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ON THE WEAK LIMIT OF MAPPINGS WITH FINITE DISTORTION

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Abstract. We give a new proof that the limit of a weakly convergent sequence of mappings with finite distortion also has finite distortion. The result has been recently proved by Gehring and Iwaniec using the biting convergence of Jacobians. We present a different proof using simply the lower semi-continuity of quasiconvex functionals.

1. Introduction

Let $f\colon \Omega \to \mathbf{R}^n$ be a mapping in the Sobolev space $W^{1,p}_{loc}(\Omega;\mathbf{R}^n)$, where Ω is a domain in $\mathbf{R}^n$. Then the Jacobian matrix $Df(x)$ and its determinant $J(x,f) = \det Df(x)$ are well-defined at almost every point $x \in \Omega$. We shall use $M^{n\times n}$ to denote the space of all $n \times n$ real matrices equipped with the operator norm

$$|\xi| = \max\{|\xi v| \mid |v| = 1\}.$$

Definition 1.1. A mapping $f \in W^{1,n}_{loc}(\Omega;\mathbf{R}^n)$ is said to have finite distortion if there exists a finite measurable function $K(x) \geq 0$ such that

$$|Df(x)|^n \leq K(x)J(x,f) \tag{1.1}$$

for almost every $x \in \Omega$.

From this definition, a mapping f with finite distortion has the property that either the Jacobian matrix $Df(x) = 0$ or its determinant $J(x,f) > 0$; in the latter case the matrix $Df(x)$ is invertible.

Definition 1.2. The outer and inner dilatation functions $K_O(x,f)$ and $K_I(x,f)$ of a mapping f with finite distortion are defined as follows:

$$\begin{cases} K_O(x,f) = K_I(x,f) = 1 & \text{if } Df(x) = 0; \\ K_O(x,f) = K_O(Df(x)), & \\ K_I(x,f) = K_I(Df(x)) & \text{if } J(x,f) > 0, \end{cases} \tag{1.2}$$

where, for any invertible matrices ξ,

$$K_O(\xi) = |\xi|^n / \det \xi, \quad K_I(\xi) = K_O(\xi^{-1}). \tag{1.3}$$

We shall prove the following theorem; see also Gehring and Iwaniec [3].

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Theorem 1.3. *Let $f_\nu\colon \Omega \to \mathbf{R}^n$ be a sequence of mappings with finite distortion which converges weakly in $W^{1,n}_{loc}(\Omega;\mathbf{R}^n)$ to a mapping f. Suppose there exists a finite measurable function $M(x)$ such that*

$$K(x,f_\nu) \le M(x) < \infty$$

almost everywhere in Ω for all $\nu = 1,2,...,$ where $K(x,f)$ is either $K_O(x,f)$ or $K_I(x,f)$. Then the limit mapping f has finite distortion. Moreover, for any subsequence f_{ν_k}, one has

$$K_O(x,f) \le \limsup_{k\to\infty} K_O(x,f_{\nu_k}) \tag{1.4}$$

and

$$K_I(x,f) \le \limsup_{k\to\infty} K_I(x,f_{\nu_k}) \tag{1.5}$$

for almost every $x \in \Omega$.

This theorem is a refinement of Reshetnyak's convergence theorem [6, Theorem 9.2] concerning mappings of *bounded* distortion, that is, mappings f with $K(x) \le K < \infty$ for a constant K in (1.1). For such mappings, the (maximum) outer and inner dilatations are defined by

$$K_O(f) = \|K_O(x,f)\|_{L^\infty(\Omega)}, \quad K_I(f) = \|K_I(x,f)\|_{L^\infty(\Omega)}.$$

In Theorem 1.3, if f_ν has bounded distortion and satisfies

$$K_O(f_\nu) \le M < \infty$$

for a constant M and all $\nu = 1,2,...$, then from (1.4) and (1.5) we can establish

$$K_O(f) \le \liminf_{\nu\to\infty} K_O(f_\nu), \quad K_I(f) \le \liminf_{\nu\to\infty} K_I(f_\nu), \tag{1.6}$$

which recovers Reshetnyak's convergence theorem [6, Theorem 9.2].

Finally, we remark that the estimate (1.4) seems weaker than the estimate given in Gehring and Iwaniec [3, Remark 1.7] in terms of biting convergence. We also refer to [3, 4] for the convergence results regarding other dilatation functions.

2. Variational approaches

In order to present our proof of Theorem 1.3, we need some variational characterizations of mappings with finite distortion.

For any given finite measurable function $K(x)$ on Ω, consider the function from $\Omega \times M^{n\times n}$ to $\mathbf{R}$ defined by

$$F_1(x,\xi) = \max\{0,\, |\xi|^n - K(x)\det\xi\}. \tag{2.1}$$

It is easy to see that

$$0 \le F_1(x,\xi) \le \rho_1(x)\,|\xi|^n,$$

where $\rho_1(x) = 1 + |K(x)|$, and the condition (1.1) in Definition 1.1 is equivalent to

$$F_1(x, Df(x)) = 0$$

for almost every $x \in \Omega$. Hence, any mapping f with finite distortion is an *absolute* minimizer of the functional

$$I_1(u,\Omega) = \int_\Omega F_1(x, Du(x))\,dx,$$

where $F_1(x,\xi)$ is defined by (2.1) with $K(x) = K_O(x,f)$.

In order to give another characterization using the inner dilatation function $K_I(x,f)$, we need some notation. Let $\xi^\#$ be the matrix of cofactors of matrix ξ such that

$$\xi\xi^\# = \xi^\#\xi = (\det\xi)\, I,$$

where I is the identity matrix. For invertible matrices ξ, this shows that $\xi^{-1} = \xi^\#/\det\xi$, thus one can easily see that

$$K_I(\xi) = |\xi^\#|^n/(\det\xi)^{n-1}. \tag{2.2}$$

Now let $P(x)$ be any finite measurable function on Ω. Consider

$$F_2(x,\xi) = \max\{0,\, |\xi^\#|^{\frac{n}{n-1}} - P(x)\det\xi\}. \tag{2.3}$$

Then $F_2(x,\xi)$ also satisfies

$$0 \le F_2(x,\xi) \le \rho_2(x)\,|\xi|^n,$$

where $\rho_2(x) = c_n + |P(x)|$. If f is a mapping with finite distortion, then

$$F_2(x, Df(x)) = 0$$

for almost every $x\in\Omega$, where $F_2(x,\xi)$ is defined as above with $P(x)=K_I(x,f)^{1/n-1}$. Therefore f is an absolute minimizer of the functional

$$I_2(u,\Omega) = \int_\Omega F_2(x, Du(x))\,dx.$$

Finally, we remark that functions F_1, F_2 defined above have the important property of *quasiconvexity* introduced by Morrey [5]; see also Ball [2].

Proposition 2.1. *Let $F(x,\xi)$ denote one of $F_1(x,\xi)$ and $F_2(x,\xi)$ defined by (2.1) and (2.3). Then, for almost every $x\in\Omega$, $F(x,\xi)$ is* quasiconvex *in ξ in the sense that*

$$F(x,\xi) \le \frac{1}{|\Omega|}\int_\Omega F(x, \xi + D\phi(y))\,dy, \quad \forall\,\phi\in C_0^\infty(\Omega;\mathbf{R}^n). \tag{2.4}$$

Proof. For any $\xi\in M^{n\times n}$, $\phi\in C_0^\infty(\Omega;\mathbf{R}^n)$, it follows that [2, 5]

$$\det\xi = \frac{1}{|\Omega|}\int_\Omega \det(\xi + D\phi(y))\,dy, \quad \xi^\# = \frac{1}{|\Omega|}\int_\Omega (\xi + D\phi(y))^\#\,dy.$$

Then, (2.4) follows from the definition of F_1, F_2 and Jensen's inequality. □

3. Lower semicontinuity

In what follows, we assume that $F(x,\xi)$ is a *Carathéodory* function from $\Omega\times M^{n\times n}$ to $\mathbf{R}$ in the sense that $F(x,\xi)$ is continuous in ξ for almost every $x\in\Omega$ and measurable in x for all $\xi\in M^{n\times n}$. Assume also that there exists a finite measurable function $\rho(x)\ge 0$ such that

$$0 \le F(x,\xi) \le \rho(x)\,|\xi|^p \tag{3.1}$$

for almost every $x\in\Omega$, where $1\le p<\infty$ is a constant. We need the following lower semicontinuity theorem mainly due to Acerbi and Fusco [1].

Theorem 3.1. *Let $F(x,\xi)$ be given as above. Suppose for almost every $x \in \Omega$ the function $F(x,\xi)$ is quasiconvex in ξ in the sense as defined in Proposition* 2.1. *For any measurable subset E of Ω and $t > 0$ let*

$$E_t = \{x \in E \,|\, \rho(x) < t\}.$$

If $f_\nu \colon \Omega \to \mathbf{R}^n$ is a sequence of mappings which converges weakly in $W^{1,p}(\Omega;\mathbf{R}^n)$ to a mapping f, then, for every $t > 0$, one has

$$\int_{E_t} F(x, Df(x))\,dx \le \liminf_{\nu\to\infty} \int_{E_t} F(x, Df_\nu(x))\,dx.$$

Proof. Consider $G(x,\xi) = F(x,\xi)\,\chi_{E_t}(x)$, where χ_S denotes the characteristic function of set S. Then $G(x,\xi)$ is a Carathéodory function and, for almost every $x \in \Omega$, $G(x,\xi)$ is quasiconvex in ξ and satisfies

$$0 \le G(x,\xi) \le t\,|\xi|^p.$$

Therefore, by the lower semicontinuity theorem of Acerbi and Fusco [1, Theorem II.4], the functional $J(u) = \int_\Omega G(x, Du)$ is (sequentially) weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbf{R}^n)$. Hence,

$$\int_{E_t} F(x, Df) = J(f) \le \liminf_{\nu\to\infty} J(f_\nu) = \liminf_{\nu\to\infty} \int_{E_t} F(x, Df_\nu).$$

The theorem is proved. □

Theorem 3.2. *Suppose a sequence of mappings $f_\nu \colon \Omega \to \mathbf{R}^n$ converges weakly in $W^{1,n}_{loc}(\Omega;\mathbf{R}^n)$ to a mapping f. Let $K(x)$ and $P(x)$ be any given finite measurable functions in Ω, and let $F_1(x,\xi)$ and $F_2(x,\xi)$ be defined by (2.1) and (2.3), respectively. Assume $F(x,\xi)$ is one of F_1 and F_2. Let E be a measurable subset of Ω such that*

$$\lim_{\nu\to\infty} \int_E F(x, Df_\nu(x))\,dx = 0.$$

Then $F(x, Df(x)) = 0$ for almost every $x \in E$.

Proof. The theorem follows easily from Proposition 2.1 and Theorem 3.1. □

Finally, we prove a result which enables us to estimate one dilatation function in terms of the other.

Lemma 3.3. *Let $f \in W^{1,n}_{loc}(\Omega;\mathbf{R}^n)$ be a mapping with finite distortion. Then*

$$K_O(x,f) \le K_I(x,f)^{n-1}, \quad K_I(x,f) \le K_O(x,f)^{n-1}$$

for almost every $x \in \Omega$.

Proof. Note that the functions $K_O(\xi)$ and $K_I(\xi)$ defined by (1.3) can be represented by the principal values of ξ, that is, the eigenvalues of the matrix $\sqrt{\xi^T\xi}$. Let $\det\xi \ne 0$ and let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the principal values of ξ. Then it is easy to see that

$$K_O(\xi) = \lambda_n^n/\lambda_1\lambda_2\cdots\lambda_n, \quad K_I(\xi) = \lambda_1\lambda_2\cdots\lambda_n/\lambda_1^n.$$

Therefore,

$$K_O(\xi) \le K_I(\xi)^{n-1}, \quad K_I(\xi) \le K_O(\xi)^{n-1},$$

and the lemma follows from the definition of $K_O(x,f)$ and $K_I(x,f)$. □

4. Proof of Theorem 1.3

By Lemma 3.3, we can assume that the function $K(x,f)$ in Theorem 1.3 is $K_O(x,f)$. Let $F_1(x,\xi)$ be defined by (2.1) with $K(x) = M(x)$ as given in the theorem. We then have $F_1(x, Df_\nu(x)) = 0$. Thus by Theorem 3.2 it follows that $F_1(x, Df(x)) = 0$, that is,

$$|Df(x)|^n \le M(x)J(x,f)$$

for almost every $x \in \Omega$. Therefore, f is a mapping with finite distortion and $K_O(x,f) \le M(x)$ for almost every $x \in \Omega$.

We need to show estimates (1.4) and (1.5). For this purpose, we may assume that the subsequence f_{ν_k} is the original full sequence f_ν. Let

$$L(x) = \limsup_{\nu\to\infty} K_O(x,f_\nu), \quad Q(x) = \limsup_{\nu\to\infty} K_I(x,f_\nu).$$

We need to show

$$K_O(x,f) \le L(x), \quad K_I(x,f) \le Q(x) \tag{4.1}$$

for almost every $x \in \Omega$.

If $J(x,f) = 0$, that is, $Df(x) = 0$, then $K_O(x,f) = 1 \le L(x)$ and $K_I(x,f) = 1 \le Q(x)$, and hence (4.1) holds. So we have only to prove (4.1) for almost every x in the measurable set $\Omega' = \{x \in \Omega \,|\, J(x,f) > 0\}$. We assume $|\Omega'| > 0$, and let

$$N = \{x \in \Omega' \,|\, L(x) < K_O(x,f)\}, \quad R = \{x \in \Omega' \,|\, Q(x) < K_I(x,f)\}.$$

We need to show $|N| = |R| = 0$. Note that

$$N = \bigcup_{m=1}^\infty \bigcup_{k=1}^\infty N_{mk}, \quad R = \bigcup_{m=1}^\infty \bigcup_{k=1}^\infty R_{mk}, \tag{4.2}$$

where

$$N_{mk} = \bigcap_{\nu=k}^\infty \left\{x \in \Omega' \,|\, K_O(x,f_\nu) \le K_O(x,f) - \frac{1}{m}\right\} \tag{4.3}$$

and

$$R_{mk} = \bigcap_{\nu=k}^\infty \left\{x \in \Omega' \,|\, K_I(x,f_\nu) \le K_I(x,f) - \frac{1}{m}\right\}. \tag{4.4}$$

We shall prove $|N_{mk}| = |R_{mk}| = 0$ for all $m,k = 1,2,...$ and hence $|N| = |R| = 0$. This will complete the proof of (4.1).

Proof of $|N_{mk}| = 0$. Note that $K_O(x,f_\nu) \le K_O(x,f) - 1/m$ for all $x \in N_{mk}$ and all $\nu \ge k$. We have

$$|Df_\nu(x)|^n \le \left(K_O(x,f) - \frac{1}{m}\right) J(x,f_\nu)$$

for all $\nu = k, k+1, ...$ and almost every $x \in E = N_{mk}$. Hence, by Theorem 3.2,

$$|Df(x)|^n \le \left(K_O(x,f) - \frac{1}{m}\right) J(x,f)$$

for almost every $x \in E = N_{mk}$. Since $J(x,f) > 0$ for $x \in E$, it follows that $K_O(x,f) \le K_O(x,f) - 1/m$ for almost every $x \in E$. Hence, $|E| = |N_{mk}| = 0$ for all $m,k = 1,2,...$; the proof is complete. □

Proof of $|R_{mk}| = 0$. This is similar to the previous one. Note that $K_I(x, f_\nu) \leq K_I(x, f) - 1/m$ for all $x \in R_{mk}$ and all $\nu \geq k$. Let $F_2(x, \xi)$ be the function defined by (2.3) with

$$P(x) = (K_I(x, f) - 1/m)^{\frac{1}{n-1}}.$$

We then have $F_2(x, Df_\nu(x)) = 0$ for almost every $x \in E = R_{mk}$ and all $\nu \geq k$. Therefore, again by Theorem 3.2, $F_2(x, Df(x)) = 0$ for almost every $x \in R_{mk}$. This implies

$$|(Df(x))^{\#}|^{\frac{n}{n-1}} \leq P(x)J(x, f)$$

and thus by (2.2)

$$K_I(x, f) \leq (P(x))^{n-1} = K_I(x, f) - 1/m$$

for almost every $x \in R_{mk}$. Thus $|R_{mk}| = 0$. The proof of Theorem 1.3 is complete. □

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