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On a Reverse Estimate for Hodge Decompositions of p-Laplacian Type Operators

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Let $\sigma(x, \xi) \approx |\xi|^{p-2}\,\xi$ be a p-Laplacian type operator and consider the Hodge decomposition $\sigma(x, Du) = D\varphi + H$, $\operatorname{div} H = 0$. A standard elliptic theory asserts that $\|D\varphi\|_{q/(p-1)} \leqslant C\,\|Du\|_q^{p-1}$ for all $q > p-1$. There has been considerable recent interest in the validity of the reverse estimate $\|Du\|_q^{p-1} \leqslant C\,\|D\varphi\|_{q/(p-1)}$ for $q > p-1$ in the regularity study of certain geometrical mappings. In this paper, we give a relatively new proof of a well-known theorem that this reverse estimate holds for all q sufficiently close to the natural power p and also prove that the estimate holds for all $q \geqslant p-1$ for certain special weak solutions u. © 2001 Academic Press

1. INTRODUCTION

Given a map u from $\mathbf{R}^n$ to $\mathbf{R}^N$ and a number $p > 1$, by the Hodge decomposition, the field $|Du|^{p-2}\,Du$ can be written as

$$|Du|^{p-2}\,Du = D\varphi + H, \qquad \operatorname{div} H = 0,$$

where the map $\varphi\colon \mathrm{R}^n \to \mathbf{R}^N$ is determined by

$$\Delta\varphi = \operatorname{div}(|Du|^{p-2}\,Du). \tag{1.1}$$

If $Du \in L^q$ and $q > p-1$, then a standard linear elliptic theory easily shows that

$$\|D\varphi\|_{L^{q/(p-1)}} \leqslant C\,\|Du\|_{L^q}^{p-1}.$$

In this paper, we are interested in the reverse estimate of this estimate. To be more precise, given φ, we are interested in whether the following estimate holds for weak solutions u of equation (1.1),

$$\|Du\|_{L^q}^{p-1} \leqslant C\,\|D\varphi\|_{L^{q/(p-1)}}. \tag{1.2}$$

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When $q=p$, one easily establishes estimate (1.2) using u as a test function in (1.1). Notice that if $p=2$ the same estimate obtains for all $q>1$ since the equation (1.1) is then linear in this case. The estimate (1.2) in the case when $p-1\leqslant q<p$ and $p\neq 2$ remains a major open problem. An outstanding difficulty in this case is that one cannot use u as a test function in the equation.

There has been considerable recent interest in studying estimate (1.2) for q below the natural power p in the study of optimal regularity and removability for weakly quasiregular mappings in higher dimensions; see e.g. [6, 7, 8]. A conjecture made in these papers, which is closely related to the optimal higher integrability for quasiregular mappings, states that the estimate (1.2) hold for all $q>p-1$. In this paper, instead of investigating this difficult conjecture, we study a similar problem for more general nonlinear systems of p-Laplacian type. We refer to [2, 5] for some recent studies of such systems in other spaces larger than the natural Sobolev space of power p. We consider the system

$$\operatorname{div}(\sigma(x, Du)) = \operatorname{div} g, \tag{1.3}$$

where $\sigma(x, \xi)$ is a function from $\Omega\times \mathbf{M}^{N\times n}$ to $\mathbf{M}^{N\times n}$ and g is a map from Ω to $\mathbf{M}^{N\times n}$; here $\mathbf{M}^{N\times n}$ denotes the space of all $N\times n$ real matrices. We assume that $\sigma(x, \xi)$ is measurable in x for all $\xi\in\mathbf{M}^{N\times n}$ and continuous in ξ for almost every $x\in\Omega$ and that

$$\sigma(x, \xi)\cdot\xi\geqslant|\xi|^p, \qquad |\sigma(x, \xi)|\leqslant a\,|\xi|^{p-1} \tag{1.4}$$

for all $x\in\Omega$, $\xi\in\mathbf{M}^{N\times n}$, where $a>0$ is a constant.

Let $q\geqslant p-1$, $g\in L^{q/(p-1)}(\Omega; \mathbf{M}^{N\times n})$. A function $u\in W^{1,q}(\Omega; \mathbf{R}^N)$ is called a (very) *weak solution* of system (1.3) if the equality

$$\int_\Omega \sigma(x, Du)\cdot D\psi\, dx = \int_\Omega g(x)\cdot D\psi\, dx \tag{1.5}$$

holds for all $\psi\in C_0^\infty(\Omega; \mathbf{R}^N)$ thus for all $\psi\in W_0^{1, q/(q-p+1)}(\Omega; \mathbf{R}^N)$.

The main result of the present paper is that for all q sufficiently close to p the estimate

$$\|Du\|^{p-1}_{L^q(\Omega)}\leqslant C\,\|g\|_{L^{q/(p-1)}(\Omega)} \tag{1.6}$$

holds for all weak solutions u in $W_0^{1,q}(\Omega; \mathbf{R}^N)$. This result has been proved by Iwaniec [6] and Iwaniec and Sbordone [8]. We present a different approach to attacking the problem in hoping that it could shed some new insights on the conjecture mentioned above about the p-Laplacian system.

Assume $u \in W_0^{1,q}(\Omega; \mathbf{R}^N)$ is a weak solution of (1.3). Notice that, since $q < \frac{q}{q-p+1}$ for $p-1 \leqslant q < p$, we cannot take $\psi = u$ as a test function in (1.5). On the other hand, in order to obtain certain useful estimates, one need to choose test functions ψ in (1.5) with the property

$$D\psi \approx |Du|^{q-p}\, Du. \tag{1.7}$$

If q is sufficiently close to p, it has been shown in [6, 8] that the gradient part $D\psi$ of the Hodge decomposition of $|Du|^{q-p}\, Du$ indeed provides a very useful test function for system (1.3) and using it one establishes (1.6).

In this paper, we use a different method to prove the main result. Our approach, which is greatly inspired by the work of Lewis [9] and a recent work of Dolzmann, Hungerbühler and Müller [2], is to construct Lipschitz test functions by truncating the gradient; see also [1, 10].

2. THE MAIN RESULTS

In the rest of this paper, we assume Ω is bounded and the complement $\Omega^c = \mathbf{R}^n \backslash \Omega$ is *of type A* (see e.g. [3]); that is, there exists a constant $A > 0$ such that $|B_r(x) \backslash \Omega| \geqslant A r^n$ for all $x \in \Omega^c$ and $r > 0$. This means that Ω cannot have "sharp inward cusps". For example, all bounded Lipschitz domains Ω satisfy this assumption.

THEOREM 2.1. *Let $p \geqslant 2$. Then, there exists a number $p^* \in [p-1, p)$ such that for all $p^* \leqslant q \leqslant p$ the estimate*

$$\int_\Omega |Du|^q\, dx \leqslant C \int_\Omega |g|^{q/(p-1)}\, dx \tag{2.1}$$

holds for any weak solution u of (1.3) *belonging to $W_0^{1,q}(\Omega; \mathbf{R}^N)$.*

COROLLARY 2.2. *There exists a number $p_* \in [p-1, p)$ such that if $u \in W_0^{1,p^*}(\Omega; \mathbf{R}^N)$ is a weak solution of the system* (1.3) *with $g = 0$ then $u \equiv 0$. Note that by Theorem* 2.1 *it follows that $p_* \leqslant p^*$.*

THEOREM 2.3. *Assume, in addition to* (1.4), *$\sigma(x, \xi)$ satisfies a Lipschitz type condition*

$$|\sigma(x, \xi) - \sigma(x, \eta)| \leqslant b\, |\xi - \eta|\, (|\xi| + |\eta|)^{p-2}.$$

Then for $p^* \leqslant q \leqslant p$ *and for any weak solution* $u \in W_0^{1,q}(\Omega; \mathbf{R}^N)$ *of the non-homogeneous system*

$$\operatorname{div} \sigma(x, h + Du) = \operatorname{div} g$$

it follows that

$$\int_\Omega |Du|^q \, dx \leqslant C \int_\Omega [|h|^q + |g|^{q/(p-1)}] \, dx.$$

Remark. It has been conjectured in [6, 7, 8] that $p_* = p - 1$ for the p-Laplacian system, that is, if $\sigma(x, \xi) = |\xi|^{p-2}\xi$. We shall prove (Theorem 6.1) that for certain special weak solutions the number p^* equals $p-1$ for all general systems $\sigma(x, \xi)$. However, the example below, based on Serrin [11], shows that the constant p_* in Corollary 2.2 may be strictly greater than $p-1$, even for linear operators $\sigma(x, \xi)$.

Example. Let $n \geqslant 2$, $N = 1$, $0 < \varepsilon < 1$, and $a = \frac{n-1}{\varepsilon(n-2+\varepsilon)}$. Let

$$\sigma(x, \xi) = \xi + (a-1) \frac{x \cdot \xi}{|x|^2} x. \tag{2.2}$$

Then the assumption (1.4) above holds with $p = 2$; that is,

$$\sigma(x, \xi) \cdot \xi \geqslant |\xi|^2, \qquad |\sigma(x, \xi)| \leqslant a\, |\xi|$$

for all $x \in \mathbf{R}^n \backslash \{0\}$, $\xi \in \mathbf{R}^n$. Let $w(x) = x_1 |x|^{1-n-\varepsilon}$. Then, from [11], $\operatorname{div}(\sigma(x, Dw)) = 0$ weakly, and $w \in W^{1,q}(B_1(0))$ only for $q < \frac{n}{n+\varepsilon-1}$. Let v be the classical solution (see [4]) of

$$\operatorname{div}(\sigma(x, Dv)) = 0, \qquad v|_{\partial B_1(0)} = x_1.$$

Let $u = w - v$. Then $u \neq 0$ is a weak solution of $\operatorname{div}(\sigma(x, Du)) = 0$ and $u \in W_0^{1,q}(B_1(0))$ for all $1 \leqslant q < \frac{n}{n+\varepsilon-1}$. This shows that the constant p_* in Corollary 2.2 must satisfy

$$p_* \geqslant \frac{n}{n+\varepsilon-1} > 1 = p - 1$$

for this particular linear operator $\sigma(x, \xi)$ defined by (2.2).

3. PRELIMINARIES

For $x \in \mathbf{R}^n$ and $\rho > 0$, we use $B_\rho(x)$ to denote the open ball of radius ρ at center x. For a measurable function h on $\mathbf{R}^n$ and a set S with Lebesgue measure $|S| > 0$, we let

$$h_S = |S|^{-1} \int_S h(z)\, dz = ⨍_S h(z)\, dz.$$

A point x in $\mathbf{R}^n$ is said to be a Lebesgue point of h provided that

$$\lim_{\rho \to 0} ⨍_{B_\rho(x)} \big| |h(x)| - |h(y)| \big|\, dy = 0.$$

By the Lebesgue differentiation theorem, almost every x is Lebesgue point of h. The Hardy–Littlewood maximal function of h is defined by

$$M(h)(x) = \sup_{\rho > 0} ⨍_{B_\rho(x)} |h(z)|\, dz.$$

If x is a Lebesgue point of h then $|h(x)| \leqslant M(h)(x)$. For each $\lambda \geqslant 0$, we define

$$E_h^\lambda = \{x \in \mathbf{R}^n \mid M(h)(x) > \lambda\}.$$

LEMMA 3.1 (Hardy–Littlewood Theorem [12]). *There exist constants $c_q > 0$ such that*

$$\begin{aligned} |\{x \in \mathbf{R}^n \mid M(h)(x) > \lambda\}| &\leqslant c_1 \lambda^{-1} \|h\|_1 && \text{if } \ q = 1, \\ \|M(h)\|_q &\leqslant c_q \|h\|_q && \text{if } \ 1 < q \leqslant \infty. \end{aligned} \tag{3.1}$$

LEMMA 3.2. *For any $1 \leqslant q < \infty$ there exists a constant N_q such that for all $h \in L^q(\mathbf{R}^n)$ and $\lambda > 0$*

$$\lambda^q |E_h^\lambda| + \int_{E_h^\lambda} |h|^q\, dz \leqslant N_q \int_{|h| > \lambda/2} |h|^q\, dz.$$

Proof. Let $A = E_h^\lambda$. Then for each $x \in A$ there exists a $\rho = \rho(x) > 0$ such that

$$\int_{B_\rho(x)} |h(z)|\, dz \geqslant \lambda\, |B_\rho(x)|. \tag{3.2}$$

By Besicovitch's covering lemma [12], there exists a sequence of disjoint balls $\{B_k = B_{\rho(x_k)}(x_k)\}$ such that

$$A \subset \bigcup_k B_{5\rho(x_k)}(x_k).$$

Since $\int_{B_k} |h|\, dz \geqslant \lambda\, |B_k|$, $\int_{B_k \cap \{|h| \leqslant \lambda/2\}} |h|\, dz \leqslant \frac{\lambda}{2}\, |B_k|$, we have

$$|B_k| \leqslant \frac{2}{\lambda} \int_{B_k \cap \{|h| > \lambda/2\}} |h|\, dz.$$

Summation over k yields

$$|A| \leqslant 5^n \sum_k |B_k| \leqslant \frac{5^n}{\lambda} \int_{|h| > \lambda/2} |h|\, dz \leqslant \frac{b_q}{\lambda^q} \int_{|h| > \lambda/2} |h|^q\, dz,$$

where $b_q = 5^n \cdot 2^{q-1}$. Therefore

$$\lambda^q\, |E_h^\lambda| = \lambda^q\, |A| \leqslant b_q \int_{|h| > \lambda/2} |h|^q\, dz. \tag{3.3}$$

It remains to prove

$$\int_{E_h^\lambda} |h|^q\, dz \leqslant N_q \int_{|h| > \lambda/2} |h|^q\, dz. \tag{3.4}$$

To this end, let $\mathscr{L}_h$ be the set of all Lebesgue points of h, and let

$$A_1 = A \cap \{|h| \leqslant \lambda\}, \qquad A_2 = \mathscr{L}_h \cap \{|h| > \lambda\}.$$

Then $A_1 \cap A_2 = \varnothing$, $|A| = |A_1| + |A_2|$. For each $x \in A_2$ there exists a sequence $\rho_k \to 0^+$ such that

$$\int_{B_{\rho_k}(x)} |h(z)|\, dz > \lambda\, |B_{\rho_k}(x)|$$

for all $k = 1, 2, \ldots$. Therefore, the family $\{B_{\rho_k}(x) \mid x \in A_2,\ \ k = 1, 2, \ldots\}$ forms a Vitali covering of A_2. Hence, there exists a sequence of disjoint balls $B_j = B_{\rho_{k_j}}(x_j)$ such that

$$\left| A_2 \Big\backslash \bigcup_j B_j \right| = 0, \quad \fint_{B_j} |h|\, dz \geqslant \lambda.$$

Since $q\geqslant 1$ we have, by Jensen's inequality,

$$\fint_{B_j} |h|^q\,dz \geqslant \left(\fint_{B_j} |h|\,dz\right)^q \geqslant \lambda^q.$$

Hence, from $1/|B_j|\int_{B_j\cap\{|h|\leqslant\lambda/2\}} |h|^q\,dz\leqslant(\frac{\lambda}{2})^q$, we have

$$\int_{B_j} |h|^q\,dz\leqslant C_q\int_{B_j\cap\{|h|>\lambda/2\}} |h|^q\,dz$$

with $C_q=(1-2^{-q})^{-1}\leqslant 2$, and summation over j yields $\int_{A_2}|h|^q dz\leqslant 2\int_{|h|>\lambda/2}|h|^q\,dz$. Finally, since $\int_{A_1}|h|^q\,dz\leqslant\lambda^q\,|A_1|\leqslant\lambda^q\,|A|\leqslant b_q\int_{|h|>\lambda/2}|h|^q\,dz$, we obtain (3.3). From the proof, it also follows that $N_q=2+5^n\cdot 2^{q-1}\leqslant 5^n\cdot 2^q$. ∎

LEMMA 3.3 (See [4, 12]). *Let (X,μ) be a measure space and $|f|^\rho\in L^1(X,\mu)$ for some $0<\rho<\infty$. Then for any $0\leqslant\varepsilon<\rho<\delta<\infty$*

$$\int_0^\infty s^{\rho-1-\varepsilon}\left(\int_{|f|>s}|f|^\varepsilon\,d\mu\right)ds=\frac{1}{\rho-\varepsilon}\int_X|f|^\rho\,d\mu, \tag{3.4}$$

$$\int_0^\infty s^{\rho-1-\delta}\left(\int_{|f|\leqslant s}|f|^\delta\,d\mu\right)ds=\frac{1}{\delta-\rho}\int_X|f|^\rho\,d\mu. \tag{3.5}$$

4. CONSTRUCTION OF LIPSCHITZ TEST FUNCTIONS

Let $1\leqslant q<\infty$ and $v\in W_0^{1,q}(\Omega;\mathbf{R}^N)$. Extend v to $\mathbf{R}^n$ by zero outside Ω and denote the new function still by v. Then $v\in W^{1,q}(\mathbf{R}^n;\mathbf{R}^N)$. Let $M(|Dv|)$ be the maximal function of $|Dv|$. For each $\lambda>0$, define

$$E^\lambda(v)=\{x\in\mathbf{R}^n\mid M(|Dv|)(x)>\lambda\}.$$

Since $v\in W^{1,q}(\mathbf{R}^n;\mathbf{R}^N)$, there exists a sequence $\{v_j\}$ in $C_0^\infty(\mathbf{R}^n;\mathbf{R}^N)$ such that $v_j\to v$ in $W^{1,q}(\mathbf{R}^n;\mathbf{R}^N)$ and $v_j(x)\to v(x)$ for almost every $x\in\mathbf{R}^n$ as $j\to\infty$. Since

$$|M(|Dv_j|)(x)-M(|Dv|)(x)|\leqslant M(|Dv_j-Dv|)(x),$$

it follows easily from Lemma 3.1 that

$$\begin{aligned}&|\{x\in\mathbf{R}^n\mid\ |M(|Dv_j|)(x)-M(|Dv|)(x)|>\lambda\}|\\ &\qquad\leqslant|\{x\in\mathbf{R}^n\mid\ |M(|Dv_j-Dv|)(x)|>\lambda\}|\leqslant c_q^q\,\lambda^{-q}\,\|Dv_j-Dv\|_q^q\to 0\end{aligned} \tag{4.1}$$

as $j \to \infty$ for all $\lambda > 0$; thus, $M(|Dv_j|) \to M(|Dv|)$ in measure. We may then assume a subsequence $M(|Dv_{j_k}|)(x) \to M(|Dv|)(x)$ for almost every $x \in \mathbf{R}^n$ as $j_k \to \infty$. Let

$$\mathscr{L}(v) = \{x \in \mathbf{R}^n \mid v_{j_k}(x) \to v(x),\ M(|Dv_{j_k}|)(x) \to M(|Dv|)(x)\}.$$

Then $|\mathbf{R}^n \backslash \mathscr{L}(v)| = 0$. Define

$$R^\lambda(v) = E^\lambda(v) \cup (\mathbf{R}^n \backslash \mathscr{L}(v)).$$

Since $E^\lambda(v) \subseteq R^\lambda(v)$, $|R^\lambda(v)| = |E^\lambda(v)|$, from Lemma 3.2, we easily obtain

$$\lambda^q\, |R^\lambda(v)| + \int_{R^\lambda(v)} |Dv|^q\, dz \leqslant N_q \int_{|Dv| > \lambda/2} |Dv|^q\, dz. \tag{4.2}$$

LEMMA 4.1 (See also [1, 9, 10]). *Let $H^\lambda(v) = \mathbf{R}^n \backslash R^\lambda(v)$. Then there exists a constant $\alpha_n > 0$ depending only on n such that*

$$\int_{B_r(x)} |v(x) - v(y)|\, dy \leqslant \alpha_n\, \lambda\, r^{n+1} \tag{4.3}$$

for all $x \in H^\lambda(v)$ and $r > 0$.

Proof. We first prove this for smooth v. Let $v \in C_0^\infty(\mathbf{R}^n; \mathbf{R}^N)$. It is easily calculated that

$$\int_{B_r(x)} |v(x) - v(y)|\, dy \leqslant \frac{r^n}{n} \int_S \int_0^r |Dv(x + t\omega)|\, dt\, d\omega,$$

where $S = S^{n-1}$ is the unit sphere in $\mathbf{R}^n$. Let $g(t) = \int_S |Dv(x + t\omega)|\, d\omega$. Note that for $\rho > 0$ and $x \in H^\lambda(v)$, since $M(|Dv|)(x) \leqslant \lambda$, it follows that

$$\int_0^{2\rho} g(t)\, t^{n-1}\, dt = \int_{B_{2\rho}(x)} |Dv(y)|\, dy \leqslant C_1\, \rho^n\, \lambda.$$

From this we deduce that $\int_\rho^{2\rho} g(t)\, dt \leqslant C_2\, \rho\, \lambda$ for all $\rho > 0$. Hence, for every $k \in \mathbb{N}$,

$$\int_{r/2^k}^r g(t)\, dt = \sum_{i=1}^k \int_{r/2^i}^{r/2^{i-1}} g(t)\, dt \leqslant \sum_{i=1}^k C_2\, \lambda\, \frac{r}{2^i} \leqslant C_2\, r\, \lambda.$$

and thus, letting $k \to \infty$, we have $\int_0^r g(t)\,dt \leqslant C_2\, r\, \lambda$. Therefore, we have

$$\int_{B_r(x)} |v(x)-v(y)|\,dy \leqslant \frac{r^n}{n} \int_S \int_0^r |Dv(x+t\omega)|\,dt\,d\omega \leqslant r^n \int_0^r g(t)\,dt \leqslant C_2\,\lambda\, r^{n+1}.$$

This proves the estimate (4.3) for smooth v. For general functions v, this estimate follows by approximation. ∎

LEMMA 4.2. *For all $x,\ y \in H^\lambda(v)$ and $r>0$, we have*

$$|v(x)-v(y)| \leqslant C_3\,\lambda\,|x-y|, \qquad \left| v(x) - \fint_{B_r(x)} v(z)\,dz \right| \leqslant C_3\,\lambda\, r. \tag{4.4}$$

Proof. The second estimate follows easily from (4.3). To prove the first, let a be the midpoint of x and y, and $r=|x-y|/2$. Then by (4.3)

$$|v(x)\,|B_r| - \int_{B_r(a)} v(z)\,dz| \leqslant \int_{B_r(a)} |v(x)-v(z)|\,dz$$

$$\leqslant \int_{B_{2r}(x)} |v(x)-v(z)|\,dz \leqslant C_4\,\lambda\, r^{n+1}.$$

Similarly, $|v(y)\,|B_r| - \int_{B_r(a)} v(z)\,dz| \leqslant C_4\,\lambda\, r^{n+1}$. Hence, $|v(x)-v(y)|\,|B_r| \leqslant 2\,C_4\,\lambda\, r^{n+1}$. Since $r=|x-y|/2$, we easily obtain $|v(x)-v(y)| \leqslant C_3\,\lambda\,|x-y|$. The proof is now completed. ∎

Assume now the domain Ω is bounded and the complement $\Omega^c = \mathbf{R}^n \backslash \Omega$ is of type A; that is, there exists a constant $A>0$ such that

$$|B_r(x) \backslash \Omega| \geqslant A r^n, \quad \forall\, x \in \Omega^c, \quad r>0. \tag{4.5}$$

LEMMA 4.3. *Let $v \in W_0^{1,q}(\Omega; \mathbf{R}^N)$, $\lambda>0$. Define v^λ on $H^\lambda(v) = \mathbf{R}^n \backslash R^\lambda(v)$ by letting $v^\lambda(x) = v(x)$ on $\Omega \backslash R^\lambda(v)$ and $v^\lambda(x)=0$ on Ω^c. Then v^λ is a Lipschitz function on $H^\lambda(v)$ and satisfies*

$$|v^\lambda(x) - v^\lambda(y)| \leqslant \beta\,\lambda\,|x-y|, \qquad |v^\lambda(x)| \leqslant \beta\,\lambda\, \operatorname{dist}(x; \Omega^c)$$

for all $x,\ y \in H^\lambda(v)$, where $\beta = \beta(n, A) > 0$ is a constant depending only on n and the constant A in (4.5).

Proof. We first prove the second estimate, following an idea in [2]. The proof is trivial if $x \in \Omega^c$. So let $x \in \Omega \backslash H^\lambda(v)$. Let $|x-\bar{x}| = \operatorname{dist}(x; \Omega^c)$ for some $\bar{x} \in \Omega^c$. Let $r = 2\,|x-\bar{x}| > 0$, $U = B_r(x)$ and $S = \{x \in U \mid v(x) = 0\}$. Then $B_{r/2}(\bar{x}) \backslash \Omega \subset S$; thus condition (4.5) implies that $|S| \geqslant C_5\, r^n$ for some constant

$C_5 > 0$ depending on the constant A in (4.5). Since $v \in W^{1,1}(U; \mathbf{R}^N)$ and $v = 0$ on S, by a Poincaré type inequality (see e.g. [4, p. 164]), we have

$$\int_{B_r(x)} |v(z)|\, dz = \int_{B_r(x)} |v(z) - v_S|\, dz$$

$$\leqslant C_6\, |S|^{-1+1/n}\, r^n \int_{B_r(x)} |Dv|\, dz \leqslant C_7\, \lambda\, r^{n+1},$$

using the fact that $M(|Dv|)(x) \leqslant \lambda$. Therefore, by (4.4),

$$|v^\lambda(x)| = |v(x)| \leqslant C_8\, \lambda\, r = 2C_8\, \lambda\, \mathrm{dist}(x; \Omega^c).$$

The first estimate of the lemma follows from this last inequality and (4.4). ∎

LEMMA 4.4. *Let $H \subset \mathbf{R}^n$ be any nonempty set and $v: H \to \mathbf{R}^N$ be a Lipschitz map with Lipschitz constant L; that is,*

$$L = \sup_{x,\, y \in H} \frac{|v(x) - v(y)|}{|x - y|} < \infty.$$

Then there exists a Lipschitz map $\tilde{v}: \mathbf{R}^n \to \mathbf{R}^N$ with Lipschitz constant $\tilde{L} \leqslant \sqrt{N}\, L$ such that $\tilde{v}(x) = v(x)$ for all $x \in H$.

Proof. Define $\tilde{v}: \mathbf{R}^n \to \mathbf{R}^N$ to be $\tilde{v} = (\tilde{v}^i)$ with $\tilde{v}^i(z) = \inf_{x \in H} \{v^i(x) + L\, |z - x|\}$ for $z \in \mathbf{R}^n$ and $i = 1, 2, ..., N$. It is then easily verified that $\tilde{v}(x) = v(x)$ for all $x \in H$ and

$$|\tilde{v}^i(x) - \tilde{v}^i(y)| \leqslant L\, |x - y|, \quad \forall\, x,\ y \in \mathbf{R}^n.$$

So $\tilde{v}$ is a Lipschitz map on $\mathbf{R}^n$ with Lipschitz constant $\tilde{L} \leqslant \sqrt{N}\, L$. ∎

THEOREM 4.5. *There exists a constant $\gamma = \gamma(n, N, A) > 0$, where A is the constant in (4.5), such that for $v \in W_0^{1,q}(\Omega; \mathbf{R}^N)$ and $\lambda > 0$ there exists a Lipschitz function $v_\lambda \in W_0^{1,\infty}(\mathbf{R}^n; \mathbf{R}^N)$ satisfying*

$$\begin{cases} \|Dv_\lambda\|_\infty \leqslant \gamma\lambda, \\ v_\lambda(x) = 0, & x \in \mathbf{R}^n \backslash \Omega, \\ v_\lambda(x) = v(x), & x \in \Omega \backslash R^\lambda(v), \end{cases} \tag{4.6}$$

where $R^\lambda(v)$ is the set defined by (4.1) above.

Proof. By Lemma 4.3, the function $v^\lambda: H^\lambda(v) \to \mathbf{R}^N$ defined above is a Lipschitz map with Lipschitz constant $L \leqslant \beta\lambda$. By Lemma 4.4, we extend v^λ to the whole $\mathbf{R}^n$ as a Lipschitz function ($v_\lambda : \mathbf{R}^n \to \mathbf{R}^N$) with Lipschitz constant

$\tilde{L} \leqslant \sqrt{N}\,\beta\lambda$. Let $\gamma = \sqrt{N}\,\beta$. Then $v_\lambda(x) = 0$ for all $x \in \Omega^c$, thus $v_\lambda \in W_0^{1,\infty}(\Omega; \mathbf{R}^N)$. We can easily verify that this function v_λ satisfies the all requirements of the theorem. ∎

5. PROOF OF THE MAIN RESULTS

In what follows, let $1 \leqslant p-1 \leqslant q \leqslant p$ and $g \in L^{q/(p-1)}(\Omega; \mathbf{M}^{N\times n})$. Assume $u \in W_0^{1,q}(\Omega; \mathbf{R}^N)$ is a weak solution of

$$\operatorname{div}(\sigma(x, Du)) = \operatorname{div} g. \tag{5.1}$$

For $\lambda > 0$, let $u_\lambda \in W_0^{1,\infty}(\mathbf{R}^n; \mathbf{R}^N)$ be the Lipschitz function constructed as in Theorem 4.5 above. Using u_λ as a test function in (5.1) yields

$$\int_{\mathbf{R}^n} \sigma(x, Du) \cdot Du_\lambda \, dx = \int_{\mathbf{R}^n} g(x) \cdot Du_\lambda \, dx.$$

Let $H^\lambda(u) = \mathbf{R}^n \backslash R^\lambda(u)$. Since $Du_\lambda = Du$ on $H^\lambda(u)$, by (1.4), Theorem 4.5, we have

$$\int_{H^\lambda(u)} |Du|^p \leqslant \gamma\lambda \int_{R^\lambda(u)} |Du|^{p-1} + \int_\Omega |g|\,|Du_\lambda|. \tag{5.2}$$

We also easily have

$$\begin{aligned}\int_{|Du| \leqslant \lambda/2} |Du|^p &= \int_{\{|Du| \leqslant \lambda/2\} \cap R^\lambda(u)} |Du|^p + \int_{\{|Du| \leqslant \lambda/2\} \cap H^\lambda(u)} |Du|^p \\ &\leqslant \lambda^p\, |R^\lambda(u)| + \int_{H^\lambda(u)} |Du|^p \\ &\leqslant \lambda\, N_{p-1} \int_{|Du| > \lambda/2} |Du|^{p-1} + \int_{H^\lambda(u)} |Du|^p,\end{aligned}$$

and, by (4.2), $\int_{R^\lambda(u)} |Du|^{p-1} \leqslant N_{p-1} \int_{|Du|>\lambda/2} |Du|^{p-1}$. Therefore, by (5.2),

$$\int_{|Du| \leqslant \lambda/2} |Du|^p \leqslant (\gamma+1)\, N_{p-1}\, \lambda \int_{|Du| > \lambda/2} |Du|^{p-1} + \int_\Omega |g|\,|Du_\lambda|.$$

Changing $\lambda/2$ to λ we have

$$\int_{|Du| \leqslant \lambda} |Du|^p \leqslant \Gamma\, \lambda \int_{|Du| > \lambda} |Du|^{p-1} + G(\lambda), \tag{5.3}$$

where

$$\Gamma = \Gamma(n, N, A, p) = 2\,(\gamma+1)\,N_{p-1}, \qquad G(\lambda) = \int_{\Omega} |g(x)|\,|Du_{2\lambda}|\,dx.$$

From (5.3), we immediately obtain the following result.

PROPOSITION 5.1. *Suppose* $u \in W_0^{1,p-1}(\Omega; \mathbf{R}^N)$ *is a weak solution of* (5.1) *with* $g=0$. *Then* $u \equiv 0$ *provided that*

$$\liminf_{\lambda\to\infty}\left(\lambda \int_{|Du|>\lambda} |Du|^{p-1}\,dx\right) = M < \infty.$$

Proof. By (5.3), there exists a sequence $\lambda_j \to \infty$ such that

$$\lim_{\lambda_j\to\infty} \int_{|Du|\leqslant\lambda_j} |Du|^p\,dx \leqslant \Gamma M < \infty$$

and hence $u \in W_0^{1,p}(\Omega; \mathbf{R}^N)$. Therefore, using u as a test function in (5.1) with $g \equiv 0$ we easily deduce that $u \equiv 0$. ▮

Remark. If $\int_{|Du|>\lambda} |Du|^{p-1}\,dx \leqslant M/\lambda$, $\forall \lambda > T$ for some M, $T>0$, then one can easily prove that $|Du| \in L^q(\Omega)$ for all $p-1<q<p$; hence, in this case, Proposition 5.1 would follow from Theorem 2.1.

PROPOSITION 5.2. *If* $p-1<q<p$ *then*

$$\frac{1}{p-q}\int_{\mathbf{R}^n} |Du|^q \leqslant \frac{\Gamma}{q-p+1}\int_{\mathbf{R}^n} |Du|^q + \int_0^\infty \lambda^{q-1-p}\,G(\lambda)\,d\lambda. \tag{5.4}$$

Proof. Multiplying (5.3) by λ^{q-1-p} and integrating over $\lambda \in (0, \infty)$, we obtain

$$\int_0^\infty \lambda^{q-1-p}\left(\int_{|Du|\leqslant\lambda} |Du|^p\right) d\lambda \leqslant \Gamma \int_0^\infty \lambda^{q-p}\left(\int_{|Du|>\lambda} |Du|^{p-1}\right) d\lambda$$
$$+ \int_0^\infty \lambda^{q-1-p}\,G(\lambda)\,d\lambda.$$

From this we easily deduce (5.4) using the formulas in Lemma 3.3. ▮

Proposition 5.3. *For $p-1<q<p$ there exists a constant $C(p,q)$ depending also on the dimensions n, N and the constant A in condition* (4.5) *such that*

$$\int_0^\infty \lambda^{q-1-p}\, G(\lambda)\, d\lambda \leqslant C(p,q)\, \|Du\|_q^{q-p+1}\, \|g\|_{q/(p-1)}.$$

Proof. Let $f(x)=M(|Du|)(x)$ be the maximal function of $|Du|$. We write $G(\lambda)=G_1(\lambda)+G_2(\lambda)$, where

$$G_1(\lambda)=\int_{f\leqslant 2\lambda} |g(x)|\,|Du_{2\lambda}|\,dx, \qquad G_2(\lambda)=\int_{f>2\lambda} |g(x)|\,|Du_{2\lambda}|\,dx.$$

First of all, we have $G_1(\lambda)\leqslant \int_{f\leqslant 2\lambda} |g(x)|\,|f(x)|\,dx=\int_{f\leqslant 2\lambda} f\,d\mu$, where $d\mu=|g(x)|\,dx$ is a measure on $X=\mathbf{R}^n$. Therefore,

$$\begin{aligned}
\int_0^\infty \lambda^{q-1-p}\, G_1(\lambda)\, d\lambda &\leqslant \int_0^\infty \lambda^{q-1-p}\left(\int_{f\leqslant 2\lambda} f\,d\mu\right) d\lambda\\
&= \frac{1}{p-q}\int_{\mathbf{R}^n} f^{q-p+1}\,d\mu\\
&\leqslant \frac{1}{p-q}\,\|g\|_{q/(p-1)}\,\|f\|_q^{q-p+1}\\
&\leqslant \frac{c_q}{p-q}\,\|g\|_{q/(p-1)}\,\|Du\|_q^{q-p+1}
\end{aligned}$$

in view of Lemma 3.1. Note that our assumption on q implies $q>1$. Next, we have

$$G_2(\lambda)\leqslant 2\,\gamma\,\lambda\int_{f>2\lambda} |g(x)|\,dx = C\,\lambda\int_{f>2\lambda} d\mu.$$

Therefore, similarly as above we deduce

$$\begin{aligned}
\int_0^\infty \lambda^{q-1-p}\, G_2(\lambda)\, d\lambda &\leqslant C\int_0^\infty \lambda^{q-p}\left(\int_{f>2\lambda} d\mu\right) d\lambda\\
&= \frac{C}{q-p+1}\int_{\mathbf{R}^n} f^{q-p+1}\,d\mu\\
&\leqslant \frac{C(q)}{q-p+1}\,\|g\|_{q/(p-1)}\,\|Du\|_q^{q-p+1}.
\end{aligned}$$

The proposition is thus proved. Also notice that the constant $C(p,q)$ can be chosen as $C(p,q)=(c_q/(p-q))+(C(q)/(q-p+1))$. ∎

THEOREM 5.4. *Let Γ be the constant in* (5.4). *Then for any q with $p-\frac{1}{\Gamma+1}<q\leqslant p$ there exists a constant $K(p,q)$ such that*

$$\int_\Omega |Du|^q\,dx \leqslant K(p,q)\int_\Omega |g|^{q/(p-1)}\,dx$$

holds for any weak solution u of equation (5.1) *in $W_0^{1,q}(\Omega;\mathbf{R}^N)$.*

Proof. Let $k(p,q)=\frac{(p-q)\Gamma}{q-p+1}$. Then, $0\leqslant k(p,q)<1$ for $p-\frac{1}{\Gamma+1}<q\leqslant p$. From (5.4) and Proposition 5.3, we have

$$\left(\int_{\mathbf{R}^n}|Du|^q\,dx\right)^{(p-1)/q}\leqslant\frac{(p-q)\,C(p,q)}{1-k(p,q)}\,\|g\|_{q/(p-1)}$$

for $p-\frac{1}{\Gamma+1}<q<p$, where $C(p,q)=(c_q/(p-q))+(C(q)/(q-p+1))$ is the constant in Proposition 5.3. Notice that the constant

$$K(p,q)=\left[\frac{(p-q)\,C(p,q)}{1-k(p,q)}\right]^{q/(p-1)}$$

does not blow up as $q\to p$; we have thus proved the theorem. ∎

Proof of Theorem 2.1. Theorem 2.1 follows from Theorem 5.4 with $p^*\leqslant p-\frac{1}{\Gamma+1}$, while its Corollary 2.2 follows easily from the estimate in the theorem. ∎

Proof of Theorem 2.3. We assume the Lipschitz condition given in the theorem; that is,

$$|\sigma(x,\xi)-\sigma(x,\eta)|\leqslant b\,|\xi-\eta|\,(|\xi|+|\eta|)^{p-2}. \tag{5.5}$$

Let $p^*\leqslant q\leqslant p$, where p^* is the constant in Theorem 2.1. Let $u\in W_0^{1,q}(\Omega;\mathbf{R}^N)$ be a weak solution of

$$\operatorname{div}\sigma(x,h+Du)=\operatorname{div}g.$$

Assume $h\in L^q(\Omega;\mathbf{M}^{N\times n})$ and $g\in L^{q/(p-1)}(\Omega;\mathbf{M}^{N\times n})$. Then u is a weak solution of

$$\operatorname{div}\sigma(x,Du)=\operatorname{div}G,$$

where $G=\sigma(x,Du)-\sigma(x,h+Du)+g$. We have thus

$$\|G\|_{q/(p-1)}\leqslant\|g\|_{q/(p-1)}+\|\sigma(x,Du)-\sigma(x,h+Du)\|_{q/(p-1)}$$

and by Hölder's inequality and condition (5.5), it follows that

$$\|\sigma(x, Du) - \sigma(x, h + Du)\|_{q/(p-1)} \leqslant C \, \| |h| \cdot |Du|^{p-2} + |h|^{p-1} \|_{q/(p-1)}$$
$$\leqslant \varepsilon \, \|Du\|_q^{p-1} + C_\varepsilon \, \|h\|_q^{p-1}.$$

Using these estimates and Theorem 2.1 we have

$$\|Du\|_q^{p-1} \leqslant K \, \|G\|_{q/(p-1)} \leqslant K \, \|g\|_{q/(p-1)} + K\varepsilon \, \|Du\|_q^{p-1} + KC_\varepsilon \, \|h\|_q^{p-1}.$$

Choosing $\varepsilon > 0$ so that $K\varepsilon < 1$ yields $\|Du\|_q^{p-1} \leqslant C \, (\|g\|_{q/(p-1)} + \|h\|_q^{p-1})$, proving Theorem 2.3. ▮

6. A SPECIAL CASE OF RADIAL SOLUTIONS

In this final section, we consider a special case where we can indeed establish the estimate (2.1) for all $p - 1 \leqslant q \leqslant p$ for a certain class of weak solutions; here again we assume $p \geqslant 2$. This is done by proving the existence of certain test functions satisfying (1.7).

Assume $\Omega = B$ is the unit ball in $\mathbf{R}^n$. We also assume $N = 1$; that is, $u \colon B \to \mathbf{R}$. We say u is a *radial function* if $u(x) = U(r)$ with $r = |x|$ for some function U.

THEOREM 6.1. *Let $p - 1 \leqslant q \leqslant p$ and $g \in L^{q/(p-1)}(B; \mathbf{R}^n)$. Then for any radial weak solutions u in $W_0^{1,q}(B)$ of the equation*

$$\operatorname{div}(\sigma(x, Du)) = \operatorname{div} g$$

one has

$$\int_B |Du|^q \, dx \leqslant \int_B |g|^{q/(p-1)} \, dx. \tag{6.1}$$

As an immediate consequence of this theorem, we have the following uniqueness result.

COROLLARY 6.2. *The only radial weak solution u of* $\operatorname{div}(\sigma(x, Du)) = 0$ *in $W_0^{1,p-1}(B)$ is $u \equiv 0$.*

Before proving Theorem 6.1, let us recall some properties of radial functions. Let $v(x) = V(r)$ with $r = |x|$ be a radial function. If V is smooth it is easily seen that v is smooth on $B \setminus \{0\}$ and

$$Dv(x) = V'(r) \, x/r, \qquad r \neq 0.$$

Let $AC(0, 1]$ be the set of all functions V on $(0, 1]$ that are absolutely continuous on $[\varepsilon, 1]$ for all $0<\varepsilon<1$ and satisfy $V(1)=0$. We have the following elementary result.

LEMMA 6.3. *Let $v(x)=V(|x|)$. (i) Assume $1<q<\infty$. Then $v\in W_0^{1,q}(B)$ if and only if $V\in AC(0, 1]$ and*

$$\int_0^1 |V'(r)|^q \, r^{n-1}\, dr<\infty. \tag{6.2}$$

(ii) *If $v\in W_0^{1,1}(B)$ then $V\in AC(0, 1]$ and (6.2) holds with $q=1$.*

(iii) *$v\in W_0^{1,\infty}(B)$ if and only if $V\in AC(0, 1]$ and $|V'(r)|\leqslant L<\infty$.*

Proof. Let $v(x)=V(|x|)$ and $v\in W^{1,q}(B)$, where $1\leqslant q\leqslant\infty$. Let $0<\varepsilon<1$. Then, for all $\varepsilon\leqslant s<r\leqslant 1$ and $|\omega|=1$,

$$|V(r)-V(s)|=\left|\int_s^r \frac{dv(t\omega)}{dt}\, dt\right|=\left|\int_s^r Dv(t\omega)\cdot\omega\, dt\right|$$

$$\leqslant\int_s^r |Dv(t\omega)|\, dt\leqslant\varepsilon^{1-n}\int_s^r t^{n-1}\, |Dv(t\omega)|\, dt;$$

thus integrating over $|\omega|=1$ and using polar coordinates yield

$$|V(r)-V(s)|\leqslant C\varepsilon^{1-n}\int_{B_r\setminus B_s} |Dv(x)|\, dx.$$

This implies V is absolutely continuous on $[\varepsilon, 1]$. By a density argument, it is also shown that if $v\in W_0^{1,q}(B)$ then $V(1)=0$. Therefore we have proved that if $v\in W_0^{1,q}(B)$ then $V\in AC(0, 1]$. In this case, we also obtain

$$\int_0^1 |V'(r)|^q\, r^{n-1}\, dr=c_n\int_B |Dv(x)|^q\, dx<\infty; \qquad 1\leqslant q<\infty,$$

$$|V'(r)|=|Dv(x)|\leqslant \|Dv\|_{L^\infty(B)}<\infty; \qquad q=\infty.$$

To complete the proof of the lemma, we assume $1<q\leqslant\infty$ and $V\in AC(0, 1]$ satisfies (6.2) if $q<\infty$ or satisfies $|V'(r)|\leqslant L<\infty$ if $q=\infty$. We need to show $v=V(|x|)\in W_0^{1,q}(B)$. Define

$$f(x)=V'(r)\, x/r, \qquad r=|x|\neq 0.$$

From the assumption, we have $f \in L^q(B; \mathbf{R}^n)$. We first show $v = V(|x|) \in L^q(B)$, which is equivalent to $\int_0^1 |V(r)|^q r^{n-1}\, dr < \infty$ if $q < \infty$ or $|V(r)| \leqslant M < \infty$ if $q = \infty$. Since $V(r) = -\int_r^1 V'(t)\, dt$, it follows that

$$|V(r)| \leqslant \int_r^1 |V'(t)|\, t^{\alpha}\, t^{-\alpha}\, dt \leqslant r^{-\alpha} \int_0^1 |V'(t)|\, t^{\alpha}\, dt$$

for all $0 < r < 1$, $\alpha \geqslant 0$. Therefore if $q = \infty$ then $|V(r)| \leqslant L$ and hence $v \in W^{1,\infty}(B)$. We now consider the case $1 < q < \infty$. In this case, with $\alpha = \frac{n-1}{q}$ in the previous inequality, we have by Hölder's inequality

$$|V(r)| \leqslant r^{-(n-1)/q} \left(\int_0^1 |V'(t)|^q\, t^{n-1}\, dt \right)^{1/q}, \tag{6.3}$$

from which $v \in L^q(B)$ follows. We now show $v \in W^{1,q}(B)$; this is proved if we show $f = Dv$ in the sense of distribution. To prove this, we observe that, for all $\phi \in C_0^{\infty}(B)$,

$$\begin{aligned}
\int_B v\, D\phi\, dx &= \lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < 1} vD\phi\, dx \\
&= -\lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < 1} \phi\, Dv\, dx + \lim_{\varepsilon \to 0^+} \int_{|x| = \varepsilon} V(r) \frac{\partial \phi}{\partial n}\, dS \\
&= -\lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < 1} \phi\, f\, dx + \lim_{\varepsilon \to 0^+} \int_{|x| = \varepsilon} V(r) \frac{\partial \phi}{\partial n}\, dS.
\end{aligned}$$

Since $f \in L^q(B; \mathbf{R}^n)$, we have

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < 1} \phi\, f\, dx = \int_B \phi\, f\, dx.$$

Also by (6.3), $|V(\varepsilon)| \leqslant C\, \varepsilon^{-(n-1)/q}$ and hence $|\int_{|x| = \varepsilon} V(r) \frac{\partial \phi}{\partial n}\, dS| \leqslant C\, \varepsilon^{n-1-((n-1)/q)} \to 0$ as $\varepsilon \to 0^+$ since $q > 1$. Therefore

$$\int_B v\, D\phi\, dx = -\int_B \phi\, f\, dx, \qquad \forall \phi \in C_0^{\infty}(B);$$

thus $Dv = f \in L^q(B; \mathbf{R}^n)$. Finally, for all $1 < q \leqslant \infty$, an easy density argument using $V(1) = 0$ shows $v \in W_0^{1,q}(B)$. ∎

Proof of Theorem 6.1. Let $p - 1 \leqslant q \leqslant p$ and let $u = U(|x|)$ be a weak solution of the equation given in the theorem. Define

$$V(r) = \int_1^r |U'(t)|^{q-p}\, U'(t)\, dt, \qquad \psi(x) = V(|x|).$$

Then by Lemma 6.3, $\psi \in W_0^{1,\, q/(q-p+1)}(B)$ and

$$D\psi = V'(r)\, x/r = |U'(r)|^{q-p}\, U'(r)\, x/r = |Du|^{q-p}\, Du.$$

Therefore, upon using this ψ as a test function in the given equation and using the hypothesis (1.4), we obtain

$$\begin{aligned}\int_B |Du|^q\, dx &\leqslant \int_B \sigma(x, Du)\, D\psi\, dx = \int_B g(x)\, |Du|^{q-p}\, Du\, dx \\ &\leqslant \left(\int_B |g|^{q/(p-1)}\, dx\right)^{(p-1)/q} \left(\int_B |Du|^q\, dx\right)^{(q-p+1)/q}.\end{aligned}$$

This proves the inequality (6.1), and thus the theorem follows. ∎

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