Optimal Capital Allocation Principles

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The allocation of capital

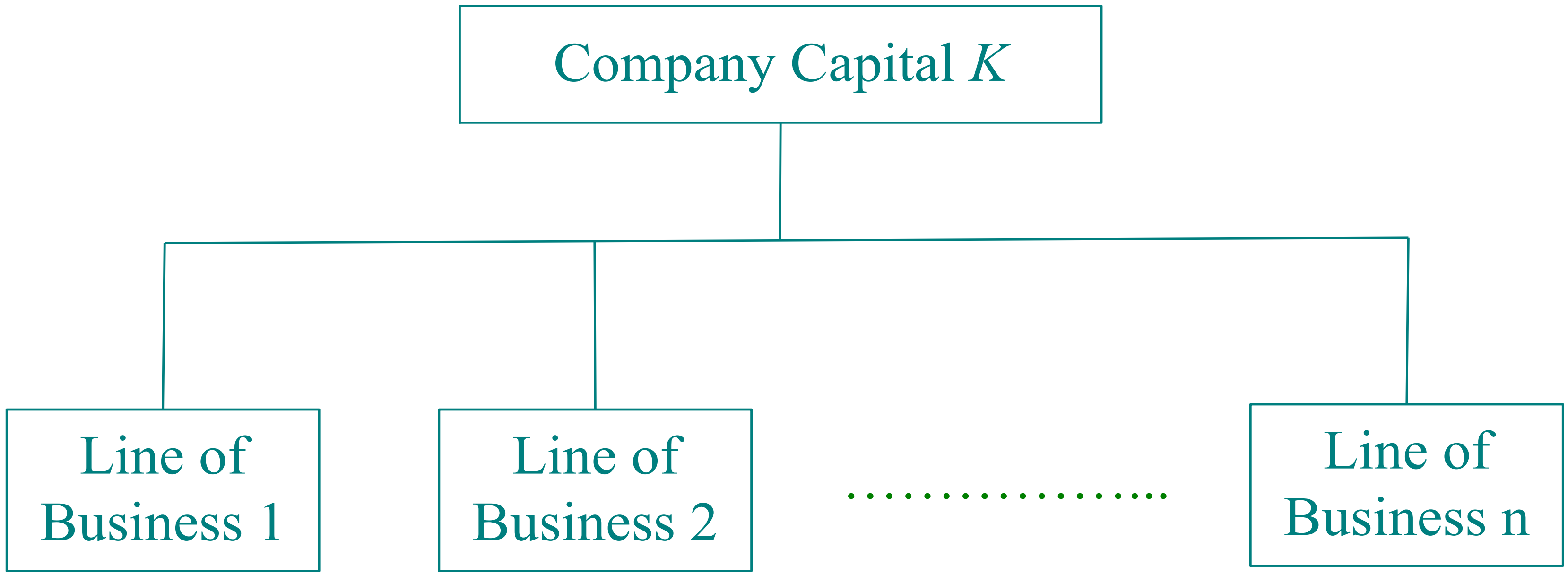
Capital allocation is the term usually referring to the subdivision of a company's aggregate capital across its various constituents:

- lines of business
- its subsidiaries
- product types within lines of business
- territories, e.g. distribution channels
- types of risks: e.g. market, credit, pricing/underwriting, operational

A very important component of *Enterprise Risk Management*:

- identifying, measuring, pricing and controlling risks

Figure: the allocation by lines of business



The literature

There are countless number of ways to allocate aggregate capital. Good overview of methods:

- Cummins (2000); Venter (2004)

Some methods based on decision making tools:

- Cummins (2000)- RAROC, EVA
- Lemaire (1984); Denault (2001) - game theory
- Tasche (2004) - marginal costs
- Kim and Hardy (2008) - solvency exchange option with limited liability

- continued

Some methods based on risk measures/distributions:

- Panjer (2001) - TVaR, multivariate normal
- Landsman and Valdez (2003) - TVaR, multivariate elliptical
- Dhaene, et al. (2008) - TVaR, lognormal
- Valdez and Chernih (2003) - covariance-based allocation, multivariate elliptical
- Tsanakas (2004, 2008) - distortion risk measures, convex risk measures
- Furman and Zitikis (2008) - weighted risk capital allocations

Methods also based on optimization principle:

- Dhaene, Goovaerts and Kaas (2003); Laeven and Goovaerts (2004); Zaks, Frostig and Levikson (2006)

The allocation problem

- Consider a portfolio of n individual losses $X_1, \ldots, X_n$ during some well-defined reference period.
- Assume these random losses have a dependency structure characterized by the joint distribution of the random vector $(X_1, \ldots, X_n)$.
- The aggregate loss is the sum $S = \sum_{i=1}^{n} X_i$.
- Assume company holds aggregate level of capital K which may be determined from a risk measure ρ such that $K = \rho(S) \in \mathbb{R}$.
- Here the capital (economic) is the smallest amount the company must set aside to withstand aggregated losses at an acceptable level.

- continued

- The company now wishes to allocate K across its various business units.

    - determine non-negative real numbers $K_1, \ldots, K_n$ satisfying:

$$\sum_{i=1}^{n} K_i = K.$$

- This requirement is referred to as "the full allocation" requirement.

- We will see that this requirement is a constraint in our optimization problem.

Our contribution to the literature

- We re-formulate the problem as minimum distance problem in the sense that the weighted sum of measure for the deviations of the business unit's losses from their respective capitals be minimized:
  - essentially distances between K_j and X_j
- Takes then into account some important decision making allocation criteria such as:
  - the purpose of the allocation allowing the risk manager to meet specific target objectives
  - the manner in which the various segments interact, e.g. legal structure
- Solution to minimizing distance formula leads to several existing allocation methods. New allocation formulas also emerge.

Risk measures

A risk measure is a mapping ρ from a set Γ of real-valued r.v.'s defined on $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$:

$$\rho : \Gamma \to \mathbb{R} : X \in \Gamma \to \rho\left[X\right].$$

Let $X, X_1, X_2 \in \Gamma$. Some well known properties that risk measures may or may not satisfy:

- *Law invariance*: If $\mathbb{P}[X_1 \leq x] = \mathbb{P}[X_2 \leq x]$ for all $x \in \mathbb{R}$, $\rho[X_1] = \rho[X_2]$.
- *Monotonicity*: $X_1 \leq X_2$ implies $\rho\left[X_1\right] \leq \rho\left[X_2\right]$.
- *Positive homogeneity*: For any $a > 0$, $\rho\left[aX\right] = a\rho\left[X\right]$.
- *Translation invariance*: For $b \in \mathbb{R}$, $\rho\left[X + b\right] = \rho\left[X\right] + b$.
- *Subadditivity*: $\rho\left[X_1 + X_2\right] \leq \rho\left[X_1\right] + \rho\left[X_2\right]$.

α–mixed inverse distribution function

For $p \in (0,1)$, we denote the Value-at-Risk (VaR) or quantile of X by $F_X^{-1}(p)$ defined by:

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}.$$

We define the inverse distribution function $F_X^{-1+}(p)$ of X as

$$F_X^{-1+}(p) = \sup \{x \in \mathbb{R} \mid F_X(x) \leq p\}.$$

The α–mixed inverse distribution function $F_X^{-1(\alpha)}$ of X is:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p).$$

It follows for any X and for all x with $0 < F_X(x) < 1$, there exists an $\alpha_x \in [0,1]$ such that $F_X^{-1(\alpha_x)}(F_X(x)) = x$.

Some important concepts

Conditional Tail Expectation (CTE): (sometimes called TailVaR)

$$\text{CTE}_p [X] = \mathbb{E}\left[X \mid X > F_X^{-1}(p)\right], \qquad p \in (0,1).$$

In general, not subadditivite, but so for continuous random variables.

Comonotonic sum: $S^c = \sum_{i=1}^n F_{X_i}^{-1}(U)$ where U is uniform on $(0,1)$.

The Fréchet bounds:

$$L_F(u_1, \ldots, u_n) \le C(u_1, \ldots, u_n) \le U_F(u_1, \ldots, u_n),$$

where

Fréchet lower bound: $L_F = \max(\sum_{i=1}^n u_i - (n-1), 0)$, and

Fréchet upper bound: $U_F = \min(u_1, \ldots, u_n)$.

Some known allocation formulas

Many well-known allocation formulas fall into a class of proportional allocations.

Members of this class are obtained by first choosing a risk measure ρ and then attributing the capital $K_i = \gamma\rho\left[X_i\right]$ to each business unit i, $i = 1, \ldots, n$.

The factor γ is chosen such that the full allocation requirement is satisfied.

This gives rise to the *proportional allocation principle*:

$$K_i = \frac{K}{\sum_{j=1}^{n} \rho[X_j]} \rho[X_i], \qquad i = 1, \ldots, n.$$

Some known allocation formulas

Allocation method	$\rho[X_i]$	K_i
Haircut allocation (no known reference)	$F_{X_i}^{-1}(p)$	$\frac{K}{\sum_{j=1}^{n} F_{X_j}^{-1}(p)} F_{X_i}^{-1}(p)$
Quantile allocation Dhaene et al. (2002)	$F_{X_i}^{-1(\alpha)}\left(F_{S^c}\left(K\right)\right)$	$F_{X_i}^{-1(\alpha)}\left(F_{S^c}\left(K\right)\right)$
Covariance allocation Overbeck (2000)	$\mathrm{Cov}[X_i, S]$	$\frac{K}{\mathrm{Var}[S]}\mathrm{Cov}\left[X_i, S\right]$
CTE allocation Acerbi and Tasche (2002), Dhaene et al. (2006)	$\mathbb{E}\left[X_i \,\middle\vert\, S > F_S^{-1}(p)\right]$	$\frac{K}{\mathrm{CTE}_p[S]}\mathbb{E}\left[X_i \,\middle\vert\, S > F_S^{-1}(p)\right]$

The optimal capital allocation problem

We reformulate the allocation problem in terms of optimization:

Given the aggregate capital $K > 0$, we determine the allocated capitals $K_i, \ i = 1, \ldots, n$, from the following optimization problem:

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} v_j \mathbb{E}\left[\zeta_j \ D\left(\frac{X_j - K_j}{v_j}\right)\right]$$

such that the full allocation is met:

$$\sum_{j=1}^{n} K_j = K,$$

and where the v_j's are non-negative real numbers such that $\sum_{j=1}^{n} v_j = 1$, the ζ_j are non-negative random variables such that $\mathbb{E}[\zeta_j] = 1$ and D is a non-negative function.

The components of the optimization

Elaborating on the various elements of the optimization problem:

- Distance measure: the function $D(\cdot)$ gives the deviations of the outcomes of the losses X_j from their allocated capitals K_j.
    - squared-error or quadratic: $D(x) = x^2$
    - absolute deviation: $D(x) = |x|$
- Weights: the random variable ζ_j provides a re-weighting of the different possible outcomes of these deviations.
- Exposure: the non-negative real number v_j measures exposure of each business unit according to for example, revenue, premiums, etc.

The case of the quadratic optimization

In the special case where

$$D(x) = x^2$$

so that the optimization is expressed as

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[\zeta_j \frac{(X_j - K_j)^2}{v_j} \right].$$

This optimal allocation problem has the following unique solution:

$$K_i = \mathbb{E}[\zeta_i X_i] + v_i \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j] \right), \qquad i = 1, \ldots, n.$$

Business unit driven allocations

Risk measure	$\zeta_i = h_i(X_i)$	$\mathbb{E}[X_i h_i(X_i)]$
Standard deviation principle Buhlmann (1970)	$1 + a\dfrac{X_i - \mathbb{E}[X_i]}{\sigma_{X_i}},\ a \geq 0$	$\mathbb{E}[X_i] + a\sigma_{X_i}$
Conditional tail expectation Overbeck (2000)	$\dfrac{1}{1-p}\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right),\ p \in (0,1)$	$\mathsf{CTE}_p\,[X_i]$
Distortion risk measure Wang (1996), Acerbi (2002)	$g'\left(\overline{F}_{X_i}(X_i)\right), g : [0,1] \mapsto [0,1],$ $g' > 0, g'' < 0$	$\mathbb{E}\left[X_i g'\left(\overline{F}_{X_i}(X_i)\right)\right]$
Exponential principle Gerber (1974)	$\displaystyle\int_0^1 \frac{e^{\gamma a X_i}}{\mathbb{E}[e^{\gamma a X_i}]} d\gamma,\ a > 0$	$\dfrac{1}{a}\ln \mathbb{E}\left[e^{aX_i}\right]$
Esscher principle Gerber (1981)	$\dfrac{e^{aX_i}}{\mathbb{E}[e^{aX_i}]},\ a > 0$	$\dfrac{\mathbb{E}[X_i e^{aX_i}]}{\mathbb{E}[e^{aX_i}]}$

Aggregate portfolio driven allocations

Reference	$\zeta_i = h(S)$	$\mathbb{E}[X_i h(S)]$
Overbeck (2000)	$1 + a\dfrac{S - \mathbb{E}[S]}{\sigma_S},\ a \geq 0$	$\mathbb{E}[X_i] + a\dfrac{\mathrm{Cov}[X_i, S]}{\sigma_S}$
Overbeck (2000)	$\dfrac{1}{1-p}\mathbb{I}\left(S > F_S^{-1}(p)\right),\ p \in (0,1)$	$\mathbb{E}[X_i \mid S > F_S^{-1}(p)]$
Tsanakas (2004)	$g'(\overline{F}_S(S)), g : [0,1] \mapsto [0,1], g' > 0,$ $g'' < 0$	$\mathbb{E}\left[X_i g'(\overline{F}_S(S))\right]$
Tsanakas (2008)	$\displaystyle\int_0^1 \frac{e^{\gamma a S}}{\mathbb{E}[e^{\gamma a S}]} d\gamma,\ a > 0$	$\displaystyle\mathbb{E}\left[X_i \int_0^1 \frac{e^{\gamma a S}}{E[e^{\gamma a S}]} d\gamma\right]$
Wang2007	$\dfrac{e^{aS}}{\mathbb{E}[e^{aS}]},\ a > 0$	$\dfrac{\mathbb{E}[X_i e^{aS}]}{\mathbb{E}[e^{aS}]}$

Market driven allocations

Let ζ_M be such that market-consistent values of the aggregate portfolio loss S and the business unit losses X_i are given by $\pi[S] = \mathbb{E}[\zeta_M S]$ and $\pi[X_i] = \mathbb{E}[\zeta_M X_i]$.

To determine an optimal allocation over the different business units, we let $\zeta_i = \zeta_M,\ i = 1, \ldots, n$, allowing the market to determine which states-of-the-world are to be regarded adverse. This yields:

$$K_i = \pi[X_i] + v_i \left(K - \pi[S]\right).$$

Using market-consistent prices as volume measures $v_i = \pi[X_i]/\pi[S]$, we find

$$K_i = \frac{K}{\pi[S]}\pi[X_i], \qquad i = 1, \ldots, n.$$

Rearranging these expressions leads to

$$\frac{K_i - \pi[X_i]}{\pi[X_i]} = \frac{K - \pi[S]}{\pi[S]}, \qquad i = 1, \ldots, n.$$

Additional items considered in the paper

- Allocation according to the default option.
  - ζ_i is suitably chosen to account for shareholders having limited liability - not obligated to pay excess $(S - K)$ in case of default.
- We also considered other optimization criterion:
  - absolute value deviation: $D(x) = |x|$
  - combined quadratic/shortfall: $D(x) = ((x)_+))^2$
  - shortfall: $D(x) = (x)_+$
- Shortfall is applicable in cases where insurance market guarantees payments out of a pooled fund contributed by all companies, e.g. Lloyd's.
- Such allocation can be posed as an optimization problem leading to formulas that have been considered by Lloyd's. [Note: views here are the authors' own and do not necessarily reflect those of Lloyd's.]

Concluding remarks

- We re-examine existing allocation formulas that are in use in practice and existing in the literature. We re-express the allocation issue as an optimization problem.
- No single allocation formula may serve multiple purposes, but by expressing the problem as an optimization problem it can serve us more insights.
- Each of the components in the optimization can serve various purposes.
- This allocation methodology can lead to a wide variety of other allocation formulas.

Thank you.