On the distortion of a copula and its margins

joint work with Yugu Xiao (Renmin University)

Emiliano A. Valdez, University of Connecticut

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University of Connecticut

Introduction and and motivation

In insurance and financial risk modeling, practitioners may be required to compute aggregate risk distribution for a portfolio of correlated risks:

- pricing or premium calculation of contingent payoffs on these multiple risks
- capital allocation among several lines of business
- analyzing diversification benefits within an enterprise
- reporting of risks to external parties, e.g. regulators

Models used to describe the correlation structure:

- multivariate distributions with correlation
- "copulas" - separates the peculiar characteristics of marginals

The concept of distortion

Apply a probability distortion to multivariate distributions:

- to adjust for risk and uncertainty in aggregating a portfolio of correlated risks
- to change probability measure to price contingent claims involving multiple risks
- a direct extension of the distortion concept in the unvariate case

Be careful in the extension because you want to preserve properties of a copula:

- three kinds of multivariate distortion - will or will not affect the dependence structure

In the paper, we show much more: numerous examples, multivariate ordering of risks, integral transform with distortion

Copulas - recipe for disaster?

Article on *Wired Magazine*, 23 Feb 2009, by F. Salmon titled "Recipe for Disaster: The Formula that Killed Wall Street"[1].

$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

- Collapse of the market on defaultable loans, collaterized debt obligations, other credit derivatives (huge $$$'s involved!!!)
- Became popular because of Li's pricing model: D.X. Li (2000), On default correlation: a copula function approach, *Journal of Fixed Income*, vol. 9, pp. 43-54.
- Pricing basis: Gaussian or normal copula.

[1] Source: P. Embrechts slides, "Did a Mathematical Formula Really Blow up Wall Street?"

Sklar's representation theorem

Sklar (1959): There exists a copula function C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the marginal for X_i, $i = 1, \ldots, n$.

Equivalently, we write

$$\mathsf{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(\mathsf{P}(X_1 \leq x_1), \ldots, \mathsf{P}(X_n \leq x_n)).$$

C need not be unique, but it is unique for continuous marginals. Else, C is uniquely determined on $\mathrm{Ran}F_1 \times \ldots \times \mathrm{Ran}F_n$.

In the continuous case, this unique copula can be expressed as

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

where F_i^{-1} are the respective quantile functions.

Examples of (implicit) copulas

Normal copula:

$$C^n_{\mathbf{R}}(\mathbf{u}) = \mathbf{\Phi}_{\mathbf{R}}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where Φ is the cdf of standard univariate normal, $\mathbf{\Phi}_{\mathbf{R}}$ is the joint cdf of $\mathbf{X} \sim \mathsf{N}_n(\mathbf{0}, \mathbf{R})$ with $\mathbf{R}$, the correlation matrix.

The case where $\mathbf{R} = \mathbf{I}_n$ results in independence, and $\mathbf{R} = \mathbf{J}_n$ gives comonotonicity.

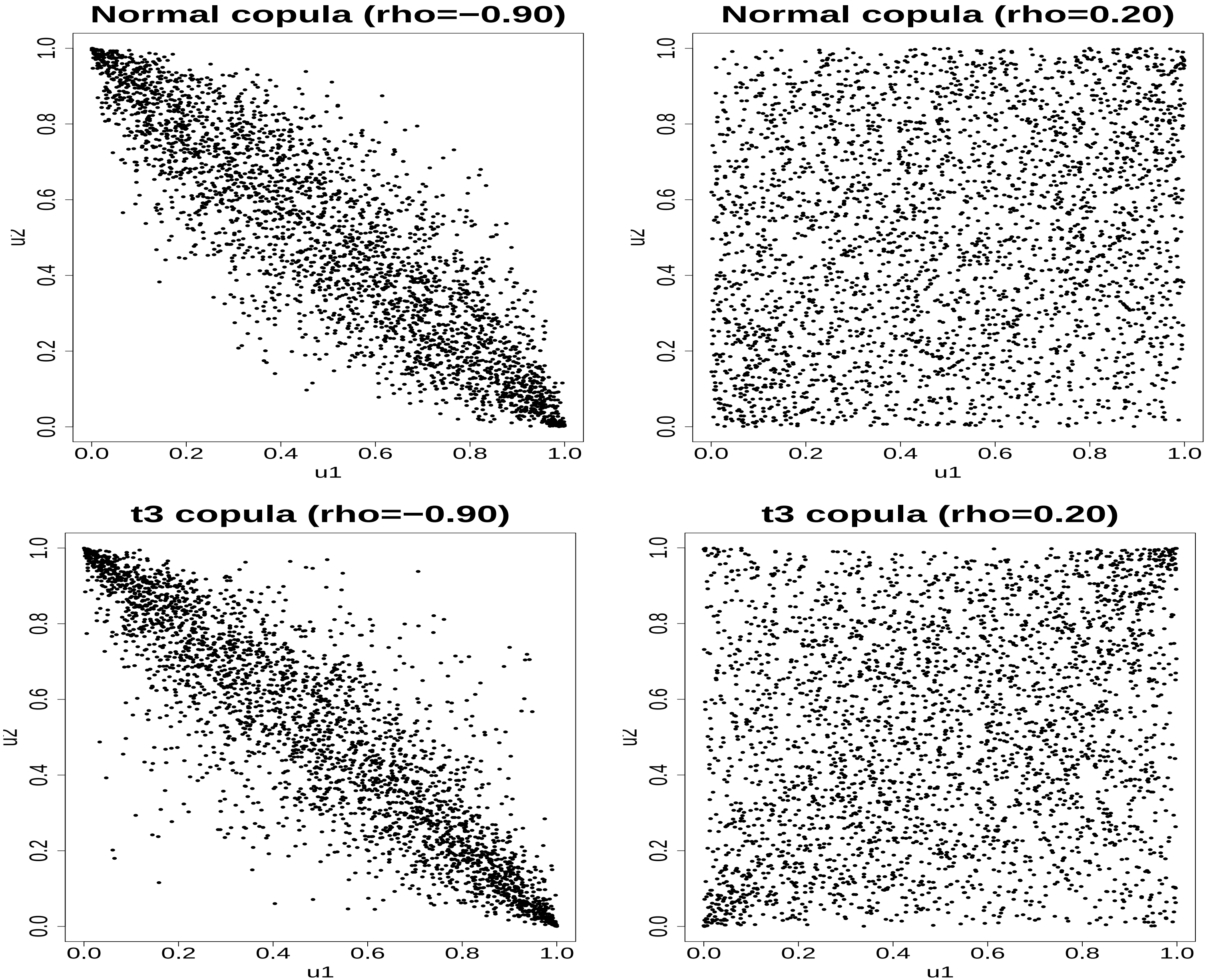
t copula:

$$C^n_{\nu,\mathbf{R}}(\mathbf{u}) = \mathbf{t}_{\nu,\mathbf{R}}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)),$$

where t_ν is the cdf of standard univariate t, $\mathbf{t}_{\nu,\mathbf{R}}$ is the joint cdf of $\mathbf{X} \sim \mathsf{t}_n(\nu, \mathbf{0}, \mathbf{R})$ with $\mathbf{R}$, the correlation matrix.

The case where $\mathbf{R} = \mathbf{J}_n$ gives comonotonicity, but $\mathbf{R} = \mathbf{I}_n$ does not result in independence.

Simulation - normal vs t copula



Some problems with multivariate normal

Some believe that there are deficiencies of the normal for multivariate modeling in finance/insurance:

- The tails of the margins may be too thin, and hence fail to generate some extreme values.
- As a consequence, in the multivariate sense, it fails to capture phenomenon of joint extreme movements. Simultaneous large values may be relatively infrequent - generally believed to lack tail dependence.
- Too much symmetry - lack of presence of skewness. Some financial/insurance data exhibits long tails.

Special class: Archimedean copulas

C is an *Archimedean* if it has the form

$$C(u_1, \ldots, u_n) = \psi^{-1}(\psi(u_1) + \cdots + \psi(u_n)),$$

for some function ψ (called the generator) satisfying:

- $\psi(1) = 0$;
- ψ is decreasing; and
- ψ is convex.

To ensure you get a legitimate copula for higher dimensions, ψ^{-1} must be completely *monotonic*, i.e. its derivatives alternate in signs.

An important source of Archimedean generators is the inverses of the Laplace transforms of distribution functions.

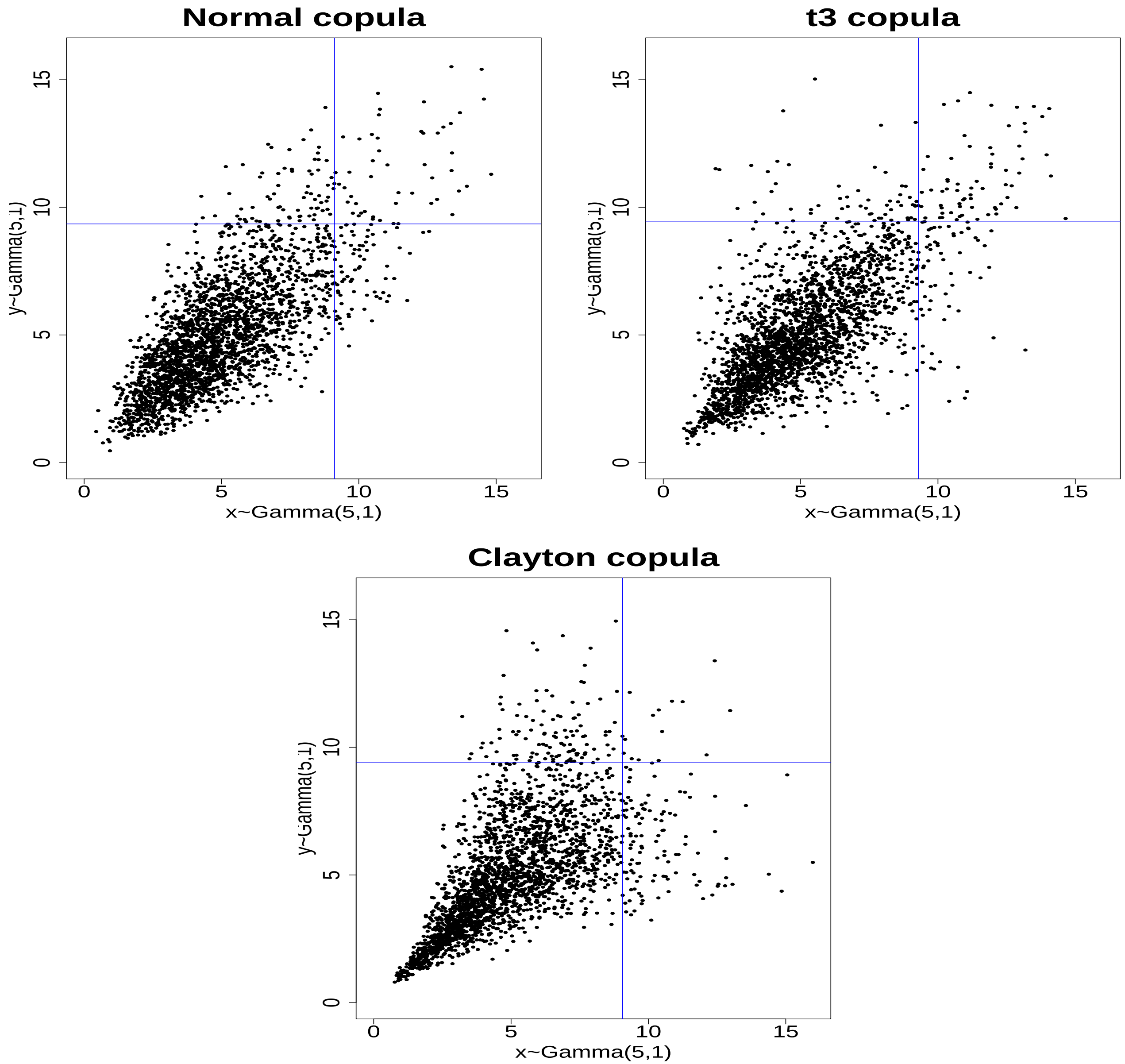
Feller (1971): A function φ on $[0, \infty]$ is the Laplace transform of a cdf F if and only if φ is completely monotonic with $\varphi(0) = 1$.

Archimedean copulas and their generators

Family	Generator $\psi(t)$	Range of α	$C(\mathbf{u})$
Independence	$-\log(t)$	na	$\prod_{i=1}^{n} u_i$
Clayton	$t^{-\alpha} - 1$	$\alpha > 0$	$\left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-1/\alpha}$
Gumbel-Hougaard	$(-\log t)^{\alpha}$	$\alpha \geq 1$	$\exp\left\{-\left[\sum_{i=1}^{n}(-\log u_i)^{\alpha}\right]^{1/\alpha}\right\}$
Frank	$-\log\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right)$	$\alpha > 0$	$-\frac{1}{\alpha}\log\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i}-1)}{(e^{-\alpha}-1)^{n-1}}\right]$

Normal, t, and Clayton copulas

Marginals: Gamma(5,1), $\rho = 0.75$, and $\nu = 3$



Review of univariate distortion

We say $g : [0, 1] \to [0, 1]$ is a *distortion function* if it satisfies the following properties:

- $g(0) = 0$ and $g(1) = 1$; and
- g is continuous and non-decreasing.

The transformation of the distribution function F_X

$$F_{X^*}(x) = g[F_X(x)] = g \circ F_X(x)$$

is the df of X^* that leads to a *probability distortion* of X to X^*.

Wang Transform: Here $g(t) = \Phi[\Phi^{-1}(t) + \gamma]$ preserves Normal and Lognormal distributions:

- $X \sim$ Normal(μ, σ^2) implies $X^* \sim$ Normal$(\mu - \gamma\sigma, \sigma^2)$
- $X \sim$ Lognormal(μ, σ^2) implies $X^* \sim$ Lognormal$(\mu - \gamma\sigma, \sigma^2)$

Some well-known distortion functions

Distortion	Functional form $g(t)$	Inverse form $g^{-1}(s)$	Convex constraints	Concave constraints
Proportional hazard	$t^{1/\gamma}$	s^{γ}	$0 < \gamma \leq 1$	$\gamma \geq 1$
Exponential	$\dfrac{1 - e^{-\gamma t}}{1 - e^{-\gamma}}$	$\log[1 - s(1 - e^{-\gamma}]$	$\gamma < 0$	$\gamma > 0$
Logarithmic	$\dfrac{1}{\gamma} \log[1 - t(1 - e^{\gamma})]$	$\dfrac{e^{\gamma t - 1}}{e^{\gamma - 1}}$	$\gamma < 0$	$\gamma > 0$
Wang transform	$\Phi[\Phi^{-1}(t) + \gamma]$	$\Phi[\Phi^{-1}(s) - \gamma]$	$\gamma \leq 0$	$\gamma \geq 0$
Dual-power	$1 - (1 - t)^{\gamma}$	$1 - (1 - s)^{1/\gamma}$	$\gamma \leq 1$	$\gamma \geq 1$

Note: The convex/concave constraints are for the function $g(t)$.

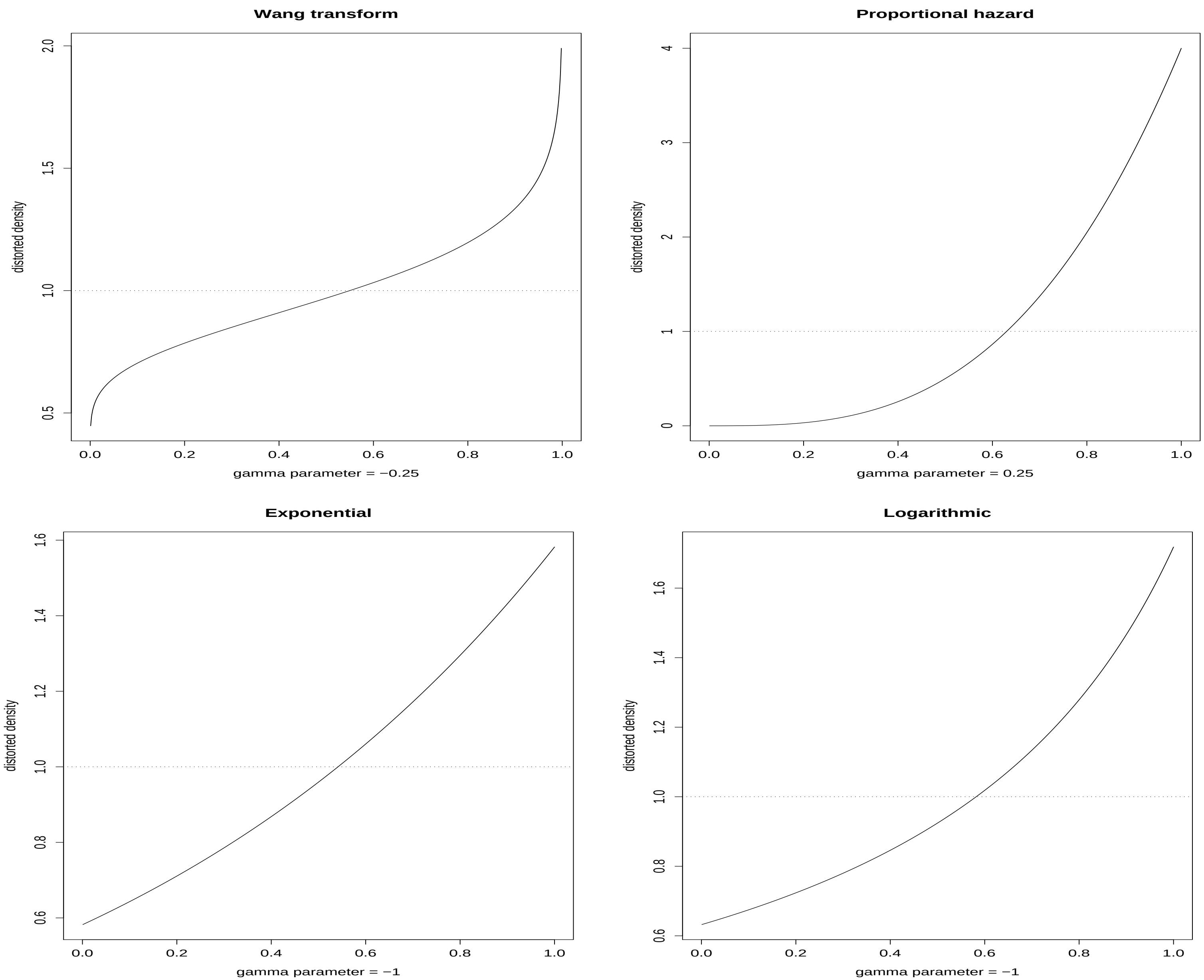
Adjustment for risk

- Wang (1996) defines premium principle based on distortion, motivated by Yaari (1987) - an alternative to utility framework.
- For a (non-negative) risk X, the premium principle associated with the distortion function:

$$\pi_g(X) = \mathsf{E}(X^*) = \int_0^\infty [1 - g[F_X(x)]]dx.$$

- The difference $\pi_g(X) - \mathsf{E}(X)$ is risk premium (or adjustment for risk), and is positive if g is convex. (Jensen's inequality)
- Distortion can also be used to price contingent payoffs, say $h(X)$, associated with an underlying asset with value X. In case of no-arbitrage, these risk-neutral (distorted) probabilities can be derived from observable prices in the market.

The effect of distortion



Parameter uncertainty

In practice, we estimate probability distributions usually based on limited data so that parameter uncertainty is always present.

To illustrate, consider the case where X, conditional on the risk parameter γ, is Exponential with: $F_X(x|\gamma) = 1 - \exp(-\gamma x)$.

If γ has a Gamma distribution with a *scale* and *shape* parameters λ and α, respectively, the unconditional distribution of X is a Pareto distribution expressed as

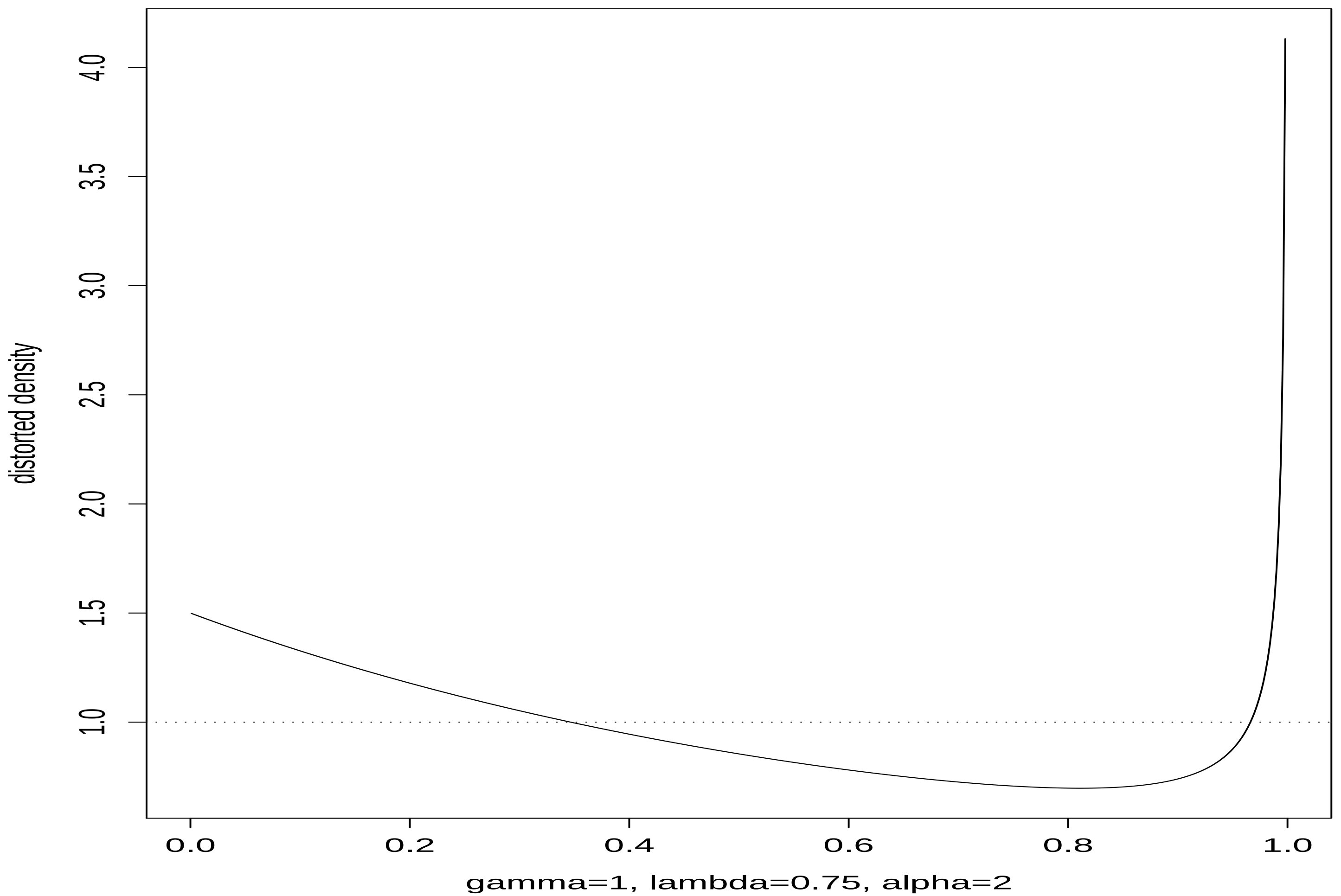
$$F_X(x) = 1 - (1 + \lambda x)^{-\alpha}.$$

Indeed, one can easily derive the corresponding distortion function in this case:

$$g(t) = 1 - (1 + \log(1 - t)^{-\lambda/\gamma})^{-\alpha}.$$

Note that this distortion function is neither strictly convex nor concave.

Effect of distortion for parameter uncertainty



Distortion of the first kind

Let $g_1, \ldots, g_n$ be n distortion functions. Then the transformation of the copula associated with $\mathbf{X}$ defined by

$$C_{\mathbf{X}}(u_1^*, \ldots, u_n^*) = C_{\mathbf{X}}(g_1(u_1), \ldots, g_n(u_n))$$

induces a multivariate probability distortion of $\mathbf{X}$ to $\mathbf{X}^*$.

This type of a distortion leads to a simple distortion of the margins while preserving the copula structure.

An example of this type is the multivariate extension of the Wang transform constructed by Kijima (2006).

Example - Multivariate Burr I

Consider the Weibull margins

$$F_i(x_i) = 1 - \exp(-x_i^k), \qquad x_i \geq 0, k > 0,$$

for $i = 1, \ldots, n$, linked with a legitimate copula, for example, a Clayton copula defined by

$$C_{\mathbf{X}}(u_1, \ldots, u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-1/\alpha}.$$

With the distortion function $g(t) = 1 - (1 - \log(1-t))^{-\gamma}$, this leads to Burr margins

$$F_i^*(x_i) = 1 - [(1 + x_i^k)]^{-\gamma}, \qquad x_i \geq 0, k > 0, \gamma > 0.$$

Distortion of the second kind

Let $g_1, \ldots, g_n$ be n distortion functions. Then the transformation of the copula associated with $\mathbf{X}$ defined by

$$\widehat{C}(u_1^*, \ldots, u_n^*) = \widehat{C}(g_1(u_1), \ldots, g_n(u_n)),$$

where $\widehat{C}$ is a copula function, induces a multivariate probability distortion of $\mathbf{X}$ to $\widehat{\mathbf{X}}$.

This leads to a simultaneous distortion of the margins and the copula structure.

Multivariate Burr II: Similarly distort margins from Weibull to Burr, but transform the copula structure to Gumbel-Hougaard

$$\widehat{C}(u_1, \ldots, u_n) = \exp\left\{-\left[\sum_{i=1}^{n}(-\log u_i)^{\alpha}\right]^{1/\alpha}\right\}.$$

Result is yet another multivariate Burr distribution.

Distortion of the third kind

Let g be a distortion function with inverse g^{-1} that is absolutely monotonic of order n on $[0, 1]$. Then the transformation of the copula associated with $\mathbf{X}$ defined by

$$C_g(u_1, \ldots, u_n) = g^{-1}(C_{\mathbf{X}}(g(u_1), \ldots, g(u_n)))$$

induces a distortion of $\mathbf{X}$ to $\widetilde{\mathbf{X}}$.

C_g induced by this distortion satisfies the necessary properties of a copula and is then the copula associated with the distorted $\widetilde{\mathbf{X}}$ and therefore can be written as

$$C_g(u_1, \ldots, u_n) = C_{\widetilde{\mathbf{X}}}(u_1, \ldots, u_n).$$

For proof, see Morillas (2005). This leads to a synchronized distortion of the margins and the copula structure, and a new method of constructing new copulas from a given one.

Interesting to note that this preserves the margins; it simply distorts the dependence structure.

An actuarial application

Consider an insurance portfolio of fire insurance policies where the loss amounts vary according to:

- buildings X_1
- contents X_2
- loss of profits X_3

To accommodate the possible large number of zeroes in each type of loss, we use a mixture model of the form:

$$f_k(x) = \begin{cases} p_k, & \text{for } x = 0 \\ (1-p_k) f_{\mathsf{LN},k}(x), & \text{for } x > 0 \end{cases}.$$

LN refers to the log-normal distribution with parameters μ and σ.

It is also easy to prove that the marginal CDF for the mixture is:

$$F_k(x) = p_k + (1-p_k) F_{\mathsf{LN},k}(x), \ \text{for } k = 1, 2, 3.$$

Marginal parameter and choice of copula

We assume the following parameter values for the margins:

Parameter	Building (X_1)	Contents (X_2)	Profits (X_3)
p	0.05	0.10	0.20
μ	0.01	-0.50	-1.25
σ	0.20	1.30	1.40

For purposes of making the illustration simple, we use a Clayton copula with

$$C(u_1, u_2, u_3) = \left(u_1^{-\alpha} + u_2^{-\alpha} + u_3^{-\alpha} - 2\right)^{-1/\alpha},$$

where the α parameter value is assumed to be 5. This translates to a Kendall's tau correlation of approximately 70%.

Valuing excess of loss reinsurance

- We apply distortion to the case where we value excess of loss reinsurance with retention d so that our variable of interest is:

$$(S-d)_+ = (X_1 + X_2 + X_3 - d)_+,$$

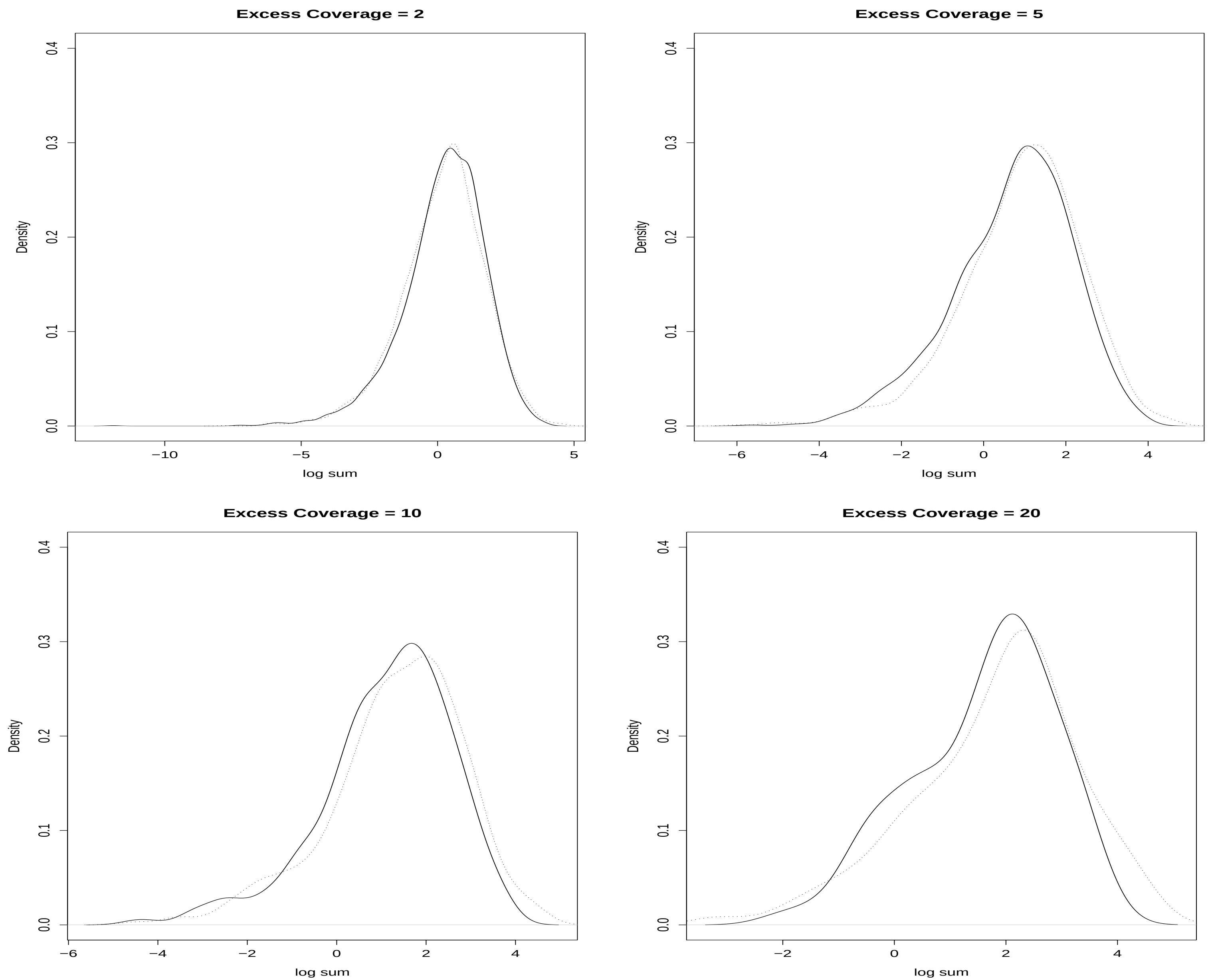
where S denotes the aggregate loss.

- To accommodate parameter uncertainty, we apply *distortion of the third kind* based on $g(t) = t^{1/\gamma}$ with $\gamma = 10$, leading to a re-parameterized Clayton copula

$$C(u_1, u_2, u_3) = \left(u_1^{-\alpha\gamma} + u_2^{-\alpha\gamma} + u_3^{-\alpha\gamma} - 2\right)^{-1/\alpha\gamma}.$$

- We then simulated values of the excess of loss and examined the resulting distribution, with and without the distortion.

Kernel density of the logarithm of sum



Summary of risk adjustments

		Excess of Loss Amount (d)			
	Expectation	2	5	10	20
without distortion	$\mathsf{E}(S-d)_+$	1.2764	0.5768	0.2372	0.0671
with distortion	$\mathsf{E}(S^*-d)_+$	1.3159	0.6667	0.3387	0.1403
	risk adjustment	0.0395	0.0899	0.1015	0.0732
	loading percentage	3.1%	15.6%	42.8%	109.0%

Additional materials in the paper

You can find additional discussion of materials in the paper:

- Multivariate ordering of risks with distortion
  - supermodular ordering
- Multivariate probability integral transform with distortion
  - extended Genest and Rivest (2001) results

Concluding remarks

- Increasingly important to assess the aggregate risk distribution of a portfolio of often correlated risks.
- Some limitations as to specifying just the correlation structures to model the dependencies of risks - users are warned of use of copulas.
- Copulas provide flexibility to allow modeling various dependence structures, allowing to separate the effects of peculiar characteristics of the margins such as thickness of tails.
- We advocate applying distortion to multivariate distributions, and hence to copulas, as a means to adjust for risk and uncertainty in the aggregation of portfolios of correlated risks.
- We caution practical users to understand the implications of distortion.

Some useful references

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