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Sharp Stability Results for Almost Conformal Maps in Even Dimensions

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ABSTRACT. *Let $\Omega \subset \mathbf{R}^n$ and $n \geq 4$ be even. We show that if a sequence $\{u^j\}$ in $W^{1,n/2}(\Omega; \mathbf{R}^n)$ is almost conformal in the sense that* dist $(\nabla u^j, \mathbf{R}^+ SO(n))$ *converges strongly to 0 in $L^{n/2}$ and if u^j converges weakly to u in $W^{1,n/2}$, then u is conformal and $\nabla u^j \to \nabla u$ strongly in L^q_{loc} for all $1 \leq q < n/2$. It is known that this conclusion fails if $n/2$ is replaced by any smaller exponent p. We also prove the existence of a quasiconvex function $f(A)$ that satisfies $0 \leq f(A) \leq C(1 + |A|^{n/2})$ and vanishes exactly on $\mathbf{R}^+ SO(n)$. The proof of these results involves the Iwaniec–Martin characterization of conformal maps, the weak continuity and biting convergence of Jacobians, and the weak-L^1 estimates for Hodge decompositions.*

1. Introduction

Let $n \geq 2$ and Ω be a domain in $\mathbf{R}^n$. We denote by $W^{1,p}(\Omega; \mathbf{R}^n)$ $(p \geq 1)$ the usual space of all Sobolev maps $u: \Omega \to \mathbf{R}^n$. A map $u \in W^{1,p}(\Omega; \mathbf{R}^n)$ is called *conformal* if

$$\nabla u(x) \in \mathbf{R}^+ SO(n) = \{\lambda Q \mid \lambda \geq 0,\ Q \in SO(n)\} \quad \text{a.e. } x \in \Omega .$$

Here, $\mathbf{R}^+$ denotes all nonnegative real numbers, and $SO(n)$ denotes the set of all rotations with determinant equal to 1. A classical Liouville's theorem asserts that if $n \geq 3$ and $p \geq n$, then a conformal map in $W^{1,p}(\Omega; \mathbf{R}^n)$ must be a restriction onto Ω of a Möbius map (see [4] and [26]). A recent result of Iwaniec and Martin [16] shows that in even dimensions Liouville's theorem is still true for conformal maps in $W^{1,p}$ if $p \geq n/2$. In odd dimensions, Liouville's theorem holds for conformal maps in $W^{1,p}$ if p is not too far below n; the minimal value of all such $p's$ is unknown (see [14] and [17]). Note that there are counterexamples in all dimensions showing that a conformal map in $W^{1,p}$ for $p < n/2$ may not be a restriction of a Möbius map (see, e.g., [16]).

In this paper, we are mainly interested in the stability of conformal maps, i.e., the question whether the weak limit of almost conformal maps is conformal. In the following, weak convergence is denoted by the half-arrow "$\rightharpoonup$" and strong convergence by the arrow "$\to$." Our main result is the following:

Theorem 1.1. *Suppose $n \geq 4$ is* even *and that $\{u^j\}$ is a sequence in $W^{1,n/2}(\Omega; \mathbf{R}^n)$ and satisfies*

$$u^j \rightharpoonup u \quad \text{in} \quad W^{1,n/2}(\Omega; \mathbf{R}^n), \tag{1.1}$$

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and

$$\operatorname{dist}\left(\nabla u^j, \mathbf{R}^+ SO(n)\right) \to 0 \quad \text{in} \quad L^{n/2}(\Omega) \,. \tag{1.2}$$

Then u is conformal (thus a Möbius map) and $\nabla u^j \to \nabla u$ in $L^q_{\text{loc}}(\Omega)$ for all $1 \le q < n/2$. In fact, $\nabla u^j \to \nabla u$ in the Marcinkiewicz space weak-$L^{n/2}_{\text{loc}} = L^{n/2,\infty}_{\text{loc}}$.

Remarks. 1. If (1.1) is replaced by

$$u^j \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbf{R}^n) \text{ for all } 1 \le p < n/2 \,, \tag{1.3}$$

the conclusion fails (see Example 4.2 below). Also, if $n/2$ in both (1.1) and (1.2) is replaced by $p > n/2$, then $\nabla u^j \to \nabla u$ in $L^p_{\text{loc}}(\Omega)$; see the remark after the proof of Theorem 1.1.

2. Hypothesis (1.2) can be replaced by the following seemingly more general condition. Let $f: \mathbf{M}^{n\times n} \to \mathbf{R}$, where $\mathbf{M}^{n\times n}$ denotes the set of all real $n \times n$ matrices, be a nonnegative $n/2$-homogeneous continuous function that vanishes exactly on $\mathbf{R}^+ SO(n)$ and we assume that

$$f\left(\nabla u^j\right) \to 0 \quad \text{in} \quad L^1(\Omega) \,.$$

This hypothesis is equivalent to the special case (1.2) under the condition (1.1) since, by homogeneity,

$$0 \le f(A) \le \epsilon\, |A|^{n/2} + C_\epsilon \operatorname{dist}^{n/2}\left(A, \mathbf{R}^+ SO(n)\right), \quad \forall \epsilon > 0, \ \forall A \in \mathbf{M}^{n\times n} \,.$$

A particular choice $f(A) = \|A\|^{n/2}(1 - r(A))$, where $\|A\|$ is the operator norm and $r(A) = \det A/\|A\|^n$, shows that in even dimensions the weak limit of *weakly K_j-quasiregular* maps u^j in $W^{1,n/2}(\Omega; \mathbf{R}^n)$ with $K_j \to 1$ is conformal and the convergence is in fact strong in $W^{1,q}_{\text{loc}}$ for all $1 \le q < n/2$ (see [14], [16], and [26]).

3. If n is odd, the similar conclusion of the theorem still holds if in both (1.1) and (1.2) one replaces $n/2$ by a number p which is not too far below n (see [34], [37]).

One key ingredient of the proof of the main theorem is the fact that in even dimensions conformality of a matrix can be (almost) characterized by a condition that involves only minors of order $n/2$ (see Lemma 2.1). This characterization, which may be viewed as a nonlinear version of the Cauchy-Riemann equations, is due to Donaldson and Sullivan [10] for $n = 4$ and to Iwaniec and Martin [16] for the general case $n = 2l$, $l \ge 2$.

The stability result stated in Theorem 1.1 is closely related to the existence of a quasiconvex function f that vanishes exactly on the set $\mathbf{R}^+ SO(n)$ and satisfies the growth condition $0 \le f(A) \le C(1 + |A|^{n/2})$. If $n/2$ is replaced by $p \ge n$, then such functions exist, as is easily seen [3], [19] by considering the function $f(A) = |A|^n - n^{n/2} \det A$, where $|A|$ is the norm defined by $|A|^2 = \operatorname{tr}(A^t A)$.

Recall that a function $f : \mathbf{M}^{n\times n} \to \mathbf{R}$ is said to be *quasiconvex* if for a bounded smooth domain $D \subset \mathbf{R}^n$,

$$\int_D f(A)\, dx \le \int_D f(A + \nabla\phi(x))\, dx, \quad \forall A \in \mathbf{M}^{n\times m}, \ \forall \phi \in C_0^\infty\left(D; \mathbf{R}^n\right) \,.$$

Furthermore, the *quasiconvexification* of f, denoted by f^{qc}, is defined by

$$f^{qc}(A) = \inf_{\phi \in C_0^\infty(D, \mathbf{R}^n)} \frac{1}{|D|} \int_D f(A + \nabla\phi(x))\, dx \,. \tag{1.4}$$

A simple covering argument shows that the definition of both quasiconvexity and quasiconvexification is independent of the domain D. Moreover, under suitable growth conditions, f^{qc} is the largest quasiconvex function on $\mathbf{M}^{n\times n}$ below f. However, most quasiconvex functions that are explicitly known are polyconvex functions introduced by Ball [2], i.e., functions that can be expressed as a convex function of all minors (subdeterminants) of A. For further information about quasiconvexity, we refer to [1], [6], [8], [20], [22], and [28].

Theorem 1.2. *Let $n \geq 2$ be even. Then there exists a quasiconvex function f which satisfies*

$$\begin{cases} 0 \leq f(A) \leq C\left(1+|A|^{n/2}\right), \ \forall A \in \mathbf{M}^{n\times n}, \\ f(A) = 0 \text{ if and only if } A \in \mathbf{R}^+ SO(n). \end{cases} \tag{1.5}$$

Furthermore, the function f may be taken to be $n/2$-homogeneous.

To put this result in perspective, we first remark that the result for $n = 2$ follows from the standard compensated compactness argument using the div-curl lemma (see [24], [30], [33], [36]). Second, as proved in [35], there are no polyconvex functions that satisfy condition (1.5), and also the growth condition $0 \leq f(A) \leq C(1+|A|^{n/2})$ in (1.5) cannot be strengthened to $0 \leq f(A) \leq C(1+\text{dist}^{n/2}(A, \mathbf{R}^+SO(n)))$ or to $0 \leq f(A) \leq C(1+|A|^p)$ for some $p < n/2$. Moreover, it follows from a result of [39] that whether there exists a quasiconvex function which vanishes exactly on a *compact* subset $\mathcal{K}$ of $\mathbf{M}^{n\times n}$ does not depend on the growth condition of the quasiconvex function (see also [36]). Finally, whether or not Theorem 1.2 holds in odd dimensions is open.

The construction of suitable quasiconvex functions plays an important rôle in the calculus of variations and nonlinear partial differential equations; see [9], [11], and [32]. Applications to phase transformations in elastic crystals have recently attracted considerable attention and we refer to Ball and James [5], Chipot and Kinderlehrer [7], Fonseca [12], Kohn [21], Müller and Šverák [25], and Šverák [29], [31] for further information.

2. Notation and preliminaries

We first recall some well-known results concerning multilinear algebra and differential forms. We follow the notation of [14] and [16].

Let $e_1, e_2, \cdots, e_n$ denote the standard basis of $\mathbf{R}^n$. For $k = 0, 1, \cdots, n$ we denote by $\Lambda^k = \wedge^k(\mathbf{R}^n)$ the linear space of all k-tensors spanned by $\{e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\}$ for all ordered k-tuples $I = (i_1, i_2, \cdots, i_k)$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Define $\Lambda^k = \{0\}$ if $k < 0$ or $k > n$. The Grassmann algebra $\Lambda = \oplus \Lambda^k$ is a graded algebra with respect to the exterior multiplication $\wedge$.

For $\alpha = \sum_I \alpha_I e_I$ and $\beta_I = \sum \beta_I e_I$ in Λ the inner product is defined by

$$\langle \alpha, \beta \rangle = \sum_I \alpha_I \beta_I,$$

where summation is taken over all k-tuples $I = (i_1, i_2, \cdots, i_k)$ and all integers $k = 0, 1, \cdots, n$. The norm of $\alpha \in \Lambda$ is defined by $|\alpha|^2 = \langle \alpha, \alpha \rangle$. The Hodge star operator $* : \Lambda \to \Lambda$ is defined by requiring

$$*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

and

$$\alpha \wedge (*\beta) = \beta \wedge (*\alpha) = \langle \alpha, \beta \rangle (*1)$$

for all $\alpha, \beta \in \Lambda$. It is obvious that $*$ maps Λ^k into Λ^{n-k}.

For a linear operator $A: \mathbf{R}^n \to \mathbf{R}^n$, the kth exterior power $\wedge^k A = A^k_\#$ of A is defined as a map $A^k_\#: \Lambda^k \to \Lambda^k$ by

$$A^k_\# (x_1 \wedge x_2 \wedge \cdots \wedge x_k) = Ax_1 \wedge Ax_2 \wedge \cdots \wedge Ax_k, \quad x_j \in \mathbf{R}^n .$$

Note that if A has the $n \times n$ matrix form in the standard basis $\{e_1, e_2, \cdots, e_n\}$, then $A^k_\#$ has the $\binom{n}{k} \times \binom{n}{k}$ matrix form in the basis $\{e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\}$ for all ordered k-tuples $I = (i_1, i_2, \cdots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$ and each element of $A^k_\#$ is a $k \times k$ minor of A.

For the purpose of this paper, we shall assume $n = 2l$ and consider only the lth exterior power $A^l_\#$ of an $n \times n$ matrix A. In this case, the Hodge $*$ operator induces a linear operator from Λ^l onto itself. Define $W: \mathbf{M}^{n \times n} \to \mathbf{R}$ as follows:

$$W(A) = \left\| * \left(A^t\right)^l_\# - \left(A^t\right)^l_\# * \right\| , \tag{2.1}$$

where the norm $\|\cdot\|$ is taken to be the operator norm. Note that $W(A)$ is polyconvex and hence quasiconvex.

The following result is the Iwaniec–Martin characterization of conformal matrices.

Lemma 2.1 [16]. *Let $W(A)$ be defined as above and $Z(W)$ denote the zero set of W. Then $W(A)$ is polyconvex and*

$$Z(W) = K_n \cup \left\{A \in \mathbf{M}^{n \times n} \mid \operatorname{rank} A \le l - 1\right\} .$$

Proof. This follows from Lemmas 2.10 and 2.13 of Iwaniec and Martin [16]. □

Lemma 2.2. *Let $I = (i_1, i_2, \cdots, i_l)$ with $1 \le i_1 < i_2 < \cdots < i_l \le n$ be given. For any $A \in \mathbf{M}^{n \times n}$ with row vectors a_j being considered as in Λ^1. Define a matrix $P_I(A) \in \mathbf{M}^{l \times n}$ such that its kth row is a_{i_k} for $k = 1, \ldots, l$. Then it follows that*

$$|P_I(A)|^l \ge l^{l/2} \left| \left(A^t\right)^l_\# (e_I) \right| , \tag{2.2}$$

and equality holds in (2.2) if and only if there exists a $\lambda_I \ge 0$ such that

$$a_{i_k} \cdot a_{i_s} = \lambda_I \, \delta_{ks}$$

for all $1 \le k,\ s \le l$. Therefore, equality in (2.2) holds for all indices I if and only if A is a conformal or an anticonformal matrix, i.e., $A^t A = \lambda\, I_n$ for some $\lambda \ge 0$.

Proof. This is a special case of the so-called Hadamards' inequality in the theory of matrices. See, for example, [15, Lemma 2.1]. □

We now review the notation of differential forms on Ω. For each $k = 0, 1, \cdots, n$, a differential form α of degree k defined on Ω (denoted by $\alpha \in \Lambda^k(\Omega)$)

$$\alpha = \sum \alpha_I(x)\, dx_I = \sum \alpha_{i_1 i_2 \ldots i_k}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

can be identified with a function $\alpha: \Omega \to \Lambda^k(\mathbf{R}^n)$ with the same coefficients $\{\alpha_I\}$. Consider the space

$$\mathcal{D}'(\Omega; \Lambda) = \oplus_k \mathcal{D}'\left(\Omega; \Lambda^k\right)$$

of all differential forms whose coefficients are distributions on Ω. Similarly, other spaces such as $L^p(\Omega; \Lambda)$, $W^{1,p}(\Omega; \Lambda)$ can be defined by requiring that all coefficients belong to the suitable function spaces.

Recall that the exterior derivative $d: \mathcal{D}'(\Omega; \Lambda^k) \to \mathcal{D}'(\Omega; \Lambda^{k+1})$ for $k = 0, 1, \ldots, n$ is defined by

$$\begin{aligned} d\alpha &= \sum_I d\alpha_I(x) \wedge dx_I \\ &= \sum_{s=1}^{n} \sum_{1 \le i_1 < i_2 < \cdots < i_l \le n} \frac{\partial \alpha_{i_1 i_2 \ldots i_k}(x)}{\partial x_s} dx_s \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} , \end{aligned}$$

and its formal adjoint operator $d^*: \mathcal{D}'(\Omega; \Lambda^{k+1}) \to \mathcal{D}'(\Omega; \Lambda^k)$ (the so-called Hodge codifferential) is given by $d^* = (-1)^{nk+1} * d *$ on $(k+1)$-forms. It follows directly from definition that the Laplace–Beltrami operator $\Delta = dd^* + d^*d$ defined on $\mathcal{D}'(\Omega; \Lambda^k)$ operates only on the coefficients, i.e.,

$$\Delta\alpha(x) = \sum_I \Delta\alpha_I(x)\, dx_I, \quad \text{where } \Delta = -\sum_i \frac{\partial^2}{\partial x_i^2} .$$

Let $u: \Omega \to \mathbf{R}^n$, $u = (u_1, u_2, \cdots, u_n)$ be a map in $W^{1,kp}_{\mathrm{loc}}$. Then u induces a homomorphism

$$u^*: C^\infty\left(\mathbf{R}^m, \Lambda^k\right) \to L^p_{\mathrm{loc}}\left(\Omega; \Lambda^k\right)$$

which is defined as follows. For each $\alpha = \sum_I \alpha_I(x)\, dx_I \in C^\infty(\mathbf{R}^m, \Lambda^k)$, let

$$\left(u^*\alpha\right)(x) = \sum_I \alpha_I(u(x))\, du_{i_1} \wedge du_{i_2} \wedge \cdots \wedge du_{i_k} .$$

If α has constant coefficients, then $(u^*\alpha)(x)$ can be identified with the kth exterior power of $\nabla^t u(x)$, where $\nabla^t u(x)$ denotes the transpose of $\nabla u(x) \in \mathbf{M}^{n \times n}$. Thus, in our notation

$$\left(u^*\alpha\right)(x) = \left(\nabla^t u(x)\right)^k_\# \alpha . \tag{2.3}$$

We use the following weak-L^1 estimates for the Hodge decomposition.

Lemma 2.3. *Let D be any smooth domain in $\mathbf{R}^n$. Suppose $\omega \in L^1(D; \Lambda^k)$ and $\alpha \in L^1_{\mathrm{loc}}(D; \Lambda^k)$. If the system*

$$d\alpha = 0, \quad d^*\alpha = d^*\omega \tag{2.4}$$

holds in the sense of distributions on D, then there exists a harmonic form h on D such that the weak-L^1 estimate

$$\|\alpha - h\|_{L^1_w(D)} \le C(n)\, \|\omega\|_{L^1(D)} \tag{2.5}$$

holds, where $\|\gamma\|_{L^1_w(D)}$ or in general $\|\gamma\|_{L^p_w(D)}$ denotes the pseudo-norm of a form γ in the Marcinkiewicz space weak-$L^p(D) = L^p_w(D)$ *defined by:*

$$\|\gamma\|^p_{L^p_w(D)} = \sup_{t>0} t^p \text{ meas}\,\{x \in D \mid |\gamma(x)| > t\} . \tag{2.6}$$

Proof. Let $\psi = N(\omega)$ be the Newton potential of ω on D (defined for each coefficient of ω)

$$\psi(x) = c_n \int_D \frac{\omega(y)}{|x-y|^{n-2}}\, dy ,$$

and define

$$\mathcal{R}_{ij}(\omega) = \frac{\partial^2 \psi}{\partial x_i \partial x_j} \quad (i, j = 1, 2, \cdots, n) . \tag{2.7}$$

The operator $\mathcal{R}_{ij} \colon \omega \to \mathcal{R}_{ij}(\omega)$ is a singular integral operator of Calderon–Zygmund type (see [13] and [27]).

From (2.4) and the identity

$$\omega = \Delta\, \psi = dd^*\psi + d^*d\psi$$

it follows that $h = \alpha - dd^*\psi$ is a harmonic form on D and hence (2.5) follows directly from the weak-L^1 estimates for Newtonian potentials (see, e.g., [13, Ch. 9]). □

Remarks. 1. If ω belongs to $L^p(D; \Lambda^k)$ for some $1 < p < \infty$, then (2.5) can be replaced by the strong L^p estimate

$$\|\alpha - h\|_{L^p(D)} \le C(n)\, \|\omega\|_{L^p(D)} . \tag{2.8}$$

2. If there is a constant form $\bar{\alpha}$ such that both ω and $\alpha - \bar{\alpha}$ are compactly supported in D, then $h \equiv \bar{\alpha}$ in both (2.5) and (2.8).

3. $W^{1,p}$-stability of almost conformal maps

In this section, we prove our main result, Theorem 1.1, about the sharp stability for almost conformal maps in even dimensions. We will frequently use the following fact.

Proposition 3.1. *Let $f(\xi)$ and $g(\xi)$ be two nonnegative continuous functions on $\mathbf{R}^N$ that are k-homogeneous. If the zero set of g contains that of f, then for every $\epsilon > 0$ there exists a constant $C_\epsilon < \infty$ such that*

$$0 \le g(\xi) \le \epsilon\, |\xi|^k + C_\epsilon\, f(\xi), \quad \forall \xi \in \mathbf{R}^N . \tag{3.1}$$

Proof. Let $c_\epsilon = \inf\{f(\xi) \,|\, |\xi| = 1,\ g(\xi) \ge \epsilon\}$. Then the infimum is attained and $c_\epsilon > 0$ by assumption. The choice $C_\epsilon = \sup_{|\xi|=1} g(\xi)/c_\epsilon$ and homogeneity yield (3.1). □

Proof of Theorem 1.1. It suffices to establish the conclusion of Theorem 1.1 for smooth bounded subdomains of Ω so we may assume that Ω itself is smooth and bounded. To simplify notation let

$$l = n/2, \quad d(A) = \operatorname{dist}\left(A, \mathbf{R}^+ SO(n)\right) ,$$

and let $W(A)$ be defined by (2.1). We proceed with the proof in several steps.

Step 1. (*Strong convergence of $l \times l$ minors*).

Since W vanishes on conformal matrices Lemma 2.1, thus, Proposition 3.1, the boundedness of $\{\nabla u^j\}$ in $L^{n/2}(\Omega)$, and (1.2) imply that

$$W\left(\nabla u^j\right) \to 0 \quad \text{in } L^1(\Omega) . \tag{3.2}$$

Let $I = \{i_1, \cdots, i_l\},\ 1 \le i_1 < \cdots < i_l \le n$, be an arbitrary l-index and define

$$\omega^j = *\left(u^j\right)^* (d\,x_I) - \left(u^j\right)^* (*d\,x_I) . \tag{3.3}$$

Since $|\omega^j(x)| \le W(\nabla u^j(x))$ a.e. (3.2) implies $\omega^j \to 0$ in $L^1(\Omega; \Lambda^l)$ as $j \to \infty$. Now by (3.3)

$$d^* \left[\left(u^j\right)^* (*d x_I) \right] = -d^* \omega^j , \quad d \left[\left(u^j\right)^* (*d x_I) \right] = 0 , \tag{3.4}$$

and by the weak-L^1 estimates for the Hodge decomposition Lemma 2.3 there exists a harmonic l-form γ^j in Ω such that

$$\left\| \left(u^j\right)^* (*d x_I) - \gamma^j \right\|_{L^1_w(\Omega)} \le C(n) \left\| \omega^j \right\|_{L^1(\Omega)} \to 0 . \tag{3.5}$$

In particular, $\{\gamma^j\}$ is bounded in $L^1_w(\Omega)$ as u^j is bounded in $W^{1,n/2}(\Omega)$. Since γ^j is harmonic in Ω, thus weak-L^1 estimates imply that $\{\gamma^j\}$ is bounded in $C^k_{\mathrm{loc}}(\Omega)$ for all $k = 1, 2, \dots$ and hence

$$\gamma^j \to \gamma \quad \text{in} \quad C^k_{\mathrm{loc}}(\Omega)$$

and γ is also harmonic in Ω. From this and (3.5) we have for $U \subset\subset \Omega$

$$\left\| \left(u^j\right)^* (*d x_I) - \gamma \right\|_{L^1_w(U)} \to 0 . \tag{3.6}$$

On the other hand, by a result of Zhang [38] there exist decreasing measurable sets E_k in Ω with $|E_k| \to 0$ such that for all $k = 1, 2, \cdots$

$$\left(u^j\right)^* (*d x_I) \rightharpoonup u^* (*d x_I) \qquad \text{in } L^1 \left(\Omega \backslash E_k\right)$$

as $j \to \infty$. This and (3.6) imply $\gamma = u^*(*d x_I)$ almost everywhere and hence

$$\left\| \left(u^j\right)^* (*d x_I) - u^* (*d x_I) \right\|_{L^1_w(U)} \to 0 \tag{3.7}$$

for all $U \subset\subset \Omega$ and all l-indices I. Since the convergence in weak-L^1_{loc} implies the strong convergence in L^s_{loc} for $s < 1$, we deduce the strong convergence of all $l \times l$ minors in $L^s_{\mathrm{loc}}(\Omega)$ for all $0 < s < 1$.

Step 2. (*Strong convergence of* $\{\nabla u^j\}$).

Let I_* be the complementary index of I, i.e., $*dx_I = dx_{I_*}$. Let $1 < q < l$, then Lemma 2.2 and Proposition 3.1, applied with $g(A) = |P_{I_*}(A)|^q - l^{q/2} \, |(A^t)^l_\#(e_{I_*})|^{q/l}$ and $f(A) = d^q(A)$, yield

$$\left| P_{I_*} \nabla u^j \right|^q \le l^{q/2} \left| \left(u^j\right)^* (*d x_I) \right|^{q/l} + \epsilon \left| \nabla u^j \right|^q + C_{\epsilon,q} \, d^q \left(\nabla u^j \right) . \tag{3.8}$$

We deduce from (1.2), (3.7), (3.8), and another application of Lemma 2.2 that

$$\limsup_{j \to \infty} \int_U \left| P_{I_*} \nabla u^j \right|^q \le \int_U \left| P_{I_*} \nabla u \right|^q .$$

Summation over all l-indices I yields the same estimate for $\|\nabla u^j\|^q_{L^q(U)}$ and $\|\nabla u\|^q_{L^q(U)}$. On the other hand, $\nabla u^j \rightharpoonup \nabla u$ in $L^q(U)$, hence,

$$\nabla u^j \to \nabla u \quad \text{strongly in} \quad L^q(U) \quad \text{for all} \quad q < l \quad \text{and all} \quad U \subset\subset \Omega . \tag{3.9}$$

In particular, $d(\nabla u^j) \to d(\nabla u)$ strongly in $L^q(U)$ which yields $\nabla u(x) \in \mathbf{R}^+ SO(n)$ and u is conformal, hence, Iwaniec–Martin's theorem shows that u is a restriction onto Ω of a Möbius map [16].

Step 3. (*Convergence of* $\{\nabla u^j\}$ *in* $L_w^{n/2}$).

In order to establish (locally) strong convergence in weak-L^l, we observe that for $q = l$ the above considerations yield

$$h^j = \left(\left|\nabla u^j\right|^l - |\nabla u|^l\right)^+ \to 0 \quad \text{in} \quad L_w^1(U)\,, \tag{3.10}$$

where f^+ denotes the positive part of a function f. Let

$$\lambda^j(t) = \operatorname{meas}\left\{x \in U \;\middle|\; \left|\nabla u^j(x) - \nabla u(x)\right| > t\right\}.$$

We have to estimate the weak-L^l pseudo-norm

$$\left\|\nabla u^j - \nabla u\right\|_{L_w^l(U)} = \left(\sup_{t\geq 0} t^l\,\lambda^j(t)\right)^{1/l}. \tag{3.11}$$

Note that, by (3.9), for all $s > 0$

$$\limsup_{j\to\infty}\ \sup_{t\leq s} t^l\lambda^j(t) \leq s^{l-q}\limsup_{j\to\infty}\ \sup_{t\geq 0} t^q\lambda^j(t) = 0\,. \tag{3.12}$$

On the other hand,

$$\begin{aligned}
\lambda^j(t) &\leq \operatorname{meas}\left(\left\{\left|\nabla u^j - \nabla u\right| > t\right\} \cap \{|\nabla u| \leq t/3\}\right) + \operatorname{meas}\{|\nabla u| > t/3\}\\
&\leq \operatorname{meas}\left\{h^j \geq (t/3)^l\right\} + \operatorname{meas}\{|\nabla u| > t/3\}\,,
\end{aligned}$$

thus

$$t^l\,\lambda^j(t) \leq C\left\|h^j\right\|_{L_w^1(U)} + C\int_{|\nabla u|>t/3} |\nabla u|^l\,.$$

Combining this inequality with (3.10) and (3.12) we obtain, for every $s > 0$

$$\limsup_{j\to\infty}\left\|\nabla u^j - \nabla u\right\|_{L_w^l(U)}^l = \limsup_{j\to\infty}\ \sup_{t\geq s} t^l\lambda^j(t) \leq C\int_{|\nabla u|>s/3} |\nabla u|^l\,.$$

Letting $s \to \infty$, we obtain

$$\limsup_{j\to 0}\left\|\nabla u^j - \nabla u\right\|_{L_w^l(U)}^l = 0\,,$$

hence Step 3 is proved. The proof of Theorem 1.1 is now complete. □

Remark. If we assume $u^j \rightharpoonup u$ in $W^{1,p}$ and $d(\nabla u^j) \to 0$ in L^p for some $p > n/2$, then by Remark (1) following Lemma 2.3 one has $(u^j)^*\,dx_I \to u^*\,dx_I$ in $L^{p/l}$ and then easily deduces L^p_{loc} convergence of $\{\nabla u^j\}$.

4. Quasiconvex functions that vanish exactly on $\mathbf{R}^+\,SO(n)$

As before, let $d(A)$ be the distance function dist $(A, \mathbf{R}^+\,SO(n))$ and $W(A)$ as defined by (2.1). Define

$$F(A) = W(A) + d^{n/2}(A), \qquad I(u) = \int_\Omega F(\nabla u(x))\,dx\,. \tag{4.1}$$

Theorem 4.1. *Let $n \geq 4$ be even. Then F^{qc} is nonnegative quasiconvex, $n/2$-homogeneous and vanishes exactly on $\mathbf{R}^+ SO(n)$. More precisely, for each $A \in \mathbf{M}^{n\times n}$ and each $\{\phi^j\} \subset C_0^\infty(\Omega; \mathbf{R}^n)$, if $I(Ax + \phi^j(x)) \to 0$ then $A \in \mathbf{R}^+ SO(n)$ and $\|\nabla\phi^j\|_{L_w^{n/2}(\Omega)} \to 0$. If $A = 0$, then one has the estimate*

$$I(\phi) \geq c\,\|\nabla\phi\|^{n/2}_{L_w^{n/2}(\Omega)} \tag{4.2}$$

for all $\phi \in C_0^\infty(\Omega; \mathbf{R}^n)$, where $c > 0$ is a constant independent of ϕ.

Remarks. 1. In (4.2) the weak-$L^{n/2}$ pseudo-norm cannot be replaced by the $L^{n/2}$ norm (see Example 4.2 below).

2. If $G(A) = (|A|^n - n^{n/2}\det A)^{1/2}$, it is easily seen that $W(A) \leq C\,G(A)$ for all $A \in \mathbf{M}^{n\times n}$. Hence, the similar proof given below also shows that the zero set $Z(G^{qc}) = \mathbf{R}^+ SO(n)$ if n is even, thus $G^{qc}(A)$ gives another quasiconvex function which satisfies the conditions of Theorem 1.2. However, all such quasiconvex functions cannot be polyconvex if $n \geq 4$ [35].

Proof of Theorem 4.1. Suppose that for some $A \in \mathbf{M}^{n\times n}$ and some sequence $\{\phi^j\} \subset C_0^\infty(\Omega; \mathbf{R}^n)$ we have $I(Ax + \phi^j(x)) \to 0$. We will show

$$A \in \mathbf{R}^+ SO(n), \qquad \left\|\nabla\phi^j\right\|_{L_w^{n/2}(\Omega)} \to 0 \qquad .$$

and that if $A = 0$ (4.2) holds. To this end, let $u^j(x) = Ax + \phi^j(x)$. Then $I(u^j) \to 0$, hence, $W(A) = 0$ as $\phi^j \in C_0^\infty(\Omega; \mathbf{R}^n)$ and W is quasiconvex. Define ω^j as in Step 1 of the proof of Theorem 1.1 by (3.3). Then ω^j is compactly supported in Ω since $W(A) = 0$. Arguing as in Step 1 of the proof of Theorem 1.1 and taking into account Remark (2) after Lemma 2.3 we deduce that (with $l = n/2$ as before)

$$\left(u^j\right)^*(*dx_I) \to \left(A^t\right)^l_\#(*e_I) \quad \text{in} \quad L^l_w(\Omega)\,. \tag{4.3}$$

Similarly, (3.8) in Step 2 of the proof of Theorem 1.1 also holds. Summation over all l-indices I in (3.8) for $q = l$ and a sufficiently small choice of $\epsilon > 0$ show that $\{\nabla u^j\}$ is bounded in $L^l_w(\Omega)$. As before one deduces from (3.8) and (4.3)

$$\limsup_{j\to\infty}\int_\Omega \left|\nabla u^j\right|^q \leq \int_\Omega |A|^q, \quad \forall q < l\,.$$

Since $A = \frac{1}{|\Omega|}\int_\Omega \nabla u^j(x)\,dx$, the last inequality and strict convexity of the map $A \to |A|^q$ imply $\nabla u^j \to A$ strongly in $L^q(\Omega)$ for all $1 < q < l$ hence $A \in \mathbf{R}^+ SO(n)$. Finally, (3.8) with $q = l$ yields

$$\left(\left|\nabla u^j\right|^l - |A|^l\right)^+ \to 0 \quad \text{in} \quad L^1_w\,.$$

For $A = 0$ this last inequality and homogeneity give (4.2). For $A \neq 0$, one can argue as in Step 3 of the proof of Theorem 1.1 or directly exploit the convexity of the map $A \to |A|^l$ to deduce $\|\nabla u^j - A\|_{L^l_w} \to 0$. The proof is complete. □

The following example will show that the coercivity in Theorem 4.1 cannot be improved from $L_w^{n/2}(\Omega)$ to $L^{n/2}(\Omega)$.

Example 4.2. Let B_r be the open ball in $\mathbf{R}^n$ with radius $r > 0$ and A be an anticonformal matrix with $\det A = -1$. For any $\sigma \geq 1$, define

$$u^\sigma(x) = \begin{cases} \left(\frac{1}{|x|}\right)^{1+\frac{1}{\sigma}} A\,x & 0 < |x| < 1 \\ (2 - |x|)\,A\,x & 1 \leq |x| < 2\,. \end{cases} \tag{4.4}$$

One easily sees that $u_\sigma(x) = 0$ on $|x| = 2$ and

$$\nabla u^\sigma(x) = \begin{cases} A\left(\frac{1}{|x|}\right)^{1+\frac{1}{\sigma}} \left(I - \frac{\sigma+1}{\sigma}\frac{x}{|x|} \otimes \frac{x}{|x|}\right), & 0 < |x| < 1 \\ A\left((2 - |x|)\,I - |x|\,\frac{x}{|x|} \otimes \frac{x}{|x|}\right), & 1 \leq |x| < 2\,. \end{cases}$$

By virtue of this, an easy calculation shows that u^σ belongs to $W_0^{1,n/2}(B_2; \mathbf{R}^n)$ and $\{\nabla u^\sigma\}$ is uniformly bounded in $L_w^{n/2}(B_2)$. Note that

$$\int_{B_2} \left(W\left(\nabla u^\sigma(x)\right) + d^{n/2}\left(\nabla u^\sigma(x)\right)\right) dx \leq C < \infty, \quad \text{for all } 1 \leq \sigma \leq 2\,,$$

however,

$$\lim_{\sigma \to 1^+} \left\|\nabla u^\sigma\right\|_{L^{n/2}(B_1)} = \infty\,.$$

This shows the weak-$L^{n/2}$ pseudo-norm cannot be replaced by $L^{n/2}$ norm in the previous theorem.

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