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L^p-Mean coercivity, regularity and relaxation in the calculus of variations

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1. Introduction

In this paper, we study the multiple integral functionals of the type

$$I_p(u;D) = \int_D [f(\nabla u(x))]^p \, \mathrm{d}x, \tag{1.1}$$

where D is a domain in $\mathbf{R}^m$, u is a map from D to $\mathbf{R}^n$, f is a given function defined on the space $\mathscr{M}^{n\times m}$ of all real $n \times m$ matrices and $p \geq 1$ is a given number. Here and throughout the paper, we use $\nabla u(x)$ to denote the Jacobian matrix of u defined by

$$(\nabla u)_{ij} = \partial u^i / \partial x_j, \quad 1 \leq i \leq n, \ \ 1 \leq j \leq m.$$

The functional $I_p(u;D)$ generalizes the classical Dirichlet p-energy (when $f(X) = |X|$) and has been encountered when one studies the variational energies with given minimum sets or *energy wells* [4,18,27]; in these cases, f is usually taken as the distance function to the energy well. Therefore, in this paper, we assume that $f \geq 0$ be a Lipschitz function on $\mathscr{M}^{n\times m}$, i.e., $|f(X) - f(Y)| \leq |X - Y|$ for $X, Y \in \mathscr{M}^{n\times m}$. This implies f grows linearly at infinity; hence, a natural class of admissible maps for $I_p(u;D)$ is the usual Sobolev space $W^{1,p}(D;\mathbf{R}^n)$.

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Among the most important properties for variational functional $I_p(u;D)$ are the conditions of (sequential, throughout this paper) *weak lower semicontinuity* and certain *coercivity* on $W^{1,p}(D;\mathbf{R}^n)$. In this paper, we attempt to study some relations between these two important issues under a condition known as the L^p-mean coercivity to be discussed later.

For many physical problems, the functional $I_p(u;D)$ is not weakly lower semicontinuous and it is important to study the *relaxation* or the envelope of $I_p(u;D)$ with respect to the weak convergence on $W^{1,p}(D;\mathbf{R}^n)$. Recall that the relaxation of $I_p(u;D)$ is the largest weakly lower semicontinuous functional on $W^{1,p}(D;\mathbf{R}^n)$ that is less than or equal to $I_p(u;D)$.

Under the assumption of the present paper, it is known (see [1,6,8]) that the relaxation of $I_p(u;D)$ is representable by another multiple integral $J_p(u;D)$ given by

$$J_p(u;D) = \int_D (f^p)^{\mathrm{qc}}(\nabla u(x))\,\mathrm{d}x, \tag{1.2}$$

where, for any given function g on $\mathscr{M}^{n\times m}$, g^{qc} denotes the (quasiconvex) *relaxation* or the quasiconvexification of g defined by

$$g^{\mathrm{qc}}(A) = \inf_{\phi\in C_0^\infty(\Omega;\mathbf{R}^n)} \fint_\Omega g(A+\nabla\phi(x))\,\mathrm{d}x, \quad A\in\mathscr{M}^{n\times m}, \tag{1.3}$$

where $\Omega\subset\mathbf{R}^m$ is any bounded open set with $|\partial\Omega|=0$ and the bar over the integral sign means taking average. Following Morrey [25], g is said to be *quasiconvex* provided that $g^{\mathrm{qc}} = g$ on $\mathscr{M}^{n\times m}$. This quasiconvexity condition turns out to be the "right" condition for the weak lower semicontinuity of multiple integral functionals on Sobolev spaces; for instance, it has been proved that (see [1]) if $0 \le g(X) \le C(|X|^p + 1)$ then the functional $G(u) = \int_D g(\nabla u)$ is weakly lower semicontinuous on $W^{1,p}(D;\mathbf{R}^n)$ if and only if g is quasiconvex. However, besides the class of *polyconvex* functions of Ball [2] which are quasiconvex, it is generally difficult to study the quasiconvex functions since, in (1.3), it involves all test functions ϕ in $C_0^\infty(\Omega;\mathbf{R}^n)$; see also [5,8,11,22,30] for some important work on quasiconvexity. Consequently, the study of the relaxation $(f^p)^{\mathrm{qc}}$ is greatly nontrivial mainly because in our case $(f^p)^{\mathrm{qc}}$ is always quasiconvex but, as many interesting examples show, not necessarily polyconvex; see Section 7 below.

Another important question concerns the coercivity of $I_p(u;D)$. This is usually dealt with by assuming f satisfies a pointwise growth condition (see [14,15,24,23])

$$f(X) \ge c_0|X| - c_1, \quad \forall X\in\mathscr{M}^{n\times m}. \tag{1.4}$$

Under this condition, some of the properties concerning the relaxation $(f^p)^{\mathrm{qc}}$ turns out independent of the power p. For example, it has been proved in Yan [35] that $\mathscr{Z}[(f^p)^{\mathrm{qc}}] = \mathscr{Z}[(f^q)^{\mathrm{qc}}]$ for all $1 \le p < q < \infty$ (see also [12,38]), where $\mathscr{Z}[g]$ denotes the zero set of function g. This result will be partially recovered later from our main results in which we replace (1.4) by a much weaker condition known as the *L^p-mean coercivity* [14,15,18]. We say that $I_p(u;D)$ or simply f satisfies the L^p-mean coercivity if

$$\int_{\mathbf{B}} f^p(\nabla\phi(x))\,\mathrm{dx} \geq \Gamma_0 \int_{\mathbf{B}} (|\nabla\phi(x)|^p - \Gamma_1)\,\mathrm{d}x \tag{1.5}$$

holds for all smooth maps ϕ with compact support in the unit open ball $\mathbf{B}$ in $\mathbf{R}^m$, where Γ_0, Γ_1 are some positive constants. Note that condition (1.5) may be satisfied even when (1.4) fails; for example, $n = m = p = 2$ and $f(X) = (|X|^2 - 2\det X)^{1/2}$.

The main purpose of the paper is to study the important relationship between the L^p-mean coercivity and certain questions regarding the relaxation and regularity issues for the functional $I_p(u; D)$. We assume, for our function f, that $\mathscr{Z}[f] \neq \emptyset$. It easily follows from the Hölder inequality that

$$\mathscr{Z}[f] \subseteq \mathscr{Z}[(f^q)^{\mathrm{qc}}] \subseteq \mathscr{Z}[(f^p)^{\mathrm{qc}}], \quad 1 \leq p < q < \infty. \tag{1.6}$$

Our first main result (Theorem 2.1) asserts that under the L^p-mean coercivity of f the zero set $\mathscr{Z}[(f^p)^{\mathrm{qc}}]$ is half-locally constant in p:

$$\mathscr{Z}[(f^p)^{\mathrm{qc}}] = \mathscr{Z}[(f^{p+\varepsilon})^{\mathrm{qc}}] \quad \text{for some } \varepsilon > 0. \tag{1.7}$$

In view of (1.6), this relation is a reverse Hölder inequality, and it relates to a higher integrability result for the first-order *Hamilton–Jacobi system* defined by

$$f(\nabla u(x)) = 0 \quad \text{a.e. } x \in \Omega. \tag{1.8}$$

It has been proved in Yan and Zhou [36] that if f satisfies the L^p-mean coercivity then $\nabla u \in W^{1,p+\varepsilon}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$ for any solution $u \in W^{1,p}(\Omega; \mathbf{R}^n)$ solving (1.8), where $\varepsilon > 0$ is some constant. This type of higher integrability results, pioneered by Gehring's celebrated work [13], has been well-known for the energy minimizers of variational integrals under certain pointwise growth conditions (see [14,15,20,24,23]). Indeed, by adapting the Caccioppoli-type estimates as in Meyers and Elcrat [24] and Giaquinta and Giusti [15], Theorem 2.1 is proved by the well-known technique of reverse Hölder inequalities of Gehring [13]. The proof here, however, requires a careful treatment since f^p does not satisfy the usual pointwise growth condition.

We now discuss a stability problem for the Hamilton–Jacobi system (1.8) in Sobolev spaces $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$, which concerns whether the weak limit of any weakly convergent sequence $\{u_j\}$ satisfying $I_p(u_j; \Omega) \to 0$ is a solution of (1.8). Let $\mathscr{K} = \mathscr{Z}[f]$. We can study this stability problem by means of the *p-quasiconvex hull* of $\mathscr{K}$ as in Yan [35]. Recall that the p-quasiconvex hull $\mathbf{Q}_p(\mathscr{K})$ of any set $\mathscr{K}$ is defined by

$$\mathbf{Q}_p(\mathscr{K}) = \bigcap \{\mathscr{Z}[g] \mid g \in \mathscr{Q}^+_p(\mathscr{K})\}, \tag{1.9}$$

where $\mathscr{Q}^+_p(\mathscr{K})$ is the set of all quasiconvex functions g with $0 \leq g(X) < C(|X|^p + 1)$ and $g|_{\mathscr{K}} = 0$. Note that our definition of p-quasiconvex hulls, motivated by the work of Šverák [31,32], is not equivalent to the one given in Zhang [39] as our p-quasiconvex hulls may be strictly smaller than those defined in [39] for certain unbounded sets; see an example in Section 7 later. We prefer this definition because it defines an *optimal* relation satisfied by the weak limits of all solutions of the Hamilton–Jacobi system (1.10) below (see [35–37]).

In Theorem 2.1, we also establish a new characterization for p-quasiconvex hulls under L^p-mean coercivity. We prove that $\mathbf{Q}_p(\mathscr{Z}[f]) = \mathscr{Z}[(f^p)^{qc}]$ if f satisfies the L^p-mean coercivity. In general, this identity does not hold for arbitrary $\mathscr{K}$ and $p > 1$; see an example in Section 2.

The second main result (Theorem 2.3) deals with a special case where f is 1-homogeneous; that is, $f(\lambda X) = \lambda f(X)$ for all $\lambda \geq 0$. In this case $\mathscr{K} = \mathscr{Z}[f]$ is a closed cone and the Hamilton–Jacobi system (1.8) takes the form

$$\nabla u(x) \in \mathscr{K} \quad \text{a.e. } x \in \Omega. \tag{1.10}$$

Assume now $\mathscr{K}$ is a given closed cone in $\mathscr{M}^{n\times m}$; that is, $\lambda\mathscr{K} \subseteq \mathscr{K}$ for all $\lambda \geq 0$. We say $\mathscr{K}$ is *L^p-mean coercive* or satisfies the L^p-mean coercivity if (1.5) is satisfied with f being the distance function $d_{\mathscr{K}}$. To study this L^p-mean coercivity, we define

$$\mu(p;\mathscr{K}) = \inf\left\{ \int_{\mathbf{B}} d^p_{\mathscr{K}}(\nabla\phi) \,\middle|\, \phi \in C_0^\infty(\mathbf{B};\mathbf{R}^n), \|\nabla\phi\|_{L^p(\mathbf{B})} = 1 \right\} \tag{1.11}$$

and

$$\mathbf{S}(\mathscr{K}) = \{p > 1 \mid \mu(p;\mathscr{K}) > 0\}. \tag{1.12}$$

Note that $\mathscr{K}$ is L^p-mean coercive if and only if $p \in \mathbf{S}(\mathscr{K})$.

In Theorem 2.3, we show that the set $\mathbf{S}(\mathscr{K})$ is an *open* set. Thus, if $\mathscr{K}$ is L^p-mean coercive for some $p > 1$ then it is L^q-mean coercive for all $q \in (p-\varepsilon, p+\varepsilon)$ for some $\varepsilon > 0$. This, to the best of our knowledge, is a new and surprising result, which assures the *near-by* mean-coercivity by establishing the L^p-mean coercivity at merely a single point $p > 1$. Some interesting applications of this result will be given in Section 7. The proof of Theorem 2.3 relies on an important technique of nonlinear Hodge decompositions of Iwaniec [17] and Iwaniec and Sbordone [20] (see also [16,21,37]).

As a consequence of Theorems 2.1 and 2.3, we prove that the p-quasiconvex hull $\mathbf{Q}_p(\mathscr{K})$ is constant for p in each connected component of $\mathbf{S}(\mathscr{K})$. Furthermore, for systems (1.10) defined by a closed cone $\mathscr{K}$, we prove a uniform higher integrability theorem (Theorem 2.4) in the sense that for any $[\alpha,\beta] \subset \mathbf{S}(\mathscr{K})$ a solution $u \in W^{1,\alpha}_{\text{loc}}(\Omega;\mathbf{R}^n)$ to (1.10) must belong to $W^{1,\beta}_{\text{loc}}(\Omega;\mathbf{R}^n)$. This can be considered as a global version of the aforementioned results on the higher integrability of energy minimizers (see, e.g., [13–15,17,20,24]).

2. Statement of the main theorems

As mentioned before, we assume that $f\colon \mathscr{M}^{n\times m} \to \mathbf{R}$ is Lipschitz continuous. Throughout the paper, we shall also assume f satisfies, for some constant $C_0 > 0$, the following condition:

$$0 \leq f(\lambda X) \leq C_0(f(X) + 1), \quad X \in \mathscr{M}^{n\times m},\ \lambda \in [0, 1]. \tag{2.1}$$

Note that (2.1) is easily satisfied if f is 1-homogeneous or if f satisfies $f(X) \geq c_0|X| - c_1$. We think that this condition may be a technical condition, but we have not been able to remove it in the proof of Theorem 2.1 given later.

L^p-Mean coercivity. We say f satisfies the L^p-mean coercivity provided that there exist constants $\Gamma_0 > 0$ and $\Gamma_1 \geq 0$ such that for the unit ball $\mathbf{B} \subset \mathbf{R}^m$

$$\int_{\mathbf{B}} f^p(\nabla\phi) \geq \Gamma_0 \int_{\mathbf{B}} (|\nabla\phi|^p - \Gamma_1) \quad \forall \phi \in C_0^\infty(\mathbf{B}; \mathbf{R}^n). \tag{2.2}$$

It is easy to see that the unit ball $\mathbf{B}$ can be replaced by any open balls in (2.2). Note also that, under the conditions (2.1) and (2.2), the zero set $\mathscr{Z}[f]$ of f is allowed to be an unbounded closed set.

One of the main results of this paper is the following important consequence of the L^p-mean coercivity.

Theorem 2.1. *Suppose f satisfies* (2.1) *and the L^p-mean coercivity for some $p > 1$. Then, there exists a constant $\bar{\varepsilon} > 0$ such that $\mathscr{Z}[(f^{p+\bar{\varepsilon}})^{\mathrm{qc}}] = \mathscr{Z}[(f^p)^{\mathrm{qc}}]$. Moreover, in this case, $\mathscr{Z}[(f^p)^{\mathrm{qc}}] = \mathbf{Q}_p(\mathscr{Z}[f])$, where $\mathbf{Q}_p(\mathscr{K})$ denotes the p-quasiconvex hull of a set $\mathscr{K}$ defined in the introduction.*

The equality $\mathscr{Z}[(f^p)^{\mathrm{qc}}] = \mathbf{Q}_p(\mathscr{Z}[f])$ may not hold without the assumption of L^p-mean coercivity of f; see an example in Section 7 later. As a corollary of Theorem 2.1, we have the following result mentioned in the introduction.

Corollary 2.2. *Suppose that $f \geq 0$ is Lipschitz and satisfies $f(X) \geq c_0|X| - c_1$ for all $X \in \mathscr{M}^{n\times m}$. Then $\mathscr{Z}[(f^p)^{\mathrm{qc}}] = \mathscr{Z}[(f^q)^{\mathrm{qc}}]$ for all $1 < p < q < \infty$.*

In fact, this result is also true for $p = 1$. However, since the proof requires some other important techniques including the Luzin type approximation of $W^{1,1}$-maps by $W^{1,\infty}$-maps which we cannot cover in this paper, we refer to Yan [35] for the proof and [1,38] for more information.

Let us consider the case where f is 1-homogeneous. Let $\mathscr{K} = \mathscr{Z}[f]$ then $\mathscr{K}$ is a closed cone. By homogeneity, it is easy to see that the L^p-mean coercivity for f is equivalent to the L^p-mean coercivity for $d_{\mathscr{K}}$, i.e.,

$$\int_{\mathbf{B}} d^p_{\mathscr{K}}(\nabla\phi) \geq \Gamma_0 \int_{\mathbf{B}} |\nabla\phi|^p \quad \forall \phi \in W_0^{1,p}(\mathbf{B}; \mathbf{R}^n). \tag{2.3}$$

In the following, we assume that $\mathscr{K}$ is a closed cone. As before, we define

$$\mu(p; \mathscr{K}) = \inf \left\{ \int_{\mathbf{B}} d^p_{\mathscr{K}}(\nabla\phi) \,\middle|\, \phi \in C_0^\infty(\mathbf{B}; \mathbf{R}^n), \|\nabla\phi\|_{L^p(\mathbf{B})} = 1 \right\}.$$

From (2.3), we say that the set $\mathscr{K}$ satisfies the L^p-mean coercivity if $\mu(p; \mathscr{K}) > 0$. Let

$$\mathbf{S}(\mathscr{K}) = \{p > 1 \mid \mu(p; \mathscr{K}) > 0\}.$$

Our second main result states that the L^p-mean coercivity for cones is in fact locally independent of the power p.

Theorem 2.3. *The set* $\mathbf{S}(\mathscr{K})$ *is an open set. Moreover,* $\mathbf{Q}_p(\mathscr{K}) = \mathscr{Z}[(d^p_{\mathscr{K}})^{\mathrm{qc}}]$ *if* $p \in \mathbf{S}(\mathscr{K})$, *and* $\mathbf{Q}_p(\mathscr{K})$ *is constant for all* p *belonging to each of the connected components of* $\mathbf{S}(\mathscr{K})$.

Finally, we have the following higher integrability result concerning the first-order Hamilton–Jacobi system defined by a cone $\mathscr{K}$

$$\nabla u(x) \in \mathscr{K} \quad \text{a.e. } x \in \Omega. \tag{2.4}$$

Theorem 2.4. *Let* $\mathscr{K}$ *be a closed cone and* $[\alpha, \beta] \subset \mathbf{S}(\mathscr{K})$. *Then, any solution* $u \in W^{1,\alpha}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$ *to system* (2.4) *must belong to* $W^{1,\beta}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$.

Proof. Suppose $[\alpha, \beta] \subset \mathbf{S}(\mathscr{K})$. From the proof of Theorem 2.3 given later (see also Theorem 6.2), we see that

$$\gamma_0 = \inf_{\alpha \le p \le \beta} \mu(p; \mathscr{K}) > 0.$$

Hence $\mathscr{K}$ satisfies a uniform L^p-mean coercivity for $p \in [\alpha, \beta]$, thus the theorem follows from a general regularity theorem of Yan and Zhou [36]. □

The proof of Theorem 2.1 will be given in Section 5 and that of Theorem 2.3 will be given in Section 6.

3. A variational principle for minimizing sequences

In this section, we study some properties concerning the relaxation of a function by constructing certain useful minimizing sequences using the Ekeland variational principle [10].

First of all, we prove the following simple but useful result.

Lemma 3.1. *Let* $g: \mathscr{M}^{n\times m} \to \mathbf{R}$ *be any continuous function. Then for any* $A \in \mathscr{M}^{n\times m}$ *there exists a sequence* $\{\psi_j\}$ *in* $W^{1,\infty}_0(\mathbf{B}; \mathbf{R}^n)$ *such that*

$$g^{\mathrm{qc}}(A) = \lim_{j\to\infty} \fint_{\mathbf{B}} g(A + \nabla \psi_j), \qquad \lim_{j\to\infty} \|\psi_j\|_{L^\infty(\mathbf{B})} = 0. \tag{3.1}$$

Proof. Given $A \in \mathscr{M}^{n\times m}$, by the definition of $g^{\mathrm{qc}}(A)$, there exists a sequence $\{\phi_j\}$ in $C^\infty_0(\mathbf{B}; \mathbf{R}^n)$ such that

$$g^{\mathrm{qc}}(A) = \lim_{j\to\infty} \fint_{\mathbf{B}} g(A + \nabla \phi_j). \tag{3.2}$$

Let $M_j = 1 + \|\phi_j\|_{L^\infty(\mathbf{B})}$ and $\varepsilon_j = 1/jM_j$. Given $j = 1, 2, \ldots$, consider the family $\mathscr{B}_j$ of all closed balls contained in $\mathbf{B}$ and of radius $< \varepsilon_j$. Then $\mathscr{B}_j$ forms a Vitali covering of $\mathbf{B}$. Therefore, there exist a sequence of disjoint closed balls $\{\bar{B}_k\}$ and a null set N such that $\mathbf{B} = \bigcup_k \bar{B}_k \bigcup N$. Let $B_k = B(x_k, r_k)$ have center $x_k \in \mathbf{B}$ and radius $r_k \in (0, \varepsilon_j)$, and hence $\sum_k r_k^m = 1$. Define

$$\psi_j(x) = \begin{cases} r_k \phi_j\left(\dfrac{x - x_k}{r_k}\right) & \text{if } x \in \bar{B}_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

It is easily seen that $\psi_j \in W_0^{1,\infty}(\mathbf{B}; \mathbf{R}^n)$. From (3.3), we have that

$$\|\psi_j\|_{L^\infty(\mathbf{B})} \le \varepsilon_j \|\phi_j\|_{L^\infty(\mathbf{B})} \le 1/j$$

and by (3.2)

$$\begin{aligned} \int_{\mathbf{B}} g(A + \nabla\psi_j(x))\,\mathrm{d}x &= \sum_k \int_{B_k} g\left(A + \nabla\phi_j\left(\frac{x - x_k}{r_k}\right)\right)\mathrm{d}x \\ &= \sum_k \int_{\mathbf{B}} g(A + \nabla\phi_j(y)) r_k^m\,\mathrm{d}y \\ &= \int_{\mathbf{B}} g(A + \nabla\phi_j(y))\,\mathrm{d}y \to g^{\mathrm{qc}}(A)|\mathbf{B}|. \end{aligned}$$

The lemma is proved. □

Using this lemma, we can characterize the set $\mathscr{Z}[(f^p)^{\mathrm{qc}}]$ as follows if f satisfies the L^p-mean coercivity.

Proposition 3.2. *Let $f \ge 0$ be Lipschitz continuous and satisfy the L^p-mean coercivity. Then $A \in \mathscr{Z}[(f^p)^{\mathrm{qc}}]$ if and only if there exists a sequence $\{\psi_j\}$ in $W_0^{1,p}(\mathbf{B}; \mathbf{R}^n)$ such that*

$$\int_{\mathbf{B}} f^p(A + \nabla\psi_j) \le j^{-2}/2, \qquad \psi_j \rightharpoonup 0 \text{ weakly in } W_0^{1,p}(\mathbf{B}; \mathbf{R}^n). \tag{3.4}$$

Proof. Note that condition (3.4) and the lower semicontinuity result mentioned in the introduction (see also [1,5]) easily imply that $A \in \mathscr{Z}[(f^p)^{\mathrm{qc}}]$. To prove the other direction, we assume $A \in \mathscr{Z}[(f^p)^{\mathrm{qc}}]$. Applying the previous lemma to $g = f^p$, we have a sequence $\{\psi_j\}$ in $W_0^{1,\infty}(\mathbf{B}; \mathbf{R}^n)$ which, via a subsequence, converges to zero in L^∞-norm and satisfies the first condition of (3.4). Finally, the L^p-mean coercivity and the Lipschitz condition of f imply that the sequence $\{\psi_j\}$ is bounded in p-norm and thus must converge weakly to zero in $W_0^{1,p}(\mathbf{B}; \mathbf{R}^n)$. □

Assuming $A \in \mathscr{Z}[(f^p)^{\mathrm{qc}}]$, consider a complete metric space $(\mathscr{V}_A, \rho)$ defined by

$$\mathscr{V}_A \equiv \{Ax + \zeta \mid \zeta \in W_0^{1,1}(\mathbf{B}; \mathbf{R}^n)\}, \qquad \rho(w, v) \equiv \int_{\mathbf{B}} |\nabla w - \nabla v|. \tag{3.5}$$

Let $\{\psi_j\}$ be the sequence in $W_0^{1,p}(\mathbf{B}; \mathbf{R}^n)$ determined by Proposition 3.2. Then,

$$w_j \equiv Ax + \psi_j \in \mathscr{V}_A, \qquad \int_{\mathbf{B}} f^p(\nabla w_j) \le j^{-2}/2. \tag{3.6}$$

We have the following version of the Ekeland variational principle (see [9,10]).

Proposition 3.3. *There exists* $b_j \in \mathscr{V}_A$ *such that* $\rho(b_j, w_j) \le 1/j$ *and*

$$\int_{\mathbf{B}} f^p(\nabla b_j) < \int_{\mathbf{B}} f^p(\nabla w) + j^{-1} \int_{\mathbf{B}} |\nabla w - \nabla b_j| \tag{3.7}$$

for all $w \in \mathscr{V}_A$ *with* $w \neq b_j$.

Proof. For convenience of the reader, we give a proof of this result, following [9, Theorem 4.2]. Let $\Phi(v) = \int_{\mathbf{B}} f^p(\nabla v)$. By Fatou's lemma, Φ is lower semicontinuous on $(\mathscr{V}_A, \rho)$. Let $v_1 = w_j$ as defined above. Define

$$S_1 = \{v \in \mathscr{V}_A \,|\, \Phi(v) \le \Phi(v_1) - j^{-1}\rho(v, v_1)\}.$$

This set is nonempty and closed, and hence there exists $v_2 \in S_1$ such that $\Phi(v_2) \le \inf_{S_1} \Phi + j^{-2}/2^2$. So, we inductively define S_k by

$$S_k = \{v \in \mathscr{V}_A \,|\, \Phi(v) \le \Phi(v_k) - j^{-1}\rho(v, v_k)\}$$

and define $v_{k+1} \in S_k$ by requiring $\Phi(v_{k+1}) \le \inf_{S_k} \Phi + j^{-2}/2^{k+1}$. Clearly, $\{S_k\}$ is a decreasing sequence of closed sets in $\mathscr{V}_A$. We now estimate the size of S_k. Let $x \in S_k$. Then $\Phi(x) \le \Phi(v_k) - j^{-1}\rho(x, v_k)$. Also, by the definition of v_k, since $x \in S_{k-1}$, it follows that $\Phi(v_k) \le \inf_{v \in S_{k-1}} \Phi(v) + j^{-2}/2^k \le \Phi(x) + j^{-2}/2^k$. Hence

$$\rho(x, v_k) \le j^{-1}/2^k, \quad \forall x \in S_k, \tag{3.8}$$

and hence $\operatorname{diam}(S_k) \le j^{-1}/2^{k-1}$, which tends to 0 as $k \to \infty$. Therefore, $\bigcap_k S_k$ contains a unique point, say, b_j. Note that this b_j satisfies that

$$\Phi(b_j) \le \Phi(v_k) - j^{-1}\rho(b_j, v_k) \quad \forall k = 1, 2, \ldots . \tag{3.9}$$

We claim that b_j satisfies the requirements in the proposition. Indeed, by (3.8),

$$\rho(v_k, w_j) = \rho(v_k, v_1) \le \sum_{i=1}^{k-1} \rho(v_i, v_{i+1}) \le \sum_{i=1}^{k-1} j^{-1}/2^i$$

and letting $k \to \infty$, we have $\rho(b_j, w_j) \le j^{-1}$. To prove (3.7), i.e.

$$\Phi(w) > \Phi(b_j) - j^{-1}\rho(w, b_j) \quad \forall w \in \mathscr{V}_A, \; w \neq b_j,$$

we assume, on the contrary, that $\Phi(w) \le \Phi(b_j) - j^{-1}\rho(w, b_j)$ for some $w \in \mathscr{V}_A$, $w \neq b_j$. Then, by (3.9), $\Phi(w) \le \Phi(v_k) - j^{-1}\rho(w, v_k)$ for all k. This implies $w \in \bigcap_k S_k$ and hence $w = b_j$, a desired contradiction. The proof is thus complete. □

We also obtain the following result using Proposition 3.2.

Corollary 3.4. $I_p(b_j; \mathbf{B}) \to 0$ *and* $b_j \rightharpoonup Ax$ *weakly in* $W^{1,p}(\mathbf{B}; \mathbf{R}^n)$ *as* $j \to \infty$, *where* $\{b_j\}$ *is the sequence defined in Proposition* 3.3.

4. Reverse Hölder inequalities and higher regularity

Let $\{b_j\}$ be determined in Proposition 3.3. In this section, we prove that the sequence $\{\nabla b_j\}$ has a uniform higher integrability.

We first prove the following uniform reverse Hölder inequalities for sequence $\{\nabla b_j\}$.

Proposition 4.1. *There exist constants N_0, β_0 and γ_0 depending on f and p such that for all $j \geq N_0$ and $B_{2R} = B(a, 2R) \Subset \mathbf{B}$*

$$\fint_{B_R} |\nabla b_j|^p \leq \beta_0 \left(\fint_{B_{2R}} |\nabla b_j|^{pm/(m+p)} \right)^{(m+p)/m} + \gamma_0. \tag{4.1}$$

Proof. The proof uses standard techniques of Caccioppoli-type estimates [14,15,20, 23,24], but requires a careful treatment since the integrand f^p does not satisfy the usual growth conditions, so we present the detail here; see also Yan and Zhou [37].

Let $(\mathscr{V}_A, \rho)$ be defined as before, and let $c_0, c_1, \dots$ denote the constants depending only on p and f. Given $B_{2R} = B(a, 2R) \Subset \mathbf{B}$ and $0 < s < t \leq 2R$, let $\eta \in C_0^\infty(\mathbf{B})$ be a cut-off function such that

$$0 \leq \eta \leq 1, \quad \eta|_{B_s} = 1, \quad \eta|_{\mathbf{B}\setminus B_t} = 0, \quad |\nabla \eta| \leq c_0 (t-s)^{-1}.$$

Let $w = \eta v + (1-\eta) b_j$ and $\phi = b_j - w$, where $v \in \mathbf{R}^n$ is a constant to be chosen later. Then $w \in \mathscr{V}_A$, $\phi \in W_0^{1,p}(B_t; \mathbf{R}^n)$ and

$$\nabla w = (1-\eta)\nabla b_j - (b_j - v) \otimes \nabla \eta, \qquad \nabla \phi = \eta \nabla b_j + (b_j - v) \otimes \nabla \eta. \tag{4.2}$$

Using this, we obtain by (2.1) and (2.2) that

$$\begin{aligned} \int_{B_s} |\nabla b_j|^p &\leq \int_{B_t} |\nabla \phi|^p \leq \Gamma_0^{-1} \int_{B_t} f^p(\nabla \phi) + \Gamma_1 |B_t| \\ &\leq c_1 \int_{B_t} f^p(\nabla b_j) + \frac{c_1}{(t-s)^p} \int_{B_t \setminus B_s} |b_j - v|^p + c_1 |B_t|. \end{aligned} \tag{4.3}$$

Since $\nabla w = \nabla b_j$ in $\mathbf{B}\setminus B_t$ and $\nabla w = 0$ in B_s, the first term in (4.3) can be estimated by (3.7) as

$$\int_{B_t} f^p(\nabla b_j) \leq \int_{B_t \setminus B_s} f^p(\nabla w) + f^p(0)|B_s| + j^{-1} \int_{B_t} |\nabla w - \nabla b_j|. \tag{4.4}$$

Using (4.2) and the inequality $f(X) \leq f(0) + |X|$, we have that

$$\int_{B_t \setminus B_s} f^p(\nabla w) \leq c_2 \int_{B_t \setminus B_s} |\nabla b_j|^p + \frac{c_2}{(t-s)^p} \int_{B_t \setminus B_s} |b_j - v|^p + c_2 |B_{2R}|. \tag{4.5}$$

Combining (4.3)–(4.5), we have

$$\begin{aligned} \int_{B_s} |\nabla b_j|^p \leq{}& c_3 \int_{B_t \setminus B_s} |\nabla b_j|^p + \frac{c_3}{(t-s)^p} \int_{B_{2R}} |b_j - v|^p \\ &+ \frac{c_3}{j} \int_{B_t} |\nabla b_j - \nabla w| + c_3 |B_{2R}|. \end{aligned} \tag{4.6}$$

Since $t \leq t^p + 1$ for all $t \geq 0$ and $p \geq 1$, it follows that

$$\begin{aligned} \int_{B_t} |\nabla b_j - \nabla w| = \int_{B_s} |\nabla b_j| + \int_{B_t \setminus B_s} |\nabla \phi| \leq \int_{B_s} |\nabla b_j|^p \\ + c_4 \int_{B_t \setminus B_s} |\nabla b_j|^p + \frac{c_4}{(t-s)^p} \int_{B_{2R}} |b_j - v|^p + c_4 |B_{2R}|. \end{aligned} \tag{4.7}$$

Let $N_0 = 2c_3$. Then, for $j \geq N_0$, by (4.6) and (4.7), we have

$$\int_{B_s} |\nabla b_j|^p \leq c_5 \int_{B_t \setminus B_s} |\nabla b_j|^p + \frac{c_5}{(t-s)^p} \int_{B_{2R}} |b_j - v|^p + c_5 |B_{2R}|. \tag{4.8}$$

Filling the hole, i.e., adding $c_5 \int_{B_s} |\nabla b_j|^p$ to both sides of (4.8), we obtain that

$$\int_{B_s} |\nabla b_j|^p \leq \frac{c_5}{1+c_5} \int_{B_t} |\nabla b_j|^p + \frac{c_6}{(t-s)^p} \int_{B_{2R}} |b_j - v|^p + c_6 |B_{2R}|.$$

With this being valid for all $0 < s < t \leq 2R$, an iteration argument [14] yields that

$$\int_{B_R} |\nabla b_j|^p \leq c_7 R^{-p} \int_{B_{2R}} |b_j - v|^p + c_7 |B_{2R}| \tag{4.9}$$

and, taking the average, hence

$$⨍_{B_R} |\nabla b_j|^p \leq \frac{c_8}{R^{m+p}} \int_{B_{2R}} |b_j - v|^p + c_8. \tag{4.10}$$

Now, choose $v = v_R = ⨍_{B_{2R}} - b_j$ and use in (4.10) the Sobolev–Poincaré inequality

$$\int_{B_{2R}} |b_j - v_R|^p \leq C_m \left(\int_{B_{2R}} |\nabla b_j|^{pm/(m+p)} \right)^{(m+p)/m}$$

we obtain (4.1). The proof is complete. □

Theorem 4.2. *There exist $\varepsilon_0 > 0$ and integer N_0 depending on f and p such that the sequence $\{b_j\}$ determined in the previous proposition satisfies*

$$\sup_{j \geq N_0} \int_D |\nabla b_j|^{p+\varepsilon_0} \leq M_D < \infty, \quad \forall D \Subset \mathbf{B}. \tag{4.11}$$

Proof. Let $h_j = 1 + |\nabla b_j|^{pm/(m+p)}$ and $r = (m+p)/m$. Then, by (2.2), $\{h_j\}$ is bounded in $L^r(D)$ and for all $j \geq N_0$ by (4.1)

$$⨍_{B_R} h_j^r \leq \kappa \left(⨍_{B_{2R}} h_j \right)^r, \quad \forall B_{2R} \Subset \mathbf{B},$$

where κ is a constant depending on f and p. By Gehring's reverse Hölder inequality estimates [13], we conclude that $\{h_j\}$ is bounded in $L^s_{\mathrm{loc}}(\mathbf{B})$ for some $s > r$ and hence $\{b_j\}$ is bounded in $W^{1,p+\varepsilon_0}_{\mathrm{loc}}(\mathbf{B})$ for some $\varepsilon_0 > 0$ depending only on f and p. We have thus proved Theorem 4.2. □

Corollary 4.3. *Let $\{b_j\}$ be determined as in Corollary* 3.4 *and $\varepsilon_0 > 0$ determined in Theorem* 4.2. *Then $I_p(b_j; \mathbf{B}) \to 0$ and $b_j \rightharpoonup Ax$ weakly in $[W^{1,p} \cap W^{1,p+\varepsilon_0}_{\mathrm{loc}}](\mathbf{B}; \mathbf{R}^n)$ as $j \to \infty$.*

5. Proof of Theorem 2.1

Let $A \in \mathscr{Z}[(f^p)^{\rm qc}]$ and let $\{b_j\}$ be defined as before. By Corollary 4.3, we have that $I_p(b_j;\mathbf{B}) \to 0$ and $b_j \rightharpoonup Ax$ in $W^{1,p+\varepsilon_0}_{\rm loc}(\mathbf{B};\mathbf{R}^n)$, where $\varepsilon_0 > 0$ is the constant determined in Theorem 4.2, which is independent of A.

Let $s = p + \varepsilon_0/2$. We claim that $A \in \mathscr{Z}[g]$ for any quasiconvex function g satisfying

$$0 \le g(X) \le C(1 + |X|^s), \quad g|_{\mathscr{K}} = 0.$$

If this is done, then, by definition, $A \in \mathbf{Q_s}(\mathscr{K})$. Therefore, $\mathscr{Z}[(f^p)^{\rm qc}] \subseteq \mathbf{Q_s}(\mathscr{K}) \subseteq \mathscr{Z}[(f^s)^{\rm qc}]$, and hence Theorem 2.1 follows.

To prove this claim, we observe that, for any given quasiconvex function g as above and $\delta > 0$, there exists a constant $C(\delta) > 0$ such that

$$g(X) \le \delta(1 + |X|^{p+\varepsilon_0}) + C(\delta) f^p(X), \quad X \in \mathscr{M}^{n\times m}. \tag{5.1}$$

This inequality and Theorem 4.2 imply that for all $D \Subset \mathbf{B}$

$$\begin{aligned}\int_D g(\nabla b_j) &\le \delta \int_D (1 + |\nabla b_j|^{p+\varepsilon_0}) + C(\delta) \int_D f^p(\nabla b_j) \\ &\le \delta(1 + M_D) + C(\delta) I_p(b_j;\mathbf{B}).\end{aligned}$$

Letting first $j \to \infty$ and then $\delta \to 0$, we have $\int_D g(\nabla b_j) \to 0$. Furthermore, as $b_j \rightharpoonup Ax$ in $W^{1,p+\varepsilon_0}(D;\mathbf{R}^n)$, the lower semicontinuity theorem mentioned earlier again yields that

$$\int_D g(A) \le \lim_{j\to\infty} \int_D g(\nabla b_j) = 0$$

hence, $g(A) = 0$, i.e., $A \in \mathscr{Z}[g]$. The claim is proved, and the proof of Theorem 2.1 is thus complete.

6. Proof of Theorem 2.3

The proof of Theorem 2.3 relies on a stability result of nonlinear Hodge decompositions due to Iwaniec [17] and Iwaniec and Sbordone [20]. We refer to [16,18,37] for other developments and to Lewis [21] for the related results using different methods involving the maximal functions in harmonic analysis.

We need the following version of the nonlinear Hodge decompositions proved in Iwaniec and Sbordone [20, Theorem 3].

Lemma 6.1. *For $r > 1$, $u \in W^{1,r}_0(\mathbf{B};\mathbf{R}^n)$ and $\varepsilon \in (-1, r-1)$, the matrix $|\nabla u|^\varepsilon \nabla u \in L^{r/(1+\varepsilon)}(\mathbf{B};\mathscr{M}^{n\times m})$ can be decomposed as*

$$|\nabla u(x)|^\varepsilon \nabla u(x) = \nabla\psi(x) + h(x) \quad \text{a.e. } x \in \mathbf{B}, \tag{6.1}$$

where $\psi \in W^{1,r/(1+\varepsilon)}_0(\mathbf{B};\mathbf{R}^n)$ and $h \in L^{r/(1+\varepsilon)}(\mathbf{R}^m;\mathscr{M}^{n\times m})$ is a divergence-free matrix field such that

$$\|h\|_{L^{r/(1+\varepsilon)}(\mathbf{R}^m)} \le C(m,n,r,\varepsilon)|\varepsilon|\,\|\nabla u\|^{1+\varepsilon}_{L^r(\mathbf{B})}. \tag{6.2}$$

Moreover, for any constants $1 < r_1 < r_2 < \infty$, *the constant* $C(m,n,r,\varepsilon)$ *satisfies that*

$$\sup_{|\varepsilon| \le (r_1-1)/(r_1+1),\, r_1 \le r \le r_2} C(m,n,r,\varepsilon) \equiv \alpha(r_1,r_2) < \infty. \tag{6.3}$$

Proof. Estimate (6.2) follows simply from the standard Hodge decompositions, but a most important part of the lemma is the estimate (6.3) on the constant $C(m,n,r,\varepsilon)$. Using a technique of complex interpolation for nonlinear commutators, Iwaniec and Sbordone have proved that [20, Estimate (2.10)]

$$C(m,n,r,\varepsilon) \le \frac{2r(s_2-s_1)c(s_1,s_2)}{(r-s_1)(s_2-r)},$$

where $c(s_1,s_2) < \infty$ and numbers $1 < s_1 < s_2 < \infty$ satisfy

$$s_1 < r < s_2, \quad s_1 \le \frac{r}{1+\varepsilon} \le s_2.$$

From this, (6.3) follows easily by choosing $s_1 = (r_1+1)/2$ and $s_2 = r_2(r_1+1)/2$. □

Proof of Theorem 2.3. The second part of the theorem follows immediately from Theorem 2.1. We only need to show that the set $\mathbf{S}(\mathscr{K})$ is open. Assume $p \in \mathbf{S}(\mathscr{K})$. Let

$$r_1 = \frac{\sqrt{8p+1}-1}{2}, \qquad \varepsilon_0 = \frac{r_1-1}{r_1+1}, \quad r_2 = (1+\varepsilon_0)p. \tag{6.4}$$

Given any ε such that $|\varepsilon| \le \varepsilon_0$, let $r = (1+\varepsilon)p$. Obviously $r_1 \le r \le r_2$. We are going to show that $r \in \mathbf{S}(\mathscr{K})$ if $|\varepsilon|$ is further sufficiently small; consequently, $\mathbf{S}(\mathscr{K})$ is open and Theorem 2.3 is proved.

Let $\phi \in C_0^\infty(\mathbf{B};\mathbf{R}^n)$. Using Lemma 6.1, for $|\nabla\phi|^\varepsilon\nabla\phi \in L^p(\mathbf{B};\mathscr{M}^{n\times m})$, we have

$$|\nabla\phi(x)|^\varepsilon\nabla\phi(x) = \nabla\psi(x) + h(x) \quad \text{a.e. } x \in \mathbf{B}, \tag{6.5}$$

where $\psi \in W_0^{1,p}(\mathbf{B};\mathbf{R}^n)$, $h \in L^p(\mathbf{R}^m;\mathscr{M}^{n\times m})$ and

$$\|h\|_{L^p(\mathbf{R}^m)} \le \alpha_p|\varepsilon|\|\nabla\phi\|^{1+\varepsilon}_{L^r(\mathbf{B})}, \qquad \|\nabla\psi\|_{L^p(\mathbf{B})} \ge (1-\alpha_p|\varepsilon|)\|\nabla\phi\|^{1+\varepsilon}_{L^r(\mathbf{B})} \tag{6.6}$$

with the constant $\alpha_p = \alpha(r_1,r_2)$ depending only on p, as defined in (6.3). From (6.5), we have that $\nabla\psi(x) = |\nabla\phi(x)|^\varepsilon\nabla\phi(x) - h(x)$. Therefore,

$$d_{\mathscr{K}}(\nabla\psi(x)) \le |\nabla\phi(x)|^\varepsilon d_{\mathscr{K}}(\nabla\phi(x)) + |h(x)| \quad \forall x \in \mathbf{B}.$$

Let $\sigma_0 = \mu(p;\mathscr{K})^{1/p}$. The estimate above and the L^p-mean coercivity of $\mathscr{K}$ imply that

$$\begin{aligned}\sigma_0\|\nabla\psi\|_{L^p(\mathbf{B})} &\le \|d_{\mathscr{K}}(\nabla\psi)\|_{L^p(\mathbf{B})}\\ &\le \||\nabla\phi|^\varepsilon d_{\mathscr{K}}(\nabla\phi)\|_{L^p(\mathbf{B})} + \|h\|_{L^p(\mathbf{B})}.\end{aligned} \tag{6.7}$$

Combining (6.6) and (6.7) yields that

$$(\sigma_0 - \alpha_p(1+\sigma_0)|\varepsilon|)\|\nabla\phi\|^{1+\varepsilon}_{L^r(\mathbf{B})} \le \||\nabla\phi|^\varepsilon d_{\mathscr{K}}(\nabla\phi)\|_{L^p(\mathbf{B})}. \tag{6.8}$$

We now claim (6.8) implies that if $|\varepsilon| \le \varepsilon_0$ is further chosen sufficiently small then

$$\int_{\mathbf{B}} d^r_{\mathscr{K}}(\nabla\phi) \ge C_p\|\nabla\phi\|^r_{L^r(\mathbf{B})}, \quad C_p > 0 \tag{6.9}$$

for $r = (1+\varepsilon)p$; that is, $r \in \mathbf{S}(\mathscr{K})$, proving that $\mathbf{S}(\mathscr{K})$ is open.

Indeed, if $\varepsilon < 0$ then, using $|\nabla\phi|^\varepsilon \le [d_{\mathscr{K}}(\nabla\phi)]^\varepsilon$, we simply have

$$\||\nabla\phi|^\varepsilon d_{\mathscr{K}}(\nabla\phi)\|_{L^p(\mathbf{B})} \le \|d_{\mathscr{K}}^{1+\varepsilon}(\nabla\phi)\|_{L^p(\mathbf{B})} = \|d_{\mathscr{K}}(\nabla\phi)\|_{L^r(\mathbf{B})}^{1+\varepsilon}$$

hence (6.9) follows from (6.8) if $|\varepsilon|$ is sufficiently small. While, if $\varepsilon > 0$, by Hölder's inequality, we have

$$\||\nabla\phi|^\varepsilon d_{\mathscr{K}}(\nabla\phi)\|_{L^p(\mathbf{B})} \le \|d_{\mathscr{K}}(\nabla\phi)\|_{L^r(\mathbf{B})}\|\nabla\phi\|_{L^r(\mathbf{B})}^\varepsilon.$$

So, we still obtain (6.9) from (6.8) for all sufficiently small $|\varepsilon|$. The proof of Theorem 2.3 is now complete. □

From the proof of Theorem 2.3, one can also see that if $p > 1$ and $\mu(p;\mathscr{K}) > 0$ then

$$\mu(p;\mathscr{K}) \le \liminf_{r\to p} \mu(r;\mathscr{K}),$$

i.e., the function $\mu(r;\mathscr{K})$ is lower semicontinuous at $r = p$ for all such p. On the other hand, from the definition, function $\mu(r;\mathscr{K})$ is easily shown to be upper semicontinuous at all $r > 1$. Therefore, we have also proved the following result.

Theorem 6.2. *The function $\mu(p;\mathscr{K})$ defined above is continuous at all $p > 1$ where $\mu(p;\mathscr{K}) > 0$.*

7. Some examples and applications

In this final section, we consider some examples for which our theorems may produce some new interesting results.

First, we consider the so-called *conformal set* C_n in the space $\mathscr{M}^{n\times n}$ for $n \ge 2$; that is, a closed cone defined by

$$C_n = \{\lambda R \mid \lambda \ge 0, R \in SO(n)\},$$

where $SO(n)$ is the set of all real $n\times n$ orthogonal matrices with determinant 1.

It has been shown in [34] that $(d_{C_n}^{n/2})^{\mathrm{qc}} \equiv 0$ for all $n \ge 3$ and shown in [28] that if $n \ge 2$ is *even* then $\mathbf{Q}_{n/2}(C_n) = C_n$. This shows that the p-quasiconvex hulls defined here are not equivalent to those given in [39] and also that the inclusion $\mathbf{Q}_p(\mathscr{K}) \subseteq \mathscr{Z}[(d_{\mathscr{K}}^p)^{\mathrm{qc}}]$ may be *strict* if a closed cone $\mathscr{K}$ is not L^p-mean coercive.

Furthermore, it can be seen that C_n is L^n-mean coercive (see below); therefore, Theorems 2.1 and 2.3 show that, for some $\varepsilon > 0$, $(d_{C_n}^p)^{\mathrm{qc}}$ is quasiconvex and

$$\mathscr{Z}[(d_{C_n}^p)^{\mathrm{qc}}] = C_n \quad \forall p \in [n-\varepsilon, n+\varepsilon].$$

However, as shown in [34], $(d_{C_n}^p)^{\mathrm{qc}}$ is not *polyconvex* in the sense of Ball [2] for all $n/2 < p < n$ with $n \ge 3$. This shows that the structure of $(d_{C_n}^p)^{\mathrm{qc}}$ is highly non-trivial if $n/2 < p < n$.

In order to illustrate some concrete examples of applications of the results proved above, let us consider a *null-Lagrangian* $N(X)$ on $\mathscr{M}^{n\times m}$; that is,

$$\int_{\mathbf{B}} N(A + \nabla\phi(x))\,\mathrm{d}x = N(A)|\mathbf{B}| \quad \forall A \in \mathscr{M}^{n\times m}$$

for all balls $\mathbf{B}$ and all $\phi \in C_0^\infty(\mathbf{B};\mathbf{R}^n)$. (We refer to [3] for more on null-Lagrangians.) Assume also that N is homogeneous of degree $k \geq 2$. Then we know k must be an integer and $k \leq \min\{n,m\}$.

Let $u \in W^{1,k}(\Omega;\mathbf{R}^n)$. Then we know that $N(\nabla u)$ belongs to the local Hardy space $\mathscr{H}^1_{\mathrm{loc}}(\Omega)$ [7] and if in addition $N(\nabla u(x)) \geq 0$ then $N(\nabla u)$ belongs locally to the Stein space $L^1 \ln L^1$ [26,29]; this last property is a higher regularity result since by scaling $N(\nabla u(x))$ only belongs to L^1. We next show that if a certain strict positivity of $N(\nabla u(x))$ holds then one could obtain some new interesting higher regularity results.

Theorem 7.1. *There exist constants $\alpha_k < k < \beta_k$ such that any map $u \in W^{1,\alpha_k}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$ satisfying*

$$N(\nabla u(x)) \geq |\nabla u(x)|^k \quad \text{a.e. } x \in \Omega \tag{7.1}$$

must belong to $W^{1,\beta_k}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$; thus $N(\nabla u(x))$ belongs to $L^p_{\mathrm{loc}}(\Omega)$ for some $p > 1$.

Proof. The crux of this theorem is that u only belongs to $W^{1,\alpha_k}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$; there is no local $L^1 \ln L^1$ regularity for $N(\nabla u)$ since it is not a priori integrable. The proof is a beautiful application of the theorems we proved above. Define

$$f(X) = (\max\{0, |X|^k - N(X)\})^{1/k}, \quad \mathscr{K} = \mathscr{Z}[f]. \tag{7.2}$$

Then condition (7.1) is equivalent to the Hamilton–Jacobi system

$$\nabla u(x) \in \mathscr{K} \quad \text{a.e. } x \in \Omega.$$

Note that f is 1-homogeneous and

$$\int_{\mathbf{B}} f^k(\nabla\phi(x))\,\mathrm{d}x \geq \int_{\mathbf{B}} (|\nabla\phi(x)|^k - N(\nabla\phi(x)))\,\mathrm{d}x = \int_{\mathbf{B}} |\nabla\phi(x)|^k\,\mathrm{d}x$$

for all balls $\mathbf{B}$ and all $\phi \in C_0^\infty(\mathbf{B};\mathbf{R}^n)$; therefore, f and hence $\mathscr{K}$ satisfy the L^k-mean coercivity. So $k \in \mathbf{S}(\mathscr{K})$, the *open* set defined before, and hence there exists a closed interval $[\alpha_k, \beta_k] \subset \mathbf{S}(\mathscr{K})$ such that $\alpha_k < k < \beta_k$. Consequently, the theorem follows from Theorem 2.4. □

From this theorem and Sobolev's embedding theorem, we easily have the following result.

Corollary 7.2. *Let $k = m$. Then any map u in $W^{1,\alpha_k}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$ satisfying* (7.1) *must be locally Hölder continuous in Ω.*

Let $\alpha_k < k < \beta_k$ be as determined in the proof given above. We prove a stability result.

Theorem 7.3. *Let* $\{u_j\}$ *converge weakly to* u *in* $W^{1,\alpha_k}(\Omega;\mathbf{R}^n)$ *and satisfy*

$$|\nabla u_j(x)|^k \le N(\nabla u_j(x)) + (g_j(x))^{k/\alpha_k} \quad \text{a.e. } x \in \Omega. \tag{7.3}$$

If $g_j \to 0$ *strongly in* $L^1(\Omega)$ *as* $j \to \infty$, *then the weak limit* u *satisfies*

$$N(\nabla u(x)) \ge |\nabla u(x)|^k \quad \text{a.e. } x \in \Omega,$$

and thus the regularity result of the previous theorem follows.

Proof. Again, the difficulty lies in that the sequence and the weak convergence are only in $W^{1,\alpha_k}(\Omega;\mathbf{R}^n)$ and in this case one cannot take any limit in the inequality (7.3). It seems necessary to use some of the results proved above to prove this theorem. Let $f, \mathscr{K}$ be defined as in the proof of the previous theorem. Note that f^k is quasiconvex since it is polyconvex in the sense of Ball [2]. Therefore, by definition, the k-quasiconvex hull $\mathbf{Q}_k(\mathscr{K}) = \mathscr{K}$. From this and Theorem 2.3 we have $\mathbf{Q}_{\alpha_k}(\mathscr{K}) = \mathscr{K} = \mathscr{Z}[f]$. Also by the assumption we have

$$\lim_{j\to\infty} \int_\Omega f^{\alpha_k}(\nabla u_j(x))\,\mathrm{d}x = 0.$$

From this and a theorem in Yan [35] it follows that the weak limit u satisfies

$$\nabla u(x) \in \mathbf{Q}_{\alpha_k}(\mathscr{K}) = \mathscr{K} = \mathscr{Z}[f]$$

and hence $N(\nabla u(x)) \ge |\nabla u(x)|^k$ for almost every $x \in \Omega$. The proof is complete. □

Remark. Let $n = m = k \ge 2$ and for $L \ge 1$ let $N(X) = Ln^{n/2} \det X$. We can then recover from Theorems 7.1 and 7.3 some of results in [17,19,33,37] concerning the regularity of the so-called *weakly* L-quasiregular mappings.

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