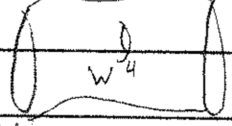
Matt Hedden — Heegaard Floer Homology — 9/7/10

C205 or A517 From now on!

Last Time:

Structure of invariants

$$(Y^3, \mathfrak{s}) \rightsquigarrow HF^\circ(Y, \mathfrak{s}) \begin{cases} \widehat{HF}(Y, \mathfrak{s}) \\ HF^-(Y, \mathfrak{s}) \\ HF^+(Y, \mathfrak{s}) \\ HF^\infty(Y, \mathfrak{s}) \end{cases}$$



$-Y_1$ $\quad$ Y_2

$$\rightsquigarrow F^\circ_{W,\mathfrak{t}} : HF^\circ(Y_1, \mathfrak{t}|_{Y_1}) \to HF^\circ(Y_2, \mathfrak{t}|_{Y_2})$$

$$K \subseteq Y^3 \rightsquigarrow \bigoplus_{i \in \mathbb{Z}} \widehat{HFK}(Y, K, i)$$

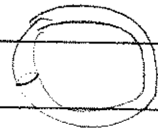
Application

Thm. (O-S) $K \in S^3$, $g(K) = \max\{i \in \mathbb{Z} : \widehat{HFK}(S^3, K, i) \neq 0\}$

Def. Call a knot $K \subseteq S^3$ fibered if $S^3 - n(K)$ is a fiber bundle over S^1, with 2-dim'l fiber

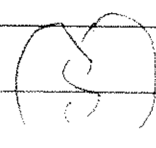
$$\begin{array}{ccc} \Sigma^2 & \hookrightarrow & S^3 - n(K) \\ & & \downarrow \pi \\ & & S^1 \end{array}$$

Ex: Unknot



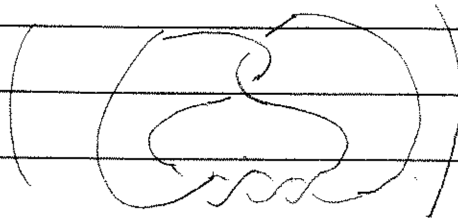
$$\begin{array}{ccc} D^2 & \hookrightarrow & S^3 - n(U) \cong D^2 \times S^1 \\ & & \downarrow \\ & & S^1 \end{array}$$

Exercise: Show that the trefoil is fibered.



Prop: If K is fibered, then $\Delta_K(T)$ is monic.
($\Delta_K(T)$: Alexander polynomial)

i.e. the highest non-vanishing power of T has coefficient ± 1.

Ex: $\Delta\Big($  $\Big) = -2T + 5 - 2T^{-1}$.

so the knot is not fibered.

Thm. (O-S) If K is fibered, then $\operatorname{rank}(\widehat{HFK}(K, g(K))) = 1$

(Ni, Ghiggini, Juhász) $\operatorname{rank}(\widehat{HFK}(K, g(K))) = 1 \Rightarrow K$ is fibered.

Thurston Norm

Y^3 closed, orientable 3-mfld, $\xi \in H_2(Y^3; \mathbb{Z})$.

Def. ξ is realized by an embedded surface $\Sigma \overset{i}{\hookrightarrow} Y$ if

$$i_*([\Sigma]) = \xi \in H_2(Y^3; \mathbb{Z}).$$

Exercise: Show that any $\xi \in H_2(Y)$ is realized by an embedded surface.

Def. Let $\Sigma \hookrightarrow Y^3$, $\quad \chi_-(\Sigma) = \sum_{\substack{\text{components} \\ \Sigma_i \text{ of } \Sigma}} \max\{-\chi(\Sigma_i), 0\}$, $\quad -\chi(\Sigma_i) = 2g(\Sigma_i) - 2$

Def $\Theta : H_2(Y; \mathbb{Z}) \longrightarrow \mathbb{Z}^{\geq 0}$, the Thurston semi-norm, is

$$\Theta(\xi) = \min_{\Sigma \hookrightarrow Y} \{\chi_-(\Sigma) : \Sigma \text{ realizes } \xi\}.$$

$\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is an oriented 2-dim'l bundle.

$\rotatebox{90}{$\leadsto$}$

$c_1(\mathfrak{s}) \in H^2(Y)$ 1st Chern class of $\mathfrak{s}$.

Thm. (O-S) $\xi \in H_2(Y)$.

$$\Theta(\xi) = \max \left\{ \langle c_1(s), \xi \rangle \;\middle|\; \begin{array}{l} s \in Spin^c(Y) \\ \widehat{HF}(Y, s) \neq 0 \end{array} \right\}$$

Gist: Floer homology solves the minimal genus problem for 3-mnflds (possibly with torus boundary)

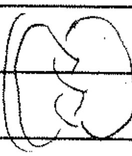
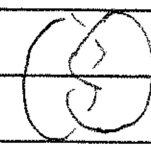
What about dim 4?

Def. $$g_4(K) = \min_{\Sigma \hookrightarrow B^4} \left\{ \text{genus}(\Sigma) : \begin{array}{l} i : \Sigma \hookrightarrow B^4, \text{ smoothly embedded} \\ i|_{\partial \Sigma} = K \end{array} \right\}$$

is called the smooth 4-ball genus, or the slice genus.

Exercise: $g_4(K) \leq g(K)$.

Def. $u(K)$ = min. # crossings necessary to change in any projection of K to unknot it.

Ex: u () $= 1$, because  is a projection of the unknot.

Exercise: $g_4(K) \leq u(K)$

Hint: The unknotting transformation gives a 'movie'. Use this to construct a surface in B^4.

Floer homology

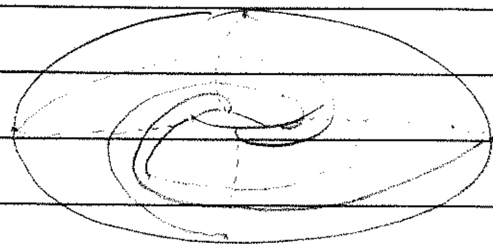
$$K \rightsquigarrow \widehat{HFK}(K)$$
$$\rightsquigarrow$$
$$\tau(K) \in \mathbb{Z}$$

Thm. (O-S, Rasmussen)

$$|\tau(K)| \leq g_4(K).$$

(standardly embedded)

Def. A torus knot $T_{p,q}$ is a knot which embeds in a (standardly embedded) torus as a curve of slope p/q.



$T_{3,4}$.

Ex:

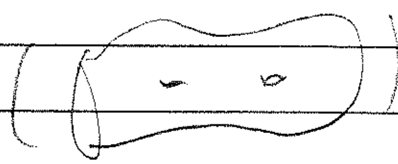
$$\underbrace{\{(z,w)\in\mathbb{C}^2 \mid z^p + w^q = 0\}}_{= V_{p,q}} \cap \underbrace{\{(z,w)\in\mathbb{C}^2 \mid |z|^2+|w|^2 = 1\}}_{\cong S^3} = K_{p,q}$$

Exercise: • Show $K_{p,q}$ is a knot (if $(p,q)=1$).

• Show $K_{p,q}$ is a torus knot $T_{p,q}$.

• Show that $\chi(V_{p,q} \cap \{|z|^2+|w|^2 \leq 1\}) = pq - p - q$.

Hint: Riemann-Hurwitz formula.

$\chi($  $) = 1 - 2g(V_{p,q}) = pq - p - q$

$$\Rightarrow g(V_{p,q}) = \frac{(p-1)(q-1)}{2}$$

• Unknot $T_{p,q}$ with $\frac{(p-1)(q-1)}{2}$ crossing changes

Milnor Conjecture: $g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2} = u(T_{p,q})$.

Thm. (Kronheimer - Mrowka, O-S-R)

$$\tau(T_{p,q}) = \frac{(p-1)(q-1)}{2} \leq \underbrace{g_4(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}}_{\text{By Exercise}}$$

Cor. (Hard)

Milnor Conjecture $\Rightarrow$ $\exists$ a 4-mfld X homeomorphic to $\mathbb{R}^4$, but not diffeomorphic to $\mathbb{R}^4$.

(X is an Exotic or Fake $\mathbb{R}^4$)

Dehn Surgery

Defn A lens space $L(p,q)$ is the mfld. obtained by $-p/q$ on the unknot.

(Note: O-S choose the opposite orientation here?).

Q: Which knots can I do Dehn surgery on and obtain a lens space?

Thm. Suppose $K \subseteq S^3$ admits a positive lens space surgery. (i.e. $S^3_{p/q}(K) = L(r,s)$ for $\frac{p}{q} > 0$, $(r,s)=1$)

Then,
- (O-S) All coefficients of $\Delta_K(T)$ are ± 1 or 0.
- (O-S) $g(K) = g_4(K)$.
- (Ni, Ghiggini, Juhasz) K is fibered.
- (Hedden) K bounds an algebraic curve $V_f \subseteq B^4$.

(At least the last 3 have no known proofs that do not involve HFH technology)

$\widehat{HFK}$ could possibly give a classification of such knots.

- (Greene) Determined which $L(p,q)$ you get by doing surgery on $K \neq$ unknot.

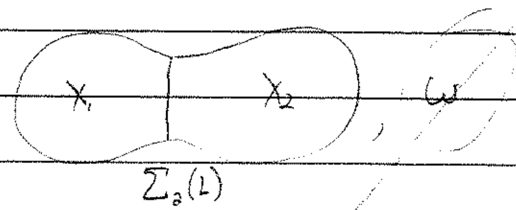
Thm. (O-S) Let $L \subseteq S^3$ be an alternating link, and let $\Sigma_2(L)$ be its branched double cover.

Then, $\Sigma_2(L) \not\hookrightarrow (X^4, \omega)$ s.t.

symplectic

$$X^4 - \Sigma_2(L) \cong X_1^4 \sqcup X_2^4$$

with $b_2^+(X_i) > 0$.



e.g. lens spaces

Also: Applications to
- Foliation Theory
- Contact geometry
- Concordance & homology cobordism

Ways to turn knots into groups

Matt Hedden HFH 9/9/10

Plan: (1) Morse homology

(2) Lagrangian Floer homology

- Milnor "Morse Theory" p. 1-27, 32-38
- Gompf - Stipsicz "4-mnflds & Kirby Calculus" p. 69-82
- Hutchings (Michael) math.berkeley.edu/~hutchings ~~Morse~~ Lectures on Morse homology p. 1-3
- McDuff "Floer Theory & low dim'l topology"

Recall: A smooth function $f: M \to \mathbb{R}$ on a mnfld M is Morse if critical pts. are isolated and have a local form:

$$f = -x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 .$$

k is called the index of the critical pt.

If $f^{-1}([a,b])$ doesn't contain a critical pt., then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeo, and $f^{-1}((-\infty,a])$ and $f^{-1}((-\infty,b])$ are diffeo.

where $\Phi: M \times \mathbb{R} \to M$ is the flow of $-\nabla f$

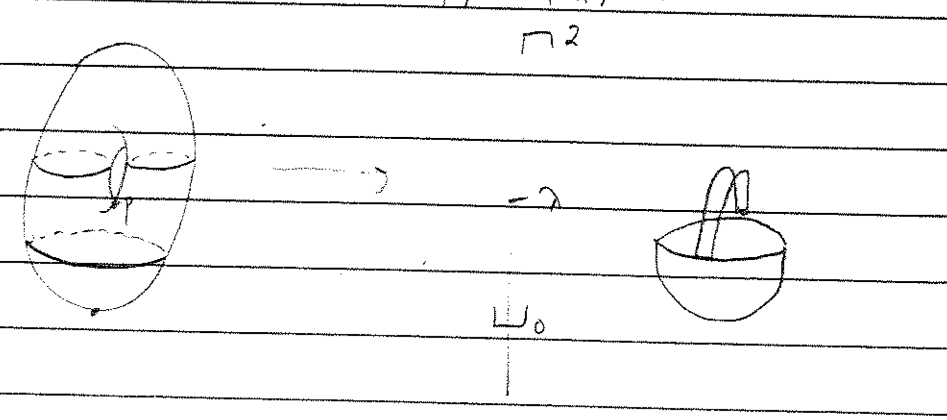
i.e. the vector field s.t. $g(-\nabla f, -) = -df$

g is the Riem. metric

Thm. If λ is a critical value of f, then $f^{-1}((-\infty, \lambda-\varepsilon]) \cup (\underbrace{D^k \times D^{n-k}}_{k\text{-handle}}) = f^{-1}((-\infty, \lambda+\varepsilon])$,

where $n = \dim(\text{mnfld})$, $k =$ index of f at p, $f(p) = \lambda$.

Ex:



Morse homology

$$\mathrm{Crit}(f) = \{\text{critical pts. of } f\}, \quad p \in \mathrm{Crit}(f).$$

$$W^u(p) = \{x \in M \mid \lim_{t\to-\infty} \Phi(x,t) = p\}$$

$$W^s(p) = \{x \in M \mid \lim_{t\to\infty} \Phi(x,t) = p\}$$

Def. Given $p, q \in \mathrm{Crit}(f)$ call a curve $\gamma(t)$ a *gradient flow connecting p to q* if $\lim_{t\to-\infty} \gamma(t) = p$; $\lim_{t\to\infty}\gamma(t) = q$ and

$$\left.\frac{d\gamma}{dt}\right|_{t'} = -\nabla f(\gamma(t')).$$

Lemma Gradient flows connecting p to q $\longleftrightarrow$ $W^u(p) \cap W^s(q)$.

Prop. $W^u(p)$ is a smooth submfld. of M diffeomorphic to an open ball of dim $\mathrm{ind}(p)$,
and $W^s(p)$ " " " " " " " " " " " dim $\dim(M) - \mathrm{ind}(p)$

Note: (1) If $W^u(p) \pitchfork W^s(q)$, then $W^u(p)\cap W^s(q)$ is a smooth mfld. of dim. $\mathrm{ind}(p) - \mathrm{ind}(q)$.

(2) If $\gamma(t)$ is a gradient flow from p to q, then $\gamma_c(t) = \gamma(t+c)$ is also a gradient flow conn. p to q.

$$W^u(p) \cap W^s(q) \times \mathbb{R} \longrightarrow W^u(p)\cap W^s(q)$$

$$(\gamma(t), c) \longmapsto (\gamma_c(t))$$

is a free $\mathbb{R}$-action provided $\gamma(t) \neq$ constant.

$$\mathcal{M}(p,q) := W^u(p)\cap W^s(q) / \mathbb{R}$$

"Moduli space of gradient flows"

" set of unparameterized gradient flows conn. p to q."

manifold of dim. $\mathrm{ind}(p) - \mathrm{ind}(q) - 1$.

Def. $\mathrm{Crit}_i(f) = \{\text{index } i \text{ critical pts.}\}$

$$C_*(f, g) := \bigoplus_{p\in \mathrm{Crit}_*(f)} \mathbb{Z}/2\langle p\rangle$$

g metric

$\partial : C_* \to C_{*-1}$ def. by $\partial(p) = \sum_{q\in \mathrm{Crit}_{i-1}} \#\mathcal{M}(p,q)\, q$, $p \in \mathrm{Crit}_i(f)$

$\#$ pts. in $\mathcal{M}(p,q)$ mod 2, i.e. $\#$ of unpar. gradient flows connecting p to q.

Thm. $\partial^2 = 0$.

Pf. $$\partial \circ \partial (p) = \partial \left(\sum_{q \in Crit_{i-1}} \# \mathcal{M}(p,q) \cdot q \right)$$

$$= \sum_{r \in Crit_{i-2}} \left(\sum_{q \in Crit_{i-1}} \# \mathcal{M}(p,q) \cdot q \right) \# \mathcal{M}(q,r) \cdot r$$

But,

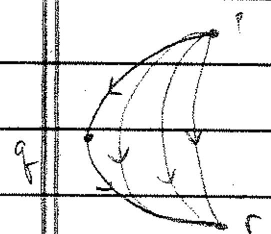
(*) $$0 = \sum_{q \in Crit_{i-1}} \# \mathcal{M}(p,q) \cdot \# \mathcal{M}(q,r) \quad \forall r \in Crit_{i-2}.$$

$\square$

Why is (*) True?

$\dim \mathcal{M}(p,r) = 1$, i.e. $\mathcal{M}(p,r)$ is a 1-dim'l mnfld.

$\mathcal{M}(p,r)$ has a natural compactification obtained by adding "broken gradient flow lines" from p to r.



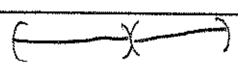
A broken flow line is a pair of flow lines connecting p to q and q to r respectively.

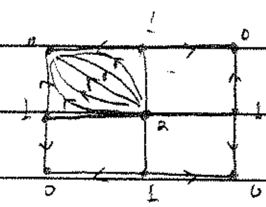
$\#$ boundary pts. of $\overline{\mathcal{M}(p,r)} = 0 \bmod 2$,

and these boundary pts. correspond to broken flow lines, which are exactly counted by the product $\# \mathcal{M}(p,q) \# \mathcal{M}(q,r)$.

- $\mathcal{M}(p,r)$ is compactified by broken flow lines "Compactness"
- Every broken flow line is added under this compactification. "Gluing"

HW#2 post online!

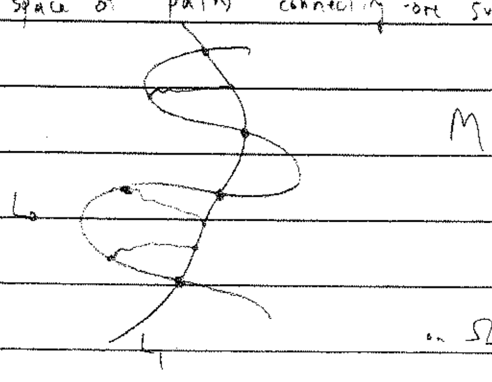
 ??



$$H_*(T^2) \cong \begin{cases} \mathbb{Z}/2 & *=2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=0 \end{cases}$$

Lagrangian Floer Homology

Want to do the same thing on an infinite dim'l space, $\Omega(L_0, L_1)$, a space of paths connecting one submanifold L_0 to another L_1.



Motivation: Get bounds for $\#$ geometric intersections of submanifolds of complimentary dimension.

Possible if we have a Morse homology for a function on $\Omega(L_0, L_1)$ whose critical pts. are the constant paths.

Context: Symplectic geometry.

Let (M^{2n}, ω) be a symplectic mnfld.

ω is a closed 2-form $\left(d\omega = 0 \;,\; \omega \in \Omega^2(M) \right)$

Nondegenerate $\underbrace{\omega \wedge \cdots \wedge \omega}_{n} > 0$

Def. If $L_0^n, L_1^n \subseteq M^{2n}$ are half-dim'l submnflds s.t. $\omega|_{L_i} \equiv 0$.

then we say L_0^n, L_1^n are Lagrangian submanifolds

Def. Let $C(M; L_0, L_1) := \bigoplus_{L_0 \cap L_1} \mathbb{Z}/2 \langle \bar{x} \rangle$

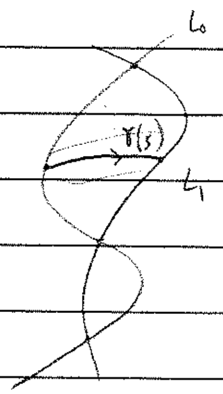
What is the Morse function? There isn't one. $\ddot{\frown}$ But we don't need a full Morse func.

Recall, $g(-\nabla f, -) = df$ ← 1-form, and critical pts. are (by Def.) 0's of df.

So maybe it suffices to have a 1-form, and some notion of a metric.

Def. $\alpha : T\Omega(L_0, L_1) \to \mathbb{R}$

$$\alpha_{\gamma(s)}(\xi(s)) = \int_0^1 \omega(\dot{\gamma}(s), \xi(s))\, ds$$



$u(s,t) : \underset{s}{[0,1]} \times [-1,1] \to M$

s.t. $u(s,0) = \gamma(s)$, $u(0,t) \in L_0$, $u(1,t) \in L_1$.

$$\left.\frac{\partial u}{\partial t}(s,t)\right|_{(s,0)}$$

$$T\Omega(L_0, L_1) = \{\text{paths}, \xi(s) \in T_{\gamma(s)} M\}$$

Exercise: Show that $\alpha_\gamma = 0 \iff \gamma(s)$ constant $\iff \gamma(s) \in L_0 \cap L_1$.

Matt Hedden HFH (class) 9/14/10

Last Time:

Morse homology

M^n, f Morse function, g $\rightsquigarrow$ $C(f) = \bigoplus_{\substack{x\in M \text{ s.t.} \\ df_x = 0}} \mathbb{Z}/2\langle x\rangle$

∂ counts gradient flows connecting x to y if $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$.

Lagrangian Floer homology

(M^{2n}, ω) $\rightsquigarrow$ $\Omega(L_0, L_1)$

$\cup\ \cup$
$L_0^n\ L_1^n$

$\alpha: T\Omega \to \mathbb{R}$

$\alpha_{\gamma(s)}(\xi(s)) = \int_0^1 -\omega\left(\frac{d\gamma}{ds}(s'), \xi(s')\right) ds'.$

$$C(L_0, L_1) = \bigoplus_{x\in L_0\cap L_1} \mathbb{Z}/2\langle x\rangle$$

What does ∂ count?

Recall: $g(-\nabla f, -) = df$

Metric on $T\Omega$:

$$g_\gamma^\Omega(\xi(s'), \eta(s')) = ?$$

Def. An almost complex structure (a.c.s.) is a bundle automorphism

$J: TM^{2n} \to TM^{2n}$ s.t. $J\circ J = -Id$.

Def. Call J an a.c.s. on (M,ω) compatible with ω if

- $\omega(Jv, v) > 0$ if $v \neq 0$.
- $\omega(Jv, Jw) = \omega(v,w)$ $\forall v,w$.

Proposition (See McDuff-Salamon Intro Book)

Compatible almost complex structures exist for any symplectic mfld., & the space of compatible a.c.s. for a given form is contractible.

$$g_\gamma^\Omega(\xi,\eta) := \int_0^1 \omega_{\gamma(s')}(J\xi(s'),\eta(s'))\,ds'$$

$$g^\Omega_{\gamma(s)}(-\mathrm{grad}_\gamma, \xi) = \alpha_\gamma(\xi)$$

Trying to find what this gradient vector field is,

This equation evaluates to

$$\int_0^1 \omega_{\gamma(s')}\left(J(-\mathrm{grad}_{\gamma(s')}), \xi(s')\right) ds' = \int_0^1 \omega_{\gamma(s')}\left(\frac{d\gamma}{ds}(s'), \xi(s')\right) ds'$$

$$\Leftrightarrow \quad \frac{d\gamma}{ds}(s') = J(-\mathrm{grad}_\gamma(s'))$$

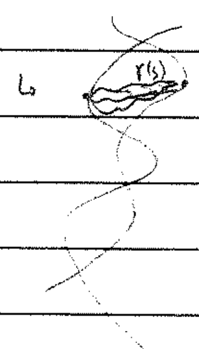
Applying J to both sides:

$$J\left(\frac{d\gamma}{ds}(s')\right) = \mathrm{grad}_\gamma(s')$$

Note: $J_{\gamma(s')}$ actually changes w/ pt in the manifold as well.

Want ∂ to count gradient flows connecting critical pts. x to y $(x,y \in L_0 \cap L_1)$.

$$u(s,t) : \underset{s}{[0,1]} \times \underset{t}{\mathbb{R}}$$

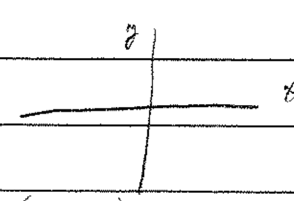


$$\frac{\partial u}{\partial t}(s',t') = -\mathrm{grad}_{u(s',t')} = -J_{u(s',t')} \frac{du}{ds}(s',t')$$

Apply J again:

$$J\left(\frac{\partial u}{\partial t}(s',t')\right) = \frac{du}{ds}(s',t') \qquad (*)$$

$(*)$ is the J-holomorphic curve Equation



Exercise: $(*) \Leftrightarrow J \circ du = du \circ i \quad (**)$

Where i is the (almost) complex structure on $[0,1] \times \mathbb{R} \subseteq (\mathbb{C}, i)$.

Note: $(**)$ means that the map u is a complex map on the tangent bundle

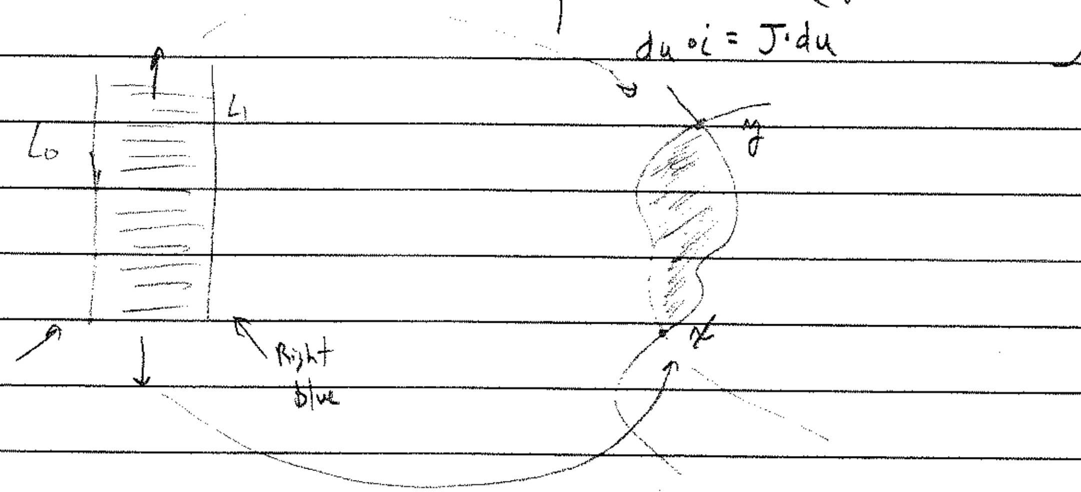
Exercise: Show (*) is equivalent to the Cauchy-Riemann equations when ($M^{2n} = \mathbb{C}$, $J = i$).

$\omega = dx \wedge dy$

Def. $\mathcal{M}(x,y) := \left\{ u : [0,1] \times \mathbb{R} \to M^{2n} \;\middle|\; \begin{array}{l} u(0,t) \in L_0 \\ u(1,t) \in L_1 \\ \lim_{t \to \square} u(s,t) = \begin{cases} x & \text{if } \square = -\infty \\ y & \text{if } \square = \infty \end{cases} \\ du \circ i = J \cdot du \end{array} \right\}$

$x, y \in L_0 \cap L_1$

Convention:



This is the

- Space of gradient trajectories (flow lines)
- Space of J-holomorphic disks (strips) connecting x to y.
  (pseudo-holomorphic)

Note: "Disk" is justified because $[0,1] \times \mathbb{R}$

$\cong$ ← conformally equivalent by Riemann mapping Thm.

Define $$\partial x = \sum_{y \in L_0 \cap L_1} \underbrace{\# \widehat{\mathcal{M}}(x,y)}_{\substack{\#\text{ of J-holomorphic} \\ \text{disks connecting} \\ x \text{ to } y \text{ (mod 2).}}} \cdot y$$

Why is this number $\# \widehat{\mathcal{M}}(x,y)$ well-defined?

For Morse theory, $\# \mathcal{M}(x,y)$ was well-defined provided $\mathrm{ind}(x) - \mathrm{ind}(y) = 1$.

What plays the role of index? Something called Maslov index.

Is $\mathcal{M}(x,y)$ a manifold (of any dimension)?

We need some sort of transversality assumption to ensure $\mathcal{M}(x,y)$ is a mnfld.

For Morse theory, we could do this because of Sard's thm.

There is an infinite dim'l. version of Sard's theorem which allows this.

(Nothing more will be said about this).

Is $\#\mathcal{M}(x,y)$ finite when this Maslov index, plus transversality, tells us we have a smooth mnfld of dimension 0 ? (We don't have compactness)

But notice: $\frac{du(s,t)}{dt} = -J\frac{du}{ds}(s,t)$ is invariant under $u(s,t) \mapsto u(s,t+c) =: u_c(s,t)$

So $\widehat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$

Finiteness is ensured, one hopes, by a Compactness Theorem for $\mathcal{M}(x,y)$.

"Gromov compactness"

$\partial^2 = 0$? This will hold because of Gromov Compactness + a "Gluing Theorem"

Thm. (Floer)

Let (M^{2n}, L_0, L_1) be a symplectic mnfld. and two Lagrangian submnflds.

This is to accommodate the fact that Gromov compactness is not actually compactness.

s.t. (1) $L_0 \pitchfork L_1$

(2) M is compact

(3) $\pi_2(M^{2n}) = \pi_2(M, L_0) = \pi_2(M, L_1) = 0$.

Then $\partial^2 = 0$ and $H_*(C(L_0, L_1), \partial)$ depends only on M up to symplectomorphism and L_i up to hamiltonian isotopy

How can we use this construction to study 3-mnflds. ?

Could try $(Y^3 \times \mathbb{R}, \omega)$ (To get an even-dim'l, symplectic mnfld.)

This is called Embedded Contact homology. (ECH)

Hutchings - Taubes

Amazingly, this gives an invariant of 3-mnflds isomorphic to our course.

Heegaard Floer Homology (Ozsváth-Szabó Floer homology)

What is a 3-mnfld, anyway?

To "see" them, we'll use a device called Heegaard diagrams.

Def. A Heegaard diagram is a 3-tuple $(\Sigma, \vec{\alpha}, \vec{\beta})$ where

(1) Σ is a closed, oriented surface of genus g

(2) $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ is a collection of g s.c.c. in Σ, pairwise disjoint, and span$\{[\vec{\alpha}]\} \subseteq H_1(\Sigma; \mathbb{R}) \cong \mathbb{R}^{2g}$ is g-dim'l.

(3) $\vec{\beta} = \{\beta_1, \ldots, \beta_g\}$

Prop. A Heegaard diagram specifies a unique (up to homeomorphism) closed, orientable 3mnfld.

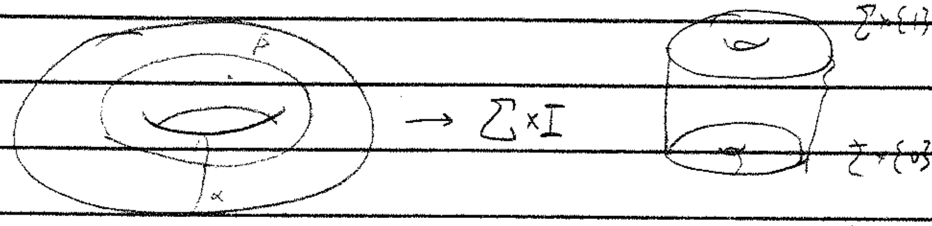
- Need Heegaard moves to go between 2 pictures for same 3-mnfld.

9/16/10 Matt Hedden HFH

Recall the Def. of a Heegaard diagram. We will also require $\vec{\alpha} \pitchfork \vec{\beta}$.

Prop. Any HD specifies a unique, oriented homeomorphism class of closed 3-mnflds, Y.

Pf.



Attach g 2-handles to $\Sigma \times \{0\}$, one for each α curve.

Note: Each α curve has a nbhd. homeomorphic to an annulus $S^1 \times I \cong nbhd(\alpha) \subseteq \Sigma \times \{0\}$.

Form the space $\Sigma \times I \underset{nbhd(\alpha_i)}{\cup} \coprod_{i=1}^{g} D^2_{\alpha_i} \times I$

$$nbhd(\alpha_i) \updownarrow \partial D^2_{\alpha_i} \times I$$

Similarly, attach g 2-handles to $\Sigma \times \{1\}$, one for each β curve.

$Y_{Disks} = \Sigma \times I \cup \{\alpha\text{ 2-handles}\} \cup \{\beta \text{ 2-handles}\}$

Exercise: $\partial_+ Y_{Disks} \cong S^2 \cong \partial_- Y_{Disks}$.

Hint! This follows from the condition that $\vec{\alpha}$ $(\vec{\beta})$ spans a g-dim'l subspace of $H_1(\Sigma \times \{0\})$ $(H_1(\Sigma \times \{1\}))$.

Glue B^3 to $\partial_+ Y_{Disks}$ along ∂B^3

" " " $\partial_- Y_{Disks}$ " "

This gives a closed 3-mnfld.

Uniqueness follows from the diffeomorphisms allowed in the handle-attachment and knowing that these are unique.

Thm. Any 3-mnfld. can be given a HD.

Pf (Idea): A HD comes from a special type of Morse function on a 3-mnfld:

(1) $f: Y^3 \to \mathbb{R}$, Morse

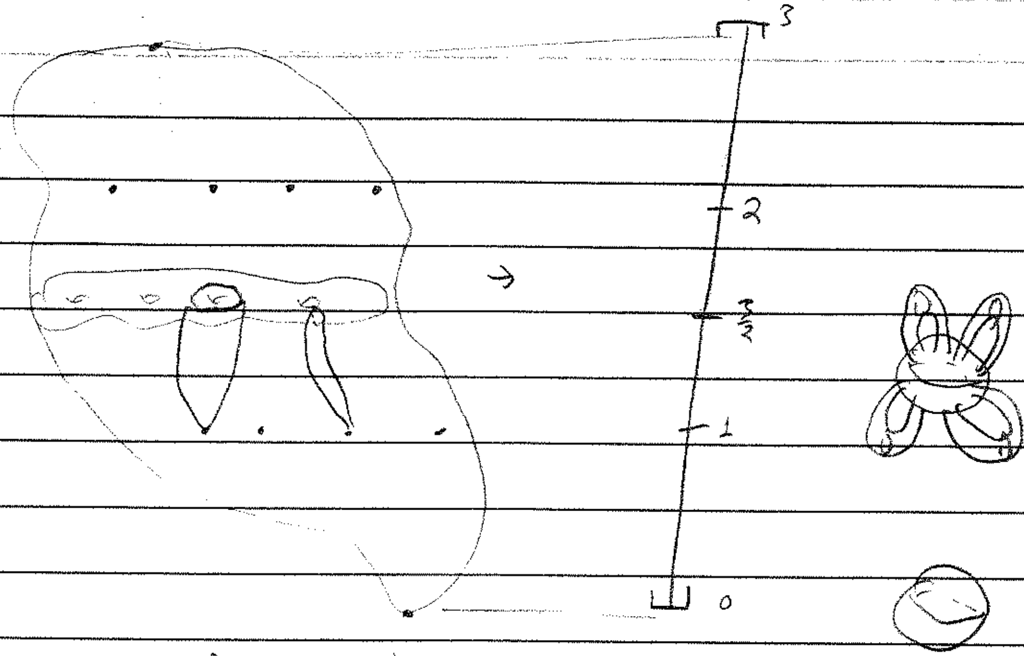
(2) f is self-indexing, i.e. $ind_f(p) = f(p)$ $\forall p \in Crit(f)$.

(Note: This implies $f(Y) = [0,3]$)

(3) f has a unique index 0 and unique index 3 critical pt.

Assuming such an f exists, we use it to construct a HD.

$\Sigma = f^{-1}(\frac{3}{2})$.



$\frac{3}{2}$ is a regular value of f, so $f^{-1}(\frac{3}{2})$ is a 2-dim'l submanifold.

Claim: $\#\{\text{index } 1 \text{ critical pts.}\} = \#\{\text{index } 2 \text{ critical pts.}\}$

Pf. Follows from the fact that there are unique index 0 and index 3 critical pts. ~~[illegible]~~

Our diagram will be:

$$\left(\Sigma = f^{-1}(\tfrac{3}{2}),\quad \vec{\alpha} = W^s(\text{ind. } 1 \text{ crit pts.}) \cap \Sigma,\ \vec{\beta} = W^u(\text{ind. } 2 \text{ crit. pts.}) \cap \Sigma\right)$$

boundaries of

Alternatively, look at the co-cores and cores of the 1-handles and 2-handles resp.

Exercise: Think about what a HD for a 3-mnfld w/ ∂ should look like.

i.e. What types of Morse functions give rise to such diagrams?

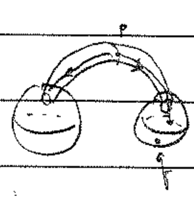
Warm-up: Draw a HD for the complement of the trefoil knot.

What about those special Morse functions?

(1) Morse functions exist (by Sard's thm.) $\quad \text{Morse}(M,\mathbb{R}) \overset{\text{open, dense}}{\subseteq} \text{Maps}(M,\mathbb{R})$.

(2) Self-indexing Morse functions exist (See Ch. 4 of Milnor "Lectures on the h-cobordism Thm.")

(3) ~~The~~ Single index 0 and 3 critical pts. requires a Handle Cancellation Lemma (Milnor)



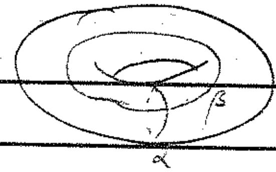
If $(\text{Descending}(p) \cap \text{Ascending}(p))/\mathbb{R} = \{pt.\}$.

Then we can "cancel" p and q, i.e. find a family of functions f_t s.t.

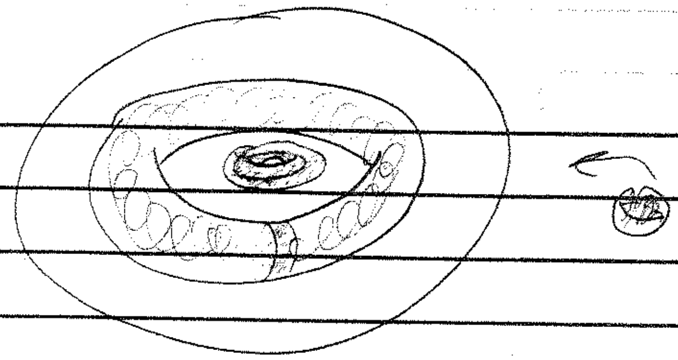
$f_0 = f$, $\quad f_1$ is a new function w/o critical pts. p & q, leaving all other crit pts. alone.

A 3-mfld has many HD's.

Ex:

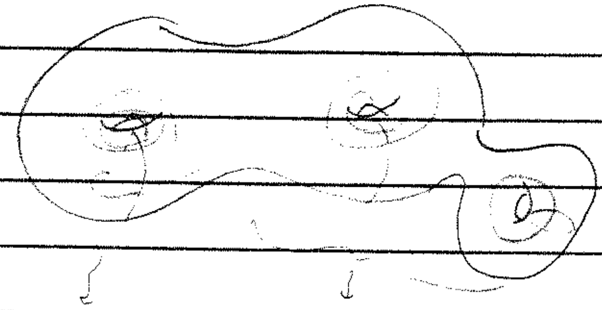


S^3



$L(2,1) = \mathbb{R}P^3$

Stabilization

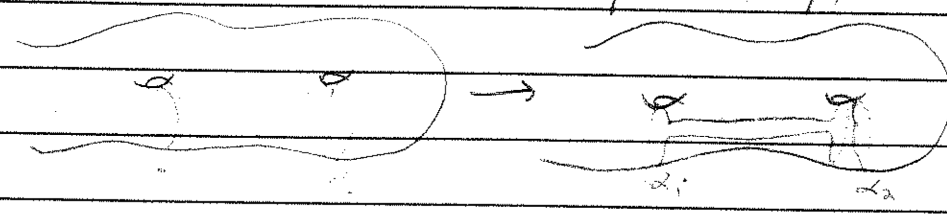


Y $\qquad Y' = (Y \setminus B^3) \cup_{S^2} (S^3 \setminus B^3) = Y \# S^3 \cong Y.$

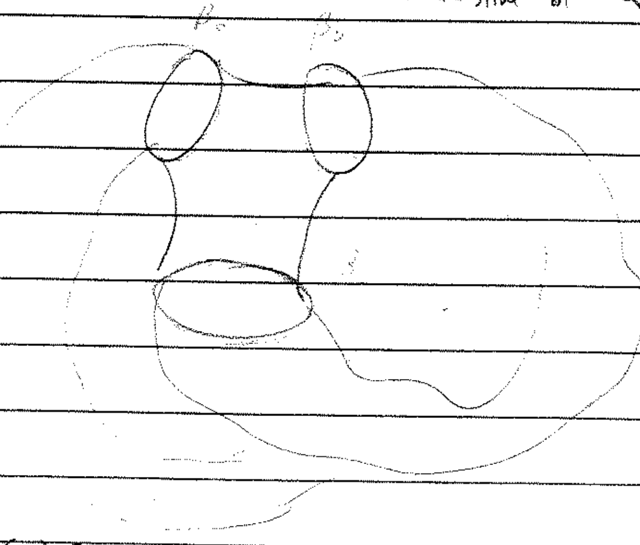
Thm. Any two HD for the same 3-mfld. can be connected by a sequence of moves:

(1) Stabilization, or its inverse (destabilization)

(2) Handle slides of α curves over α (or β curves over β)



Handle slide of α_1 över α_2.



β_a, β_b and β_c are not homologically independent, so β_a, β_b can be replaced by β_a, β_c.

(3) Isotopy of α curves (β curves) keeping $\vec{\alpha}$ ($\vec{\beta}$) pairwise disjoint.

Idea of Pf.

$\mathrm{Map}(M^n, \mathbb{R}) \longleftarrow \gamma$

$\gamma(0) = f_0$ Morse functions giving rise to HDS 1

$\gamma(1) = f_1$ 2

At only finitely many pts. in this path are the functions not Morse,

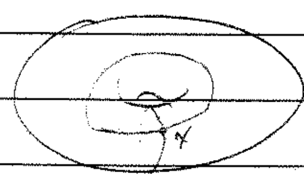
Understanding what can happen there gives rise to the 3 moves listed.

Heegaard Floer Homology

Try:

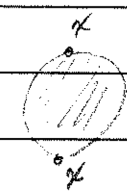
$$(\Sigma, \vec{\alpha}, \vec{\beta}) \rightsquigarrow C(\vec{\alpha}, \vec{\beta}) = \bigoplus_{x \in \vec{\alpha} \cap \vec{\beta}} \mathbb{Z}/2\langle \vec{x}\rangle$$

This is symplectic! These are Lagrangian!



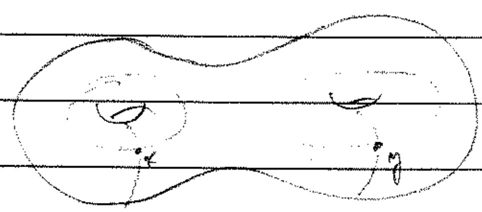
$\mathbb{Z}/2\langle x\rangle$

$\partial x = 0$. No disks except the constant map



$\Rightarrow H_*(C(\vec{\alpha}, \vec{\beta}), \partial) \cong \mathbb{Z}/2$.

stabilize



$\mathbb{Z}/2\langle x\rangle \oplus \mathbb{Z}/2\langle y\rangle$

$\partial x = 0, \ \partial y = 0$

$\Rightarrow H_*(C(\vec{\alpha}, \vec{\beta}), \partial) \cong (\mathbb{Z}/2)^2$

So this isn't invariant under the moves.

That's not to say it's not interesting - just that it doesn't give a manifold invariant.

Here's what we do:

$$(\Sigma, \vec{\alpha}, \vec{\beta}) \rightsquigarrow \mathrm{Sym}^g(\Sigma) = \overbrace{\Sigma \times \cdots \times \Sigma}^{g\text{-fold}} / S_g \leftarrow$$ Symmetric group on g letters acting by permutation.

<u>Lemma:</u> $\mathrm{Sym}^g(\Sigma)$ is smooth, complex, and symplectic!

Pf. Smooth & Complex follow from FTA! $(\mathrm{Sym}^g(\mathbb{C}) \cong \mathbb{C}^g)$

$$\mathbb{T}_\alpha := \alpha_1 \times \alpha_2 \times \cdots \times \alpha_g / S_g$$
$$\mathbb{T}_\beta := \beta_1 \times \beta_2 \times \cdots \times \beta_g / S_g$$

Lagrangian submanifolds.

$$C(\Sigma, \vec{\alpha}, \vec{\beta}) = \bigoplus_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subseteq \mathrm{Sym}^g(\Sigma)} \mathbb{Z}/2 \langle \vec{x} \rangle$$

∂ counts J-holomorphic disks connecting $\vec{x}$ to $\vec{y}$.

Thm. (O·S) The homology of $C_*(\Sigma, \vec{\alpha}, \vec{\beta})$ is a 3-manifold invariant.
i.e. it doesn't depend on the diagram.

Thm. (O·S). $\dim \overset{\text{"e.g."}}{H_*}(Y^3) = \text{order } |H_1(Y)|$
if $H_1(Y; \mathbb{Z})$ is finite.