ does not make its finer structure as readily apparent, such as the fact that the generating function only depends on the type of u. For completeness, we record this observation again in the language of pattern avoidance.

Proposition 7.4. *Let* π *and* σ *be two layered permutations whose layer lengths are equal as multisets*. *Then* $\{231, 312, \pi\}$ *and* $\{231, 312, \sigma\}$ *are Wilf equivalent*.

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References

[1] M.H. Albert, M.D. Atkinson, Simple permutations and pattern restricted permutations, Discrete Math. 300 (1–3) (2005) 1–15.
[2] F. Bergeron, M. Bousquet-Mélou, S. Dulucq, Standard paths in the composition poset, Ann. Sci. Math. Québec 19 (2) (1995) 139–151.
[3] J. Berstel, C. Reutenauer, Rational series and their languages, EATCS Monographs on Theoretical Computer Science, Vol. 12, Springer, Berlin, 1988.
[4] A. Björner, The Möbius function of subword order, in: Invariant Theory and Tableaux, Minneapolis, MN, 1988, IMA Volumes of Mathematical Applications, Vol. 19, Springer, New York, 1990, pp. 118–124.
[5] A. Björner, C. Reutenauer, Rationality of the Möbius function of subword order, Theoret. Comput. Sci. 98 (1) (1992) 53–63 Second Workshop on Algebraic and Computer-theoretic Aspects of Formal Power Series, Paris, 1990.
[6] A. Björner, R. Stanley, An analogue of young's lattice for compositions, preprint at arXiv:math.CO/0508043.
[7] M. Bóna, Combinatorics of permutations, Discrete Mathematics and its Applications, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[8] S. Eilenberg, Automata, Languages, and Machines, Vol. B, Academic Press, Harcourt Brace Jovanovich Publishers, New York, 1976, With two chapters ("Depth decomposition theorem" and "Complexity of semigroups and morphisms") by Bret Tilson, Pure and Applied Mathematics, Vol. 59.
[9] J.B. Kruskal, The theory of well-quasi-ordering: a frequently discovered concept, J. Combin. Theory Ser. A 13 (1972) 297–305.
[10] P. Paule, M. Schorn, A mathematica version of Zeilberger's algorithm for proving binomial coefficient identities, J. Symbolic Comput. 20 (5–6) (1995) 673–698 Symbolic Computation in Combinatorics Δ_1, Ithaca, NY, 1993.
[11] B. Sagan, V. Vatter, The möbius function of the composition poset, preprint at arXiv:math.CO/0507485.
[12] J. Snellman, Saturated chains in composition posets, preprint at arXiv:math.CO/0505262.
[13] J. Snellman, Standard paths in another composition poset, Electron. J. Combin. 11 (1) (2004) Research Paper 76, 8pp. (electronic).

[14] R.P. Stanley, The Fibonacci lattice, Fibonacci Quart. 13 (3) (1975) 215–232.
[15] R.P. Stanley, Differential posets, J. Amer. Math. Soc. 1 (4) (1988) 919–961.
[16] R.P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Studies in Advanced Mathematics, Vol. 49, Cambridge University Press, Cambridge, 1997. (With a foreword by Gian-Carlo Rota, corrected reprint of the 1986 original.)
[17] H.S. Wilf, The patterns of permutations, Discrete Math. 257 (2–3) (2002) 575–583.
[18] D. Zeilberger, A fast algorithm for proving terminating hypergeometric identities, Discrete Math. 80 (2) (1990) 207–211.
[19] D. Zeilberger, A holonomic systems approach to special functions identities, J. Comput. Appl. Math. 32 (3) (1990) 321–368.