# RELAXATION AND ATTAINMENT RESULTS FOR AN INTEGRAL FUNCTIONAL WITH UNBOUNDED ENERGY-WELL

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ABSTRACT. Consider functional $I(u) = \int_\Omega ||Du|^n - L \det Du| \, dx$, whose energy-well consists of matrices satisfying $|\xi|^n = L \det \xi$. We show that the relaxations of this functional in various Sobolev spaces are significantly different. We also make several remarks concerning various $p$-growth semiconvex hulls of the energy-well set and prove an attainment result for a special Hamilton-Jacobi equation, $|Du|^n = L \det Du$, in the so-called *grand* Sobolev space $W^{1,q)}(\Omega; \mathbf{R}^n)$ with $q = \frac{nL}{L+1}$.

## 1. Introduction and main results

Given $L \geq 1$, consider function $f_L(\xi) = |\xi|^n - L \det \xi$ on the space $\mathbb{M}^{n \times n}$ of $n \times n$ matrices and integral functional

$$(1.1) \qquad I(u) = \int_\Omega |f_L(Du(x))| \, dx = \int_\Omega ||Du(x)|^n - L \det Du(x)| \, dx,$$

where $u$ is a mapping from domain $\Omega \subset \mathbf{R}^n$ to $\mathbf{R}^n$ and $Du(x)$ is the Jacobi matrix of $u$. In general, $|\xi|$ denotes the operator norm of $m \times n$ matrix $\xi \in \mathbb{M}^{m \times n}$ defined by $|\xi| = \max_{h \in \mathbf{R}^n, |h|=1} |\xi h|$.

The absolute energy minimizers of $I(u)$ can be characterized as mappings satisfying the Hamilton-Jacobi equation:

$$(1.2) \qquad Du(x) \in Z_L = \{\xi \in \mathbb{M}^{n \times n} \mid |\xi|^n = L \det \xi\} \quad a.e. \ x \in \Omega.$$

In the terminology used for phase transition problems (see, e.g., [3, 9, 10, 11]), the set $Z_L$ is the energy-well of energy functional $I(u)$. Note that $Z_L$ is also the boundary of the so-called $L$-quasiconformal set $K_L$ defined by

$$K_L = \{\xi \in \mathbb{M}^{n \times n} \mid |\xi|^n \leq L \det \xi\}.$$

When $L=1$, it is easily seen that the set $K_1 = Z_1$ coincides with the set of conformal matrices, that is,

$$K_1 = Z_1 = \{\lambda R \,|\, \lambda \geq 0,\ R \in SO(n)\}.$$

Weakly $L$-quasiregular mappings are the mappings $u \in W^{1,p}_{loc}(\Omega;\mathbf{R}^n)$ satisfying $Du(x) \in K_L$ almost everywhere in $\Omega$ (see Iwaniec [6]). Many regularity and stability properties of weakly quasiregular mappings have been studied in connection with the certain *semi*-convex hulls and the attainment result of the quasiconformal sets $K_L$ in Yan [13, 14, 15]. In this paper we study similar problems related to the set $Z_L$.

The natural space for $I(u)$ is the Sobolev space $W^{1,n}(\Omega;\mathbf{R}^n)$. In this note we are interested in minimization of functional $I(u)$ in different Sobolev spaces $W^{1,p}(\Omega;\mathbf{R}^n)$. For this purpose, we define the $p$-growth relaxation functions of $I(u)$ as follows:

$$g_p(\xi) = \inf_{u \in W^{1,p}(\Omega;\mathbf{R}^n),\, u|_{\partial\Omega}=\xi x} \frac{1}{|\Omega|}\int_\Omega |f_L(Du(x))|\,dx. \tag{1.3}$$

Clearly the function $g_p(\xi)$ is non-decreasing with respect to $p \geq 1$. For any given function $f\colon \mathbb{M}^{m\times n} \to \mathbf{R}$, we define the *quasiconvexification* of $f$ to be

$$f^{\#}(\xi) = \inf_{\phi \in C_0^\infty(\Omega;\mathbf{R}^m)} \frac{1}{|\Omega|}\int_\Omega f(\xi + D\phi(x))\,dx, \quad \forall\, \xi \in \mathbb{M}^{m\times n}, \tag{1.4}$$

where $C_0^\infty(\Omega;\mathbf{R}^m)$ stands for all smooth functions with compact support in $\Omega$. A density argument easily shows that $g_\infty = |f_L|^{\#}$ and hence the function $g_\infty$ is also *quasiconvex* in the sense of Morrey. Recall that, in general, a function $f\colon \mathbb{M}^{m\times n} \to \mathbf{R}$ is said to be quasiconvex (in the sense of Morrey) if it satisfies

$$\frac{1}{|\Omega|}\int_\Omega f(\xi + D\phi(x))\,dx \geq f(\xi), \quad \forall\, \xi \in \mathbb{M}^{m\times n},\ \phi \in C_0^\infty(\Omega;\mathbf{R}^m). \tag{1.5}$$

It is well-known that any finite quasiconvex function $f$ is always *rank-one convex*; that is, $f(\xi + t\eta)$ is convex in $t \in \mathbf{R}$ for any $\xi, \eta \in \mathbb{M}^{m\times n}$ with $\operatorname{rank}\eta = 1$. A function $f\colon \mathbb{M}^{m\times n} \to \mathbf{R}$ is called *polyconvex* if it can be written as a convex function of all subdeterminants; therefore all convex functions are polyconvex. It can be shown that a polyconvex function is always quasiconvex. We refer to Acerbi & Fusco [1], Ball [2], Ball & Murat [4], Dacorogna [5], Morrey [8], and Müller [9] for the proofs and more properties of these semiconvex functions.

In what follows, we denote by $f^{-1}(0)$ the zero set of any scalar function $f$, that is, $f^{-1}(0) = \{\xi \mid f(\xi) = 0\}$. We now state our main result.

**Theorem 1.1.** *Let $n \geq 3$ and $L \geq 1$. Then*
(a) $g_p = |f_L|^\# = \max\{f_L, 0\}$ *for all* $p \geq n$;
(b) *for some* $\epsilon > 0$, $g_p^{-1}(0) = K_L$ *for all* $p \geq n - \epsilon$;
(c) $g_p \equiv 0$ *and the minimum is attained for all* $1 \leq p < \frac{nL}{L+1}$.

Part (c) of the theorem is equivalent to the following

**Theorem 1.2.** *Let $n \geq 3$, $L \geq 1$. Then, for any $1 \leq p < \frac{nL}{L+1}$ and $\xi \in \mathbb{M}^{n\times n}$, there exists a map $u \in W^{1,p}(\Omega; \mathbf{R}^n)$ with $u|_{\partial\Omega} = \xi x$ satisfying*

$$|Du(x)|^n = L \det Du(x) \quad a.e.\ x \in \Omega. \tag{1.6}$$

*Remark.* This result sharpens an earlier result in Yan [14]. Note that, if $\xi = 0$, the trivial map $u \equiv 0$ is a required solution of (1.6); but, in this case, we can show existence of nontrivial solutions. In fact, we shall prove a sharper attainment result for a special class of nontrivial solutions of (1.6) in a space strictly smaller than any $W^{1,p}(\Omega; \mathbf{R}^n)$ for $1 \leq p < \frac{nL}{L+1}$. □

For a given $q > 1$, we define the *grand* Sobolev space $W^{1,q)}(\Omega; \mathbf{R}^m)$ (see *e.g.* Iwaniec & Sbordone [7]) to be the space of all functions $u \in \cap_{1\leq p<q} W^{1,p}(\Omega; \mathbf{R}^m)$ that satisfy

$$[Du]_{q,\Omega} \equiv \sup_{1\leq p<q} \left[ (q-p) \frac{1}{|\Omega|} \int_\Omega |Du(x)|^p\, dx \right]^{1/p} < \infty. \tag{1.7}$$

Given any number $\alpha \in \mathbf{R}$, we define the set

$$S_\alpha = \{R(I + \alpha\omega \otimes \omega) \mid R \in Z_1,\ R \neq 0,\ \omega \in \mathbf{R}^n,\ |\omega| = 1\}, \tag{1.8}$$

where $I$ is $n \times n$ identity matrix and $a \otimes b$ stands for the rank-one matrix $(a_i b_j)$. In the following, we shall use $\delta(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ to denote the $n \times n$ diagonal matrix with diagonal elements $\epsilon_1, \ldots, \epsilon_n$; that is, we denote

$$\delta(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = \begin{pmatrix} \epsilon_1 & & & 0 \\ & \epsilon_2 & & \\ & & \ddots & \\ 0 & & & \epsilon_n \end{pmatrix}.$$

Let $J = I - 2e_1 \otimes e_1 = \delta(-1, 1, \cdots, 1)$. Then $J^2 = I$ and $\det J = -1$. Define $JS_\alpha = \{J\xi \mid \xi \in S_\alpha\}$. Note that any matrix $\xi \in S_\alpha$ can be written as $\xi = \lambda Q(I+\alpha\omega\otimes\omega)$ for some $\lambda>0$, $Q\in SO(n)$ and $\omega\in\mathbf{R}^n$ with $|\omega|=1$. Therefore, it follows easily that $\det\xi=\lambda^n(1+\alpha)$ and $|\xi|=\lambda$ if $|1+\alpha|\le 1$ or $|\xi|=\lambda|1+\alpha|$ if $|1+\alpha|\ge 1$. From this, one easily proves the following

**Proposition 1.3.** *For $\alpha = L^{\frac{1}{n-1}}-1$ or $L^{-1}-1$, $S_\alpha\subset Z_L$; for $\alpha = -L^{\frac{1}{n-1}}-1$ or $-L^{-1}-1$, $JS_\alpha\subset Z_L$.*

In this paper, we shall also prove the following much stronger attainment result.

**Theorem 1.4.** *Let $n\ge 3$, $L\ge 1$ be given. Then, for any $\xi\in\mathbb{M}^{n\times n}$, there exists a map $u$ in the grand Sobolev space $W^{1,\frac{nL}{L+1})}(\Omega;\mathbf{R}^n)$ such that*

$$u|_{\partial\Omega}=\xi x,\qquad Du(x)\in S_{\alpha_1}\cup JS_{\alpha_2}\subset Z_L\quad a.e.\ x\in\Omega,$$

*for $\alpha_1=L^{\frac{1}{n-1}}-1$ or $L^{-1}-1$ and $\alpha_2=-L^{-1}-1$, where the boundary condition $u|_{\partial\Omega}=\xi x$ is satisfied in the $W^{1,p}$-sense for all $1\le p<\frac{nL}{L+1}$.*

Given any subset $K$ of $\mathbb{M}^{m\times n}$ and a number $p\ge 1$, let $C_p^+(K)$ be the set of continuous functions $f\colon\mathbb{M}^{m\times n}\to\mathbf{R}$ satisfying

$$0\le f(\xi)\le C(|\xi|^p+1),\quad f|_K=0.$$

Let $QC_p^+(K)$, $RC_p^+(K)$ and $PC_p^+(K)$ be the class of quasiconvex, rank-one convex, and polyconvex functions in $C_p^+(K)$, respectively.

**Definition 1.1.** The *p-growth quasiconvex, rank-one convex* and *polyconvex hulls* of set $K$ are defined, respectively, as follows:

$$Q_p(K)=\cap\{f^{-1}(0)\mid f\in QC_p^+(K)\};\tag{1.9}$$

$$R_p(K)=\cap\{f^{-1}(0)\mid f\in RC_p^+(K)\};\tag{1.10}$$

$$P_p(K)=\cap\{f^{-1}(0)\mid f\in PC_p^+(K)\}.\tag{1.11}$$

*Remark.* These $p$-growth semiconvex hulls have been referred to as the $W^{1,p}$- or, simply, $p$-semiconvex hulls in Yan [12, 15] and Yan & Zhou [16, 17], following the definition of usual semiconvex hulls without growth restriction given, for example, in Müller & Šverák [10] and Šverák [11]. We adopt the present definition here to stress the growth condition and to distinguish with the different and not equivalent definition of $p$-semiconvex hulls given in Zhang [18, 19]. □

The $p$-growth semiconvex hulls of the set $K_L$ have been studied in Yan [15]. Concerning the set $Z_L$, we have the following

**Theorem 1.5.** *A $p$-growth semiconvex hull of $Z_L$ is the same as that of $K_L$. For example, $R_p(Z_L) = \mathbb{M}^{n\times n}$ for $p < \frac{nL}{L+1}$ and $R_p(Z_L) = K_L$ for $p \geq \frac{nL}{L+1}$; moreover, $Q_p(Z_L) = K_L$ for $p \geq n - \epsilon$ with some $\epsilon > 0$.*

## 2. Relaxations and semiconvex hulls

In this section, we prove Theorem 1.1, parts (a) and (b), and Theorem 1.5. Part (c) of Theorem 1.1 and Theorem 1.2 follow from Theorem 1.4, which will be proved later in the paper.

First of all, we observe the following useful result.

**Lemma 2.1.** $\lim_{|t|\to\infty} f_L(\xi + t\eta) = +\infty$ *for any given $\xi, \eta \in \mathbb{M}^{n\times n}$ with* $\operatorname{rank}\eta = 1$.

*Proof.* It suffices to note that $\det(\xi + t\eta) = \det\xi + ct$ for a constant $c$ if $\operatorname{rank}\eta = 1$, and hence

$$f_L(\xi + t\eta) \geq (|t||\eta| - |\xi|)^n - |c||t| - |\det\xi| \to \infty$$

as $|t| \to \infty$. □

**Proof of Theorem 1.1(a).** Using a well-known relaxation principle for integral functionals (see, e.g., [1, 5]) one can easily show that $g_n = |f_L|^{\#}$ and hence, by definition,

$$0 \leq g_p \leq g_q \leq |f_L|^{\#} \leq |f_L|, \quad \forall\, p \leq q. \tag{2.1}$$

Since $f_L = |\xi|^n - L\det\xi$ is $W^{1,n}$-quasiconvex [4], we easily have $g_n \geq f_L$ and hence $g_n \geq \max\{f_L, 0\}$. (This can be also derived directly from property of the determinant.) Therefore, by (2.1), to prove Theorem 1.1(a), it is sufficient to show

$$|f_L|^{\#} = \max\{f_L, 0\}.$$

But this follows from Lemma 2.1 and the following general result:

**Proposition 2.2.** *Let $f$ be a quasiconvex function. Suppose that for each $\xi \in \mathbb{M}^{n\times n}$ there exists a rank-one matrix $\eta$ such that*

$$\lim_{|t|\to\infty} f(\xi + t\eta) = +\infty.$$

*Then* $|f|^{\#} = \max\{f, 0\}$.

*Proof.* Obviously, since $f$ is quasiconvex, $|f|^\# \geq \max\{f,\, 0\}$. We prove the equality. Suppose, for the contrary, $|f|^\#(\xi_0) > \max\{f(\xi_0),\, 0\}$ for some $\xi_0$. Since $|f|^\#(\xi_0) \leq |f(\xi_0)|$ we must have $f(\xi_0) < 0$. Now consider $h(t) = f(\xi_0 + t\eta_0)$ where $\eta_0$ is a rank-one matrix such that $h(t) \to +\infty$ as $|t| \to \infty$. Then, since $h(0) = f(\xi_0) < 0$, we would have $t_1 < 0 < t_2$ such that $h(t_1) = h(t_2) = 0$. This implies $|f|^\#(\xi_0 + t_i\eta_0) = 0$ for $i = 1, 2$. From this, since $|f|^\#$ is rank-one convex, we would arrive at the conclusion that $|f|^\#(\xi_0) \leq 0$, a contradiction. Therefore, $|f|^\# \equiv \max\{f,\, 0\}$. □

**Proof of Theorem 1.1(b).** Let $f = (\max\{f_L,\, 0\})^{1/n}$. Then $|f_L| \geq |f_L|^\# = f^n$. Using Hölder's inequality in definition (1.3), it follows that, for $1 \leq p < n$,

$$[(f^p)^\#]^{n/p} \leq g_p \leq |f_L|^\# = f^n. \tag{2.2}$$

Since $f$ is a homogeneous function of degree 1 which vanishes exactly on the set $K_L$ and is also $L^n$-mean coercive in the sense defined in Yan & Zhou [17], it follows from [17, Theorem 2.1] that $(f^p)^\#(\xi) = 0$ if and only if $\xi \in K_L$ for $p \geq n - \epsilon$ with some $\epsilon > 0$, and therefore, by (2.2), $g_p(\xi) = 0$ if and only if $\xi \in K_L$. This proves the result.

**Proposition 2.3.** *Let $g \geq 0$ be a rank-one convex function. If $g(\xi) = 0$ for all $\xi \in Z_L$ then $g(\xi) = 0$ for all $\xi \in K_L$.*

*Proof.* Let $\xi \in K_L \setminus Z_L$. Then $f_L(\xi) < 0$. In a similar way as above, we have $f_L(\xi + t_1\eta) = f_L(\xi + t_2\eta) = 0$ for some $t_1 < 0 < t_2$ with a given rank-one matrix $\eta$. Thus $g(\xi + t_i\eta) = 0$ for $i = 1, 2$, which, by the rank-one convexity of $g$, implies $g(\xi) = 0$ and proves the result. □

**Proof of Theorem 1.5.** From definition of semiconvex hulls, to prove the theorem, it is sufficient to prove the equalities:

$$QC_p^+(Z_L) = QC_p^+(K_L), RC_p^+(Z_L) = RC_p^+(K_L), PC_p^+(Z_L) = PC_p^+(K_L).$$

Since the functions in $QC_p^+(Z_L)$, $RC_p^+(Z_L)$, and $PC_p^+(Z_L)$ are all rank-one convex, these equalities follow easily from Proposition 2.3. The proof is completed.

## 3. The lamination hull and attainment results

Given any set $K \subset \mathbb{M}^{m\times n}$ and number $p \in [1, \infty]$, define $\beta_p(K)$ to be the set of matrices $\xi \in \mathbb{M}^{m\times n}$ such that there exists a map

$u = u_\xi \in W^{1,p}(\Omega; \mathbf{R}^m)$ satisfying

$$Du(x) \in K \quad a.e.\ x \in \Omega, \quad u|_{\partial\Omega} = \xi x. \tag{3.1}$$

Note that Lemma 3.1 in Yan [14] shows that the set $\beta_p(K)$ is independent of the domain $\Omega$. From this definition, Theorem 1.2 is equivalent to the following

**Theorem 3.1.** *$\beta_p(Z_L) = \mathbb{M}^{n\times n}$ for all $1 \le p < \frac{nL}{L+1}$.*

From this theorem, a similar argument as used for [14, Theorem 4.4] also derives the following shaper result; we refer to [14] for details.

**Theorem 3.2.** *Let $n \ge 3$, $L \ge 1$ and $1 \le p < \frac{nL}{L+1}$. Then, for any piece-wise affine map $\varphi \in W^{1,p}(\Omega; \mathbf{R}^n)$ and $\epsilon > 0$, there exists a map $u_\epsilon \in \varphi + W_0^{1,p}(\Omega; \mathbf{R}^n)$ satisfying $Du_\epsilon(x) \in Z_L = \partial K_L$ a.e. in $\Omega$ and $\|u_\epsilon - \varphi\|_{L^p(\Omega)} < \epsilon$.*

Let $S_\alpha$ be the set defined by (1.8) in the introduction and define a unbounded two-well set

$$W_L = S_{\alpha_1} \cup J S_{\alpha_2}, \tag{3.2}$$

where $\alpha_2 = -L^{-1} - 1$ and $\alpha_1$ is either $L^{\frac{1}{n-1}} - 1$ or $L^{-1} - 1$.

Notice that Proposition 1.3 implies $W_L \subset Z_L$; therefore, Theorem 3.1 follows from an attainment result for set $W_L$, which is a direct corollary of Theorem 1.4.

**Theorem 3.3.** *$\beta_p(W_L) = \mathbb{M}^{n\times n}$ for all $1 \le p < \frac{nL}{L+1}$.*

To prove Theorem 3.3, or the more general Theorem 1.4, we need some techniques in convex integration theory; we refer to [10, 11, 14] for more references on this theory.

**Definition 3.1.** Let $\mathcal{L}_j(K)$ be defined for $j = 0, 1, 2, \cdots$ inductively as follows: $\mathcal{L}_0(K) = K$ and, for $j = 0, 1, \cdots$,

$$\mathcal{L}_{j+1}(K) = \{t\xi + (1-t)\eta \,\big|\, t \in [0,1],\ \xi, \eta \in \mathcal{L}_j(K),\ \mathrm{rank}(\xi - \eta) \le 1\}.$$

The *lamination hull* of set $K$, $\mathcal{L}(K)$, is then defined to be

$$\mathcal{L}(K) = \cup_{j=0}^{\infty} \mathcal{L}_j(K). \tag{3.3}$$

The following useful attainment result has been given in Yan [14]; see also Yan [13].

**Theorem 3.4** ([14, Theorem 3.2]). *Let $K \subset \mathbb{M}^{m\times n}$ be a closed set and let $A \subset \beta_p(K)$ be a set satisfying*

$$c_0 = \sup_{\xi\in A} \frac{1}{|\Omega|}\int_\Omega |Du_\xi(x)|^p\,dx < \infty, \tag{3.4}$$

*where $u_\xi \in W^{1,p}(\Omega;\mathbf{R}^n)$ is some map satisfying (3.1). Suppose the lamination hull $B = \mathcal{L}(A)$ is open and bounded. Then $B \subset \beta_p(K)$.*

*Remarks.* 1) It follows from [14, Lemma 3.1] and the remark following the proof of [14, Theorem 3.2] that, for any bounded domain $\Sigma \subset \mathbf{R}^n$ and any $\xi \in B = \mathcal{L}(A)$, there exists a map $u = u_\xi \in W^{1,p}(\Sigma;\mathbf{R}^m)$ such that

$$Du(x) \in K \quad a.e.\ x \in \Sigma, \quad u|_{\partial\Sigma} = \xi x \tag{3.5}$$

and

$$\frac{1}{|\Sigma|}\int_\Sigma |Du(x)|^p\,dx \le \max\{c_0, \sup_{\eta\in B}|\eta|^p\}. \tag{3.6}$$

2) A closer look at the proof of [14, Theorem 3.2] also shows that the map $u = u_\xi$ satisfying (3.5), (3.6) above depends only on the family $\{u_\xi \,|\, \xi \in A\}$ in (3.4) and any fixed number $q \ge p$; in particular, the solution $u = u_\xi$ for any $\xi \in B$ can be made independent of power $p$ as long as $p < q$ for some $q < \infty$. This can be seen from the choice of sequence $\{\epsilon_k\}$ in the proof of [14, Theorem 3.2]. We notice that the construction of $u = u_\xi$ depends on power $p$ only through this sequence $\{\epsilon_k\}$ that is required to satisfy $\epsilon_k \to 0^+$ and $\sum_k \epsilon_k^{1/p} < \infty$. However, if $p < q$, we can fix such a sequence $\epsilon_k$ satisfying $\epsilon_k \to 0^+$ and $\sum_k \epsilon_k^{1/q} < \infty$. Then the solution $u = u_\xi$ constructed there is seen only depending on this sequence $\{\epsilon_k\}$ and the family $\{u_\xi \,|\, \xi \in A\}$ given in (3.4). Note that the estimate (3.6) is independent of the sequence $\{\epsilon_k\}$. □

The main theorem of this section is the following

**Theorem 3.5.** *Let $K \subset \mathbb{M}^{m\times n}$ be a closed set and $q > 1$ a number given. Suppose $A \subset \mathbb{M}^{m\times n}$ is a set such that for each $\xi \in A$ there exists a map $v = v_\xi \in W^{1,q)}(\mathbf{B};\mathbf{R}^m)$ satisfying $Dv(x) \in K$ a.e. $x \in \mathbf{B}$ and $v|_{\partial\mathbf{B}} = \xi x$, where $\mathbf{B}$ is the unit ball in $\mathbf{R}^n$, and suppose that*

$$C_q = \sup_{\xi\in A}[Dv_\xi]_{q,\mathbf{B}} < \infty. \tag{3.7}$$

*If the lamination hull $B = \mathcal{L}(A)$ is open and bounded, then, for any bounded domain $\Omega \subset \mathbf{R}^n$ and any $\xi \in B = \mathcal{L}(A)$, there exists a map*

$u = u_\xi \in W^{1,q)}(\Omega; \mathbf{R}^m)$ *satisfying* (3.1) *and, therefore,* $B = \mathcal{L}(A) \subset \beta_p(K)$ *for all* $1 \le p < q$.

*Proof.* Condition (3.7) implies that, for any $1 \le p < q$,

$$c_p = \sup_{\xi \in A} \frac{1}{|\mathbf{B}|} \int_{\mathbf{B}} |Dv_\xi(x)|^p \, dx \le \frac{C_q^p}{q - p} < \infty. \tag{3.8}$$

From this, using Theorem 3.4 quoted above and its remarks, we obtain, for each given $\xi \in B = \mathcal{L}(A)$, a map $u = u_\xi$ depending only on the power $q$ and the family $\{v_\xi \,|\, \xi \in A\}$ in (3.7) that belongs to $W^{1,p}(\Omega; \mathbf{R}^n)$ for all $1 \le p < q$ and satisfies (3.1) and

$$\frac{1}{|\Omega|} \int_\Omega |Du|^p \, dx \le \max\{c_p, \sup_{\eta \in B} |\eta|^p\}.$$

Multiplying this estimate by $q - p$, taking the $1/p$ power, and maximizing over $1 \le p < q$ show that

$$[Du]_{q,\Omega} \le \max\{C_q, \gamma_q \sup_{\eta \in B} |\eta|\} < \infty,$$

where $\gamma_q = \sup_{1 \le p \le q} (q - p)^{1/p} < \infty$ is a constant. This shows $u = u_\xi \in W^{1,q)}(\Omega; \mathbf{R}^n)$ satisfying (3.1) and completes the proof. $\square$

## 4. Proof of Theorem 1.4

In this final section, we apply Theorem 3.5 to prove the main attainment theorem, Theorem 1.4. To do so, as in Yan [13, 14], we further introduce certain subsets in $\mathbb{M}^{n \times n}$. Let $W_L$ be the set defined by (3.2). Given a number $\lambda > 0$, define

$$R(n) = \{\xi \in \mathbb{M}^{n \times n} \,|\, |\xi|^n = |\det \xi|\}; \tag{4.1}$$

$$A_\lambda = \{\xi \in R(n) \,|\, |\xi| < \lambda\}, \quad \tilde{A}_\lambda = A_\lambda \setminus \{0\}; \tag{4.2}$$

$$B_\lambda = \{\xi \in \mathbb{M}^{n \times n} \,|\, |\xi| < \lambda\}. \tag{4.3}$$

**Proposition 4.1.** $\mathcal{L}_n(\tilde{A}_\lambda) = B_\lambda$. *Therefore* $B_\lambda = \mathcal{L}(\tilde{A}_\lambda)$ *is a bounded open set.*

*Proof.* Let $\xi \in B_\lambda$. If $\xi \ne 0$ then the proof of Proposition 4.1 in [14] has essentially shown that $\xi \in \mathcal{L}_{n-1}(\tilde{A}_\lambda)$. Therefore one has only to show $0 \in \mathcal{L}_n(\tilde{A}_\lambda)$. To prove this, let $0 < t < \lambda$ be fixed. Note that, since the diagonal matrices $\delta(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ and $\delta(-\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$, where $\epsilon_i =$

$\pm t$, are in $\tilde{A}_\lambda$ and their difference is rank-one, by Definition 3.1, we have

$$\delta(0, \epsilon_2, \cdots, \epsilon_n) = \frac{\delta(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) + \delta(-\epsilon_1, \epsilon_2, \cdots, \epsilon_n)}{2}$$

belongs to $\mathcal{L}_1(\tilde{A}_\lambda)$ for all $\epsilon_i = \pm t$. Now that matrices $\delta(0, \epsilon_2, \cdots, \epsilon_n)$ and $\delta(0, -\epsilon_2, \cdots, \epsilon_n)$ are in $\mathcal{L}_1(\tilde{A}_\lambda)$ and differ by rank-one, hence

$$\delta(0, 0, \epsilon_3, \cdots, \epsilon_n) = \frac{\delta(0, \epsilon_2, \epsilon_3, \cdots, \epsilon_n) + \delta(0, -\epsilon_2, \epsilon_3, \cdots, \epsilon_n)}{2}$$

is in $\mathcal{L}_2(\tilde{A}_\lambda)$ for all $\epsilon_i = \pm t$. Repeating this argument finitely many steps, we eventually arrive at $0 = \delta(0, 0, \cdots, 0) \in \mathcal{L}_n(\tilde{A}_\lambda)$. □

**Proposition 4.2.** *Let $\lambda > 0$, $n \geq 3$, $L \geq 1$ and $q = \frac{nL}{L+1}$. Then, for each $\xi \in \tilde{A}_\lambda$, there exists a map $v = v_\xi \in W^{1,q)}(\mathbf{B}; \mathbf{R}^n)$ satisfying $Dv(x) \in W_L$ a.e. $x \in \mathbf{B}$ and $v|_{\partial\mathbf{B}} = \xi x$, such that*

$$c_0 \equiv \sup_{\xi \in \tilde{A}_\lambda} [Dv_\xi]_{q,\mathbf{B}} \leq C(n, L)\lambda < \infty. \tag{4.4}$$

*Proof.* Let $\xi \in \tilde{A}_\lambda$ be given. We look for map $v = v_\xi$ in the form of radial maps: $v(x) = |x|^\alpha \xi x$, where $\alpha$ is a number to be chosen later. Note that all such maps satisfy $v|_{\partial\mathbf{B}} = \xi x$ in the $W^{1,p}$-sense if $v \in W^{1,p}(\mathbf{B}; \mathbf{R}^n)$ for some $p \geq 1$. Let $r = |x|$, $\omega = r^{-1}x$; then a simple calculation shows that

$$Dv(x) = r^\alpha \xi (I + \alpha\, \omega \otimes \omega), \tag{4.5}$$

$$\det Dv(x) = (1 + \alpha) r^{\alpha n} \det \xi, \tag{4.6}$$

$$|Dv(x)| = \begin{cases} r^\alpha |\xi|, & |1 + \alpha| \leq 1, \\ |1 + \alpha| r^\alpha |\xi|, & |1 + \alpha| > 1. \end{cases} \tag{4.7}$$

If $\det \xi > 0$, we choose $\alpha = \alpha_1$ to be one of the two numbers defined above, and then one easily sees that $Dv(x) \in S_{\alpha_1}$ for all $x \neq 0$ in $\mathbf{B}$ and, a computation also shows that this function $v = v_\xi$ belongs to $W^{1,q)}(\mathbf{B}; \mathbf{R}^n)$ and satisfies

$$[Dv_\xi]_{q,\mathbf{B}} \leq C_1(n, L)|\xi|. \tag{4.8}$$

If $\det \xi < 0$, we let $\alpha = \alpha_2 = -L^{-1} - 1$ and in this case one easily sees that $Dv_\xi(x) \in JS_{\alpha_2}$ for all $x \neq 0$. Furthermore, we compute to get

$$\frac{1}{|\mathbf{B}|} \int_{\mathbf{B}} |Dv_\xi(x)|^p \, dx = |\xi|^p \int_0^1 r^{\alpha_2 p + n - 1} \, dr = \frac{|\xi|^p}{\alpha_2 p + n} = \frac{L}{L+1} \frac{|\xi|^p}{q - p},$$

which shows that $v = v_\xi$ belongs to $W^{1,q)}(\mathbf{B};\mathbf{R}^n)$ and satisfies

$$[Dv_\xi]_{q,\mathbf{B}} \le C_2(n,L)|\xi|. \tag{4.9}$$

Combining (4.8), (4.9), one proves (4.4). The proof is completed. □

**Proof of Theorem 1.4.** Let $\xi \in \mathbb{M}^{n\times n}$ be given. Let $\lambda > 0$ be a number such that $|\xi| < \lambda$. Then $\xi \in B_\lambda$. From Proposition 4.1, $\mathcal{L}(\tilde{A}_\lambda) = B_\lambda$ is open and bounded. Also, from Proposition 4.2, the set $A = \tilde{A}_\lambda$ satisfies the condition (3.7) in Theorem 3.5 with $K = W_L$ and $q = \frac{nL}{L+1}$. Therefore, Theorem 3.5 implies that, there exists a map $u = u_\xi \in W^{1,q)}(\Omega;\mathbf{R}^n)$ satisfying $Du(x) \in W_L$ for a.e. $x \in \Omega$ and $u|_{\partial\Omega} = \xi x$. Note also that $[Du]_{q,\Omega} \le C_0\,\lambda < \infty$. We complete the proof.

## References


[1] E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations,* Arch. Rational Mech. Anal., **86** (1984), 125–145.

[2] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity,* Arch. Rational Mech. Anal., **63** (1977), 337–403.

[3] J. M. Ball and R. D. James, *Proposed experimental tests of a theory of fine microstructures and the two well problem,* Phil. Trans. Roy. Soc. London, **338A** (1992), 389–450.

[4] J. M. Ball and F. Murat, *$W^{1,p}$-Quasiconvexity and variational problems for multiple integrals,* J. Funct. Anal., **58** (1984), 225–253.

[5] B. Dacorogna, "Direct Methods in the Calculus of Variations." Springer-Verlag; Berlin, 1989.

[6] T. Iwaniec, *An approach to Cauchy-Riemann operators in $\mathbf{R}^n$,* preprint.

[7] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses,* Arch. Rational Mech. Anal., **119** (1992), 129–143.

[8] C. B. Morrey, "Multiple Integrals in the Calculus of Variations." Springer-Verlag; Berlin, 1966.

[9] S. Müller, *Variational models for microstructure and phase transitions,* in "Calculus of Variations and Geometric Evolution Problems," (Cetraro, 1996), pp. 85–210, Lecture Notes in Math. 1713, Springer, 1999.

[10] S. Müller and V. Šverák, *Attainment results for the two-well problem by convex integration,* Geometric analysis and the calculus of variations, 239–251, Internat. Press; Cambridge, MA, 1996.

[11] V. Šverák, *On the problem of two wells,* in "Microstructure and Phase Transition," pp. 183–190, (D. Kinderlehrer *et al* eds.), Springer-Verlag; New York, 1993.

[12] B. Yan, *On rank-one convex and polyconvex conformal energy functions with slow growth,* Proc. Roy. Soc. Edinburgh, **127A** (1997), 651–663.

[13] B. Yan, *Linear boundary values of weakly quasiregular mappings,* C. R. Acad. Sci. Paris Ser. I Math., **331** (2000), 379–384.

[14] B. Yan, *A linear boundary value problem for weakly quasiregular mappings,* Cal. Var. Partial Differential Equations, to appear.

[15] B. Yan, *Semiconvex hulls of quasiconformal sets,* J. Convex Analysis, to appear.

[16] B. Yan and Z. Zhou, *Stability of weakly almost conformal mappings,* Proc. Amer. Math. Soc., **126** (1998), 481–489.

[17] B. Yan and Z. Zhou, *$L^p$-mean coercivity, regularity and relaxation in the calculus of variations,* Nonlinear Analysis, to appear.

[18] K. Zhang, *On various semiconvex hulls in the calculus of variations,* Cal. Var. Partial Differential Equations, **6** (1998), 143–160.

[19] K. Zhang, *Rank-one connections at infinity and quasiconvex hulls,* J. Convex Analysis, **7** (2000), 19–45.

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