Matt Hedden HFH 9/21/10

Last Time: Heegaard Diagrams $\rightsquigarrow$ Chain Complex

$(\Sigma, \vec{\alpha}, \vec{\beta})$ $\cdot$ $\quad C(\Sigma, \vec{\alpha}, \vec{\beta}) = \bigoplus_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}/2 \langle x \rangle$

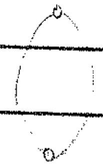
$Sym^g(\Sigma)$

$\cup \quad \cup$

$\mathbb{T}_\alpha \quad \mathbb{T}_\beta$

∂ is supposed to count J-holomorphic disks (strips) connecting intersection pts.

Recall: from Morse homology, we only counted gradient flows for critical pts satisfying $\text{ind}\, p - \text{ind}\, q = 1$.

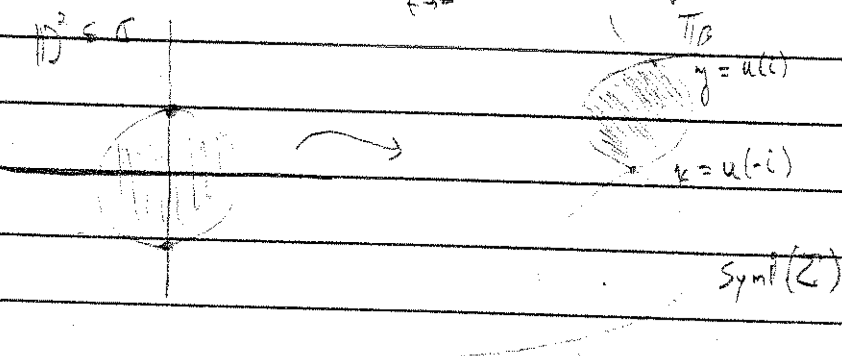


There's no natural index to associate to our critical pts.

Instead, we should think about indices of paths connecting the pts.

<u>Def.</u> A Whitney disk connecting $\vec{x}$ to $\vec{y}$ is a map $u : [0,1] \times \mathbb{R} \to Sym^g(\Sigma)$ satisfying

$$u(0 \times \mathbb{R}) \subseteq \mathbb{T}_\beta \qquad \lim_{t \to -\infty} u(s,t) = \vec{x}$$
$$u(1 \times \mathbb{R}) \subseteq \mathbb{T}_\alpha \qquad \lim_{t \to \infty} u(s,t) = \vec{y}$$



<u>Def.</u> Let $\pi_2(\vec{x}, \vec{y}) = \{\text{Whitney disks connecting } \vec{x} \text{ to } \vec{y}\} \big/ \text{homotopy Rel } \partial$ $\quad \mathbb{T}_\alpha$

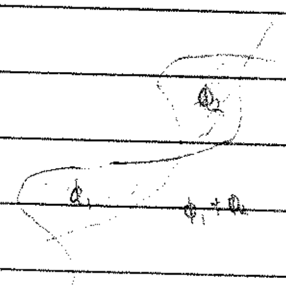
Call these homotopy classes of Whitney disks connecting $\vec{x}$ to $\vec{y}$

$\sim$

Note: ∂ need not be ptwise fixed.

<u>Note</u>: There is a natural way to concatenate paths:

$$\pi_2(\vec{x}, \vec{y}) \times \pi_2(\vec{y}, \vec{z}) \longrightarrow \pi_2(\vec{x}, \vec{z})$$
$$(\phi_1, \phi_2) \longmapsto \phi_1 * \phi_2.$$

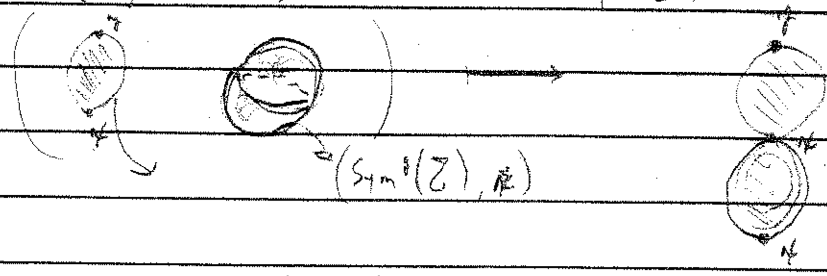


Further, we have inverses:

$$\pi_2(\vec{x}, \vec{y}) \to \pi_2(\vec{y}, \vec{x})$$
$$\phi \longmapsto \phi^{-1}$$
$$u(s,t) \longmapsto u(s,-t)$$

Finally, we have Sphere Addition ← 2nd Homotopy Group

$$\pi_2(\vec{x},\vec{y}) \times \pi_2(\mathrm{Sym}^g(\Sigma),\vec{x}) \longrightarrow \pi_2(\vec{x},\vec{y})$$
$$(\phi, \quad S) \longmapsto \phi * S$$



Exercise: Show that $\pi_2(\vec{x},\vec{x})$ is a group,

and $\pi_2(\vec{x},\vec{x})$ (or $\pi_2(\vec{y},\vec{y})$) acts on $\pi_2(\vec{x},\vec{y})$ from the left (right).

Proposition $\exists$ function $\mu : \pi_2(\vec{x},\vec{y}) \to \mathbb{Z}$ called the Maslov index satisfying

(1) Additivity $\quad \mu(\phi_1 * \phi_2) = \mu(\phi_1) + \mu(\phi_2)$, $\phi_1 \in \pi_2(\vec{x},\vec{y})$, $\phi_2 \in \pi_2(\vec{y},\vec{z})$

(2) Inverses $\quad \mu(\phi^{-1}) = -\mu(\phi)$

(3) Sphere Additivity $\quad \mu(\phi * S) = \mu(\phi) + 2\langle c_1(J), H_*(S)\rangle$

$H_* : \pi_2(\mathrm{Sym}^g(\Sigma)) \to H_2(\mathrm{Sym}^g(\Sigma))$

is the Hurwitz map,

and c_1 is the Chern class.

From (2), it follows that

(4) (constant) suppose $\phi \in \pi_2(\vec{x},\vec{x})$ is the constant map. Then $\mu(\phi) = 0$.

The Maslov index is the expected dimension of the space $\mathcal{M}(\phi)$ of J-holomorphic disks representing $\phi \in \pi_2(\vec{x},\vec{y})$.

$$\mathcal{M}(\phi) = \left\{ u : [0,1]\times\mathbb{R} \to \mathrm{Sym}^g(\Sigma) \;\middle|\; \begin{array}{l} [u] = \phi \text{ — homotopy class w/ appropriate } \partial \text{ conditions} \\ du \circ i = J\, du \end{array} \right\}$$

"This is the space of J-holomorphic disks connecting $\vec{x}$ to $\vec{y}$ in homotopy class ϕ."

$\widehat{\mathcal{M}(\phi)} = \mathcal{M}(\phi)/\mathbb{R}$ is the space of unparametrized J-holo - - -

infinite dim'l space $\qquad$ Thinking to figure out what these spaces should be

$$\bar\partial_J : \mathcal{B} \longrightarrow \mathcal{L}$$
$$u \longmapsto \bar\partial_J(u) = du \circ i - J\,du$$

Perturbed J-holomorphic operator

Want 0 to be a regular value for this map, so that $\bar\partial_J^{-1}(0)$ will be a smooth manifold.

(D, Implicit Function Thm in some infinite dim'l setting)

We can check for regular values at the level of derivatives:

$$D_u \bar{\partial}_J : T_u \mathcal{B} \to T_{\bar{\partial}_J(u)} \mathcal{L}$$

Want this to be surjective $\forall u \in \bar{\partial}_J^{-1}(0)$.

If we can do this, then $\mathcal{M}(\phi)$ is a mnfld. of dimension $\dim(\ker(D_u \bar{\partial}_J))$, $u \in \bar{\partial}_J^{-1}(0)$.

Define the index $(D\bar{\partial}_J) = \dim(\ker(D\bar{\partial}_J)) - \dim(\operatorname{coker}(D\bar{\partial}_J))$.

Want $\dim(\operatorname{coker}(D\bar{\partial}_J)) = 0$, in which case $\mathcal{M}(\phi)$ is mnfld. of dim. $\operatorname{index}(D\bar{\partial}_J)$.

For us, Maslov index $=$ index $D\bar{\partial}_J$.

Define: $\partial : C(\Sigma, \vec{\alpha}, \vec{\beta}) \to C(\Sigma, \vec{\alpha}, \vec{\beta})$ by

$$\partial \vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1}} \# |\widehat{\mathcal{M}(\phi)}| \pmod 2 \cdot \vec{y}.$$

Issues

- Ensure $\mathcal{M}(\phi)$ is smooth and expected dimension equals $\mu(\phi)$. (Achieving transversality)
  We do this, usually, by varying the almost complex structure, $J : T\mathrm{Sym}^g(\Sigma) \to T\mathrm{Sym}^g(\Sigma)$.
- Is the count finite? (compactness)
- Is $\partial^2 = 0$?

Thm. (OS) We can achieve transversality and compactness so that ∂ is well-defined, and $\partial^2 = 0$. Moreover, $H_*(C(\Sigma, \vec{\alpha}, \vec{\beta}), \partial) =: HF^{\text{wrong}}(Y)$ is independent of $(\Sigma, \vec{\alpha}, \vec{\beta})$, depends only on the 3-mnfld.

But, for Y s.t. $H_*(S^3; \mathbb{Q}) = H_*(Y; \mathbb{Q})$,

$$\operatorname{rank}(HF^{\text{wrong}}(Y)) = |H_1(Y; \mathbb{Z})|.$$

Idea: Let's refine this invariant by introducing a basepoint.

Def. A pointed Heegaard Diagram (PHD) is a HD $(\Sigma, \vec{\alpha}, \vec{\beta})$ with a distinguished base point $z \in \Sigma - \{\vec{\alpha} \cup \vec{\beta}\}$.

$(\Sigma, \vec{\alpha}, \vec{\beta}, z)$.

The basepoint produces a codimension 2 submanifold of $Sym^g(\Sigma)$, denoted

$$V_z = \underbrace{\{z\} \times Sym^{g-1}(\Sigma)}_{\text{unordered } g\text{-tuples of pts. on } \Sigma, \text{ at least one of which is } z.} \subseteq Sym^g(\Sigma).$$

$$\begin{array}{c} V_z^{2g-2} \subseteq Sym^g(\Sigma) \\ \cup\!| \\ Image(u), \text{ where } u \in \mathcal{M}(\phi). \end{array}$$

Consider $\underset{\text{Algebraic}}{\#} Im(u) \cap V_z$. This depends only on $[u] = \phi$.

Let $n_z(\phi) = \underset{\text{Alg.}}{\#} Im(u) \cap V_z$, where $[u] = \phi$.

The 4 different flavors treat the basepoint in slightly different ways

$$\widehat{CF}(Y) = \bigoplus \mathbb{Z}/2 \langle x \rangle$$

$$CF^-(Y) = \bigoplus \mathbb{Z}_2[U]\langle x \rangle$$

$$CF^\infty(Y) = \bigoplus \mathbb{Z}_2[U, U^{-1}]\langle x \rangle$$

$$CF^+(Y) = \bigoplus \left(\mathbb{Z}_2[U,U^{-1}] \big/ \mathbb{Z}[U] \right) \langle x \rangle$$

$$\{a_{-1} U^{-1} + a_{-2} U^{-2} + \cdots + a_{-n} U^{-n}\}$$

$$\partial \vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1 \\ n_z(\phi) = 0}} \# \widehat{\mathcal{M}(\phi)} \cdot \vec{y}$$

$$\partial^- \vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1}} U^{n_z(\phi)} \# \widehat{\mathcal{M}(\phi)} \cdot \vec{y}$$

$$\partial^\infty \vec{x} = \text{'' \quad '' \quad '' \quad '' \quad ''}$$

$$\partial^+ \vec{x} = \text{'' \quad '' \quad '' \quad '' \quad ''}$$

Ex: $\partial^+ x = U \cdot y = 0$

$\partial^- x = U \cdot y \neq 0$. B/c the Rings are different

(In general, we could take homology w/ twisted coefficients, $C(X; M)$, M module over $\mathbb{Z}[\pi_1(X)]$.

See e.g. Davis & Kirk, Lectures in Algebraic Topology)

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Last Time: $\pi_2(x,y)$

$\widehat{\mathcal{M}(\phi)}$ for $\phi \in \pi_2(x,y)$.

$\mu(\phi) \in \mathbb{Z}$ maslov index

ptd. HD $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$

$n_z(\phi) = \underset{\text{Alg.}}{\#} V_z \cap \operatorname{Im} u$, $[u] = \phi \in \pi_2(x,y)$

Ground Group

$\mathbb{Z}_2$ — $\widehat{CF}$, ∂ requires $u \in \widehat{\mathcal{M}}(\phi)$ to miss V_z, ie. $n_z(\phi) = 0$.

$\mathbb{Z}_2[U]$ — CF^-,

$\mathbb{Z}_2[U, U^{-1}]$ — CF^∞, "allows" $n_z(\phi)$ to be arbitrary, but keeps track by $U^{n_z(\phi)}$

$\mathbb{Z}_2[U, U^{-1}]/\mathbb{Z}[U]$ — CF^+

(Positivity Principle)

Lemma. If $\mathcal{M}(\phi) \neq \emptyset$, then $n_z(\phi) \geq 0$.

Pf. Recall; $\operatorname{Sym}^g(\Sigma)$ is a complex mnfld w/ a complex structure $\operatorname{Sym}^g(j)$, where j is a complex structure on Σ.

Now, a $(J = \operatorname{Sym}^g(j))$-holomorphic map from $[0,1] \times \mathbb{R} \xrightarrow{u} \operatorname{Sym}^g(\Sigma)$ is a holomorphic map.

$u \in \mathcal{M}(\phi)$ gives a holomorphic map $\mathbb{D} \to \operatorname{Sym}^g(\Sigma)$

$n_z(\phi) = \operatorname{Im} u \cap V_z$, but $\operatorname{Im}(u)$ and V_z are complex submanifolds of complimentary dimension. (Unless u = constant, in which case $n_z = 0$ anyway)

Exercise: Holomorphic submanifolds of a complex mnfld. intersect non-negatively when transverse.

Condition that $\partial\mathbb{D} \subseteq \mathbb{T}_\alpha \cap \mathbb{T}_\beta \Rightarrow \mathbb{D} \not\subset V_z$.

$\Rightarrow n_z \geq 0$.

(Now, this is a homotopy-theoretic condition. So perturbing the complex structure (to an almost complex structure) preserves $n_z \geq 0$.)

Alternatively, pick an almost-cpx structure J on $\operatorname{Sym}^g(\Sigma)$ which agrees with $\operatorname{Sym}^g(\Sigma)$ in nbhd(V_z).

Then the calculation is the same.

Lemma $\exists$ a short exact sequence of chain complexes

$$0 \longrightarrow CF^- \xrightarrow{i} CF^\infty \xrightarrow{p} CF^+ \longrightarrow 0$$

coming from the s.e.s.

$$0 \longrightarrow \mathbb{Z}_2[U] \to \mathbb{Z}_2[U,U^{-1}] \to \mathbb{Z}_2[U,U^{-1}]/\mathbb{Z}_2[U] \to 0.$$

Remark: The real content of this Lemma is that i, p are chain maps,

i.e. $i \circ \partial(\alpha) = \partial \circ i(\alpha)$.

Pf. Say $\alpha = 1\langle x\rangle + 1 \cdot U\langle y\rangle$

$$\begin{aligned} i\circ\partial(\alpha) &= i(\partial\langle x\rangle + \partial U\langle y\rangle) \\ &= i(\partial\langle x\rangle + U\,\partial\langle y\rangle) \\ &= \sum_w \sum_{\substack{\phi \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}(\phi)} \cdot U^{n_z(\phi)} \cdot w + \sum_{w'} \sum_{\phi'} \#\widehat{\mathcal{M}(\phi')} \cdot U^{n_z(\phi')} w' \end{aligned}$$

If $n_z(\phi) < 0$ for some ϕ with $\widehat{\mathcal{M}(\phi)} \neq \emptyset$, say $\phi \in \pi_2(x, w)$

then $U^{n_z(\phi)} \cdot w \in \partial x$ would not be defined.

But preceding Lemma ensures $n_z(\phi) \geq 0$ $\forall\, \phi$ with $\widehat{\mathcal{M}(\phi)} \neq \emptyset$.

$\partial^- : CF^- \to CF^-$, but $\partial^- = \partial^\infty = \partial^+$, but between different groups.

Argument above:

$$\begin{array}{ccc} \partial^\infty : CF^- & \to & CF^- \\ \cap| & & \cap| \\ \partial^\infty : CF^\infty & \to & CF^\infty \end{array}$$

so CF^- is a sub complex of CF^∞.

Any time we have a subcomplex A of chain complex B, there is a s.e.s.

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & \underbrace{B/A} & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & CF^- & \to & CF^\infty & \to & CF^+ & \to & 0 \end{array}$$

$\square$

Exercise: $0 \to \ker \cdot U \to CF^+ \xrightarrow{\cdot U} CF^+ \to 0$

$$\begin{array}{c} \ker \cdot U \\ \| \\ \widehat{CF} \end{array}$$

(1) Define the kernel of a chain map

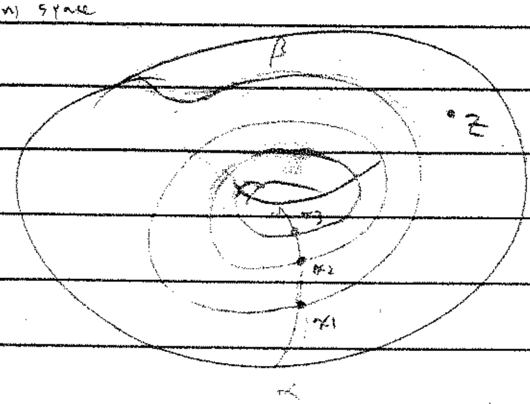
(2) Show that $\widehat{CF} = \ker U$.

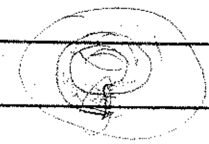
This exercise implies that

$$\begin{tikzcd} \widehat{HF} \arrow[r] & HF^- \arrow[dl] \\ HF^- \arrow[u] & \end{tikzcd}$$

Want to understand CF° in terms of H.D.

Ex: Lens space $\qquad L(3,1)$ or $L(-3,1)$.

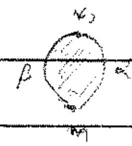




$$\mathcal{T}^+ := \mathbb{Z}_2[U,U^{-1}] \big/ \mathbb{Z}_2[U].$$

$$CF^+(L(3,1)) = \mathcal{T}^+\langle x_1\rangle + \mathcal{T}^+\langle x_2\rangle + \mathcal{T}^+\langle x_3\rangle$$

$\pi_2(x_1, x_2) = ?$



$= u(\partial D)$

But $[\gamma] \neq 0$ in $H_1(T^2)$.

Checking every possible ∂ of a disk connecting x_1 to x_2,

we find $\pi_2(x_1, x_2) = \emptyset$.

Path from x_1 to x_2 along α followed by a path from x_2 to x_1 along β.

Similarly, $\pi_2(x_2, x_3) = \pi_2(x_3, x_2) = \emptyset$.

$\pi_2(x_i, x_i) = \{\text{constant}\}$

$$\pi_2(T^2) \to \pi_2(T^2, \alpha\cup\beta) \xrightarrow{\ \to 0\ } \pi_1(\alpha\cup\beta) \hookrightarrow \pi_1(T^2)$$

$$\pi_2(T^2) = 0, \quad \pi_2(T^2,\alpha\cup\beta) = 0, \quad \pi_1(\alpha\cup\beta) = \mathbb{Z}\oplus\mathbb{Z}, \quad \pi_1(T^2) = \mathbb{Z}\oplus\mathbb{Z}$$

So $\partial^+ : CF^+(L(3,1)) \to CF^+(L(3,1))$ is trivial,

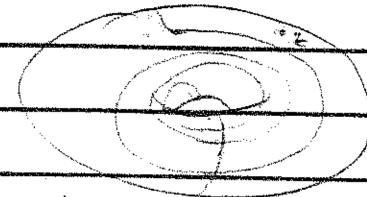
$\partial^+ \equiv 0$, so $CF^+ \cong HF^+$

Similarly, $CF^\circ \cong HF^\circ$, $\circ = +, -, \wedge, \infty$.

9/28/10 Matt Hedden HFH

Mistake at end of Last Time:

Calculation of $\pi_2(x, x)$ for



~~$\pi_2(\alpha \vee \beta) \to \pi_2(T^2) \to \pi_2(T^2, \alpha \cup \beta) \to \pi_1(\alpha \cup \beta)$~~

$$\pi_2(\vec{x}, \vec{x}) = \pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \vec{x})$$

$$\Omega(\mathrm{Sym}^g(\Sigma)) \hookrightarrow \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \ni \gamma$$

$$\downarrow \qquad \downarrow$$

$$\mathbb{T}_\alpha \times \mathbb{T}_\beta \qquad (\gamma(0), \gamma(1))$$

$$\begin{array}{ccccc} \pi_2(\mathrm{Sym}^g(\Sigma)) & & \pi_2(\vec{x}, \vec{x}) & & \mathbb{Z}^{2g} \\ \cong & & \cong & & \cong \\ \pi_1(\Omega(\mathrm{Sym}^g \Sigma)) & \to & \pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)) & \to & \pi_1(\mathbb{T}_\alpha \times \mathbb{T}_\beta) \end{array}$$

easy to see

$\downarrow \ \partial$ for $g=1$.

$$\pi_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \vec{x}) \leftarrow \pi_0(\Omega)$$

$$\cong$$

$$\pi_1(\mathrm{Sym}^g(\Sigma))$$

When $g=1$, $\pi_2(\mathrm{Sym}^g(\Sigma)) = \pi_2(T^2) = 0$.

$$0 \to \pi_2(\vec{x}, \vec{x}) \xrightarrow{n} \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^2$$

$$\cong$$

$$\pi_1(T^2)$$

independent of $\vec{y}$

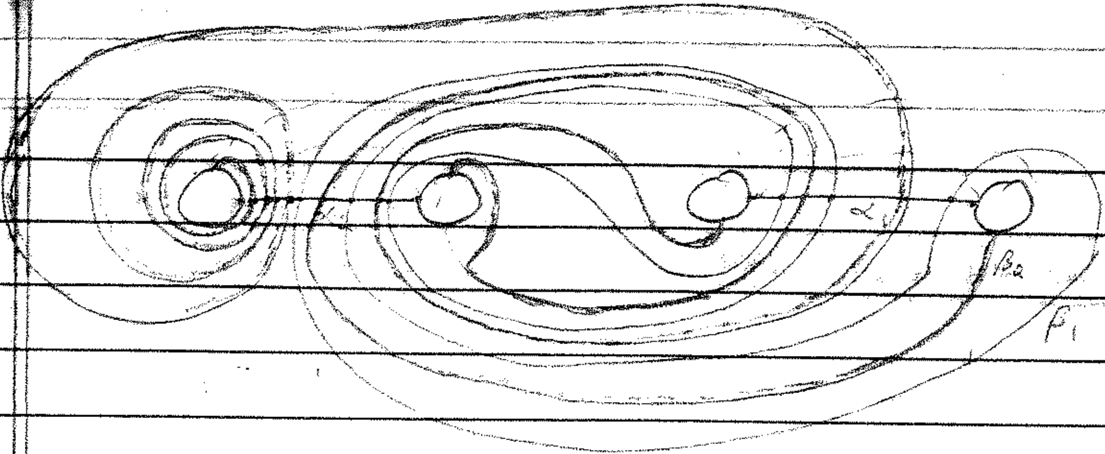
So ∂ operator decomposes along $\pi_2(\vec{x}, \vec{y}) \overset{1-1}{\longleftrightarrow} \pi_2(\vec{x}, \vec{x})$

Provided that $\pi_2(\vec{x}, \vec{y}) \neq \emptyset$.

$$\partial \vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}(\phi)} \cdot U^{n_z(\phi)} \cdot \vec{y}$$

(1) How to tell if $\pi_2(\vec{x}, \vec{y}) \neq \emptyset$, and, if so, how many $\phi \in \pi_2(\vec{x}, \vec{y})$? (i.e. what is $\pi_2(\vec{x}, \vec{x})$?)

(0) What are $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$?



$\alpha_2 \cap \beta_2$

__Exercise:__ (1) Show that $(\Sigma, (\alpha_1, \alpha_2), \beta_1)$ is a Heegaard diagram for the complement of a nbhd. of the trefoil knot.

(2) Show that $(\Sigma, (\alpha_1, \alpha_2), (\beta_1, \beta_2))$ is a Heegaard diagram for n-surgery on the trefoil, and determine n.

What are $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$?   They are g-tuples of intersection points between α + β curves, where every α & β curve is used exactly once.

$\mathbb{T}_\alpha$: $\alpha_1 \times \alpha_2 / S_2$ $\mathbb{T}_\beta$: $\beta_1 \times \beta_2 / S_2$

$\alpha_2 \cap \beta_2 = \{x_1, \dots, x_6\}$
$\alpha_2 \cap \beta_1 = \{y_1, \dots, y_3\}$ ~~$(3 \times 2) + (6 \times 3) = 6 + 18 = 24$ pts. in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.~~
$\alpha_1 \cap \beta_2 = \{m_1, m_2\}$
$\alpha_1 \cap \beta_1 = \{n_1, \dots, n_3\}$

(3) Calculate H_1(Mnfld. specified by HD ; $\mathbb{Z}$)
Calculate $H_1(S^3_n(K); \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ (if $n \neq 0$, then $\mathbb{Z}$).

$\pi_2(\vec{x}, \vec{y})$? Φ $\tilde{\Phi}$

Sps. $\pi_2(\vec{x}, \vec{y}) \neq \emptyset$, then $\exists$

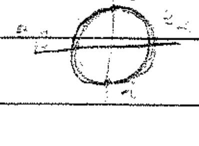
$\tilde{\Phi}: (F^2, a, b, x, y) \to \Sigma \times \mathrm{Sym}^{g-1}(\Sigma) \xrightarrow{pr_1} \Sigma$

$u^*(\pi_{g-1})$, $\tilde{\pi}_{g-1}$, $proj_1$

$(\hat{F}^2, \hat{e}_\alpha, \hat{e}_\beta, \hat{-i}, \hat{i}) \to (\Sigma \times \dots \times \Sigma, \alpha_1 \times \dots, \beta_1 \times \dots \times \beta_g, \vec{x}, \vec{y})$

$u^*(\pi)$ (pull-back) π Branched $g!$-fold covering map

$\Phi = pr_1 \circ \tilde{\Phi}$ $u: (\mathbb{D}^2, e_\alpha, e_\beta, -i, i) \xrightarrow{u} (\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta, \vec{x}, \vec{y})$



Notice that $\pi|_{\alpha_1\times\cdots\times\alpha_g} : \alpha_1\times\cdots\times\alpha_g \to \mathbb{T}_\alpha$ (resp. β) is an actual (non-branched) covering map.

So this pulls back:

When restricted to $(\hat{e}_\alpha, \hat{e}_\beta, -\hat{i}, \hat{i})$, $u^*(\pi)$ is a $g!$-fold covering map.

The diagram leads to

$$\{u : \mathbb{D}^2 \to Sym^g(\Sigma)\} \longrightarrow \left\{ \begin{array}{c} F^2 \xrightarrow{\Phi} \Sigma \\ \big\downarrow \pi/S_{g-1} \\ \mathbb{D}^2 \end{array} \right\} \qquad *$$

Where in $*$, (F^2, a, b, x, y)
$$\big\downarrow \pi/S_{g-1}$$
$(\mathbb{D}^2, e_\alpha, e_\beta, -i, i)$ is a branched g-fold covering which is a covering on a, b, x, y.

In particular, a is just g disjoint intervals.
b " " " " "
x " " " " pts.
y " " " " "

So F^2 is a surface with boundary, with its boundary divided into $2g$ arcs, meeting along $2g$ pts., with g arcs labeled a.
" " " b.

Φ is a map from $F^2 \to \Sigma$, sending a to g arcs on the g α curves
b " " " " " β "
x to the g-tuple of pts. $\{\alpha_i \cap \beta_{\sigma(i)}\}_{i=1}^{g}$ $\sigma \in S_g$.
y " " " " "

Conversely,

Sps. we have $\left\{ \begin{array}{c} F^2 \xrightarrow{\Phi} \Sigma \\ \downarrow p \\ \mathbb{D}^2 \end{array} \right\}$ satisfying all the conditions (1) p is a branched g-fold covering, honest on ∂
(2) Φ sends a to α, b to β
x to $\vec{x}$, y to $\vec{y}$

$\Rightarrow$ A Whitney disk $u : \mathbb{D}^2 \to Sym^g(\Sigma)$ connecting $\vec{x}$ to $\vec{y}$

To see this, define: for $m \in \mathbb{D}^2$, $u(m) := \Phi(p^{-1}(m))$ $\subseteq Sym^g(\Sigma)$

g points w/ multiplicity

~~Bill Breslin — Short Geodesics ... Hyperbolic 3-Mflds~~

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Last Time: Prop. $\left\{u:(\mathbb{D}^2, e_\alpha, e_\beta, -i, i) \to (\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta, \vec{x}, \vec{y})\right\}/\sim$

$$\updownarrow$$

$$\left\{\begin{array}{l}(F^2, a, b, x, y) \xrightarrow{\Phi} (\Sigma, \vec{\alpha}, \vec{\beta}, \vec{x}, \vec{y}) \\ \quad\downarrow p \\ (\mathbb{D}^2, e_\alpha, e_\beta, -i, i)\end{array}\right\}/\sim \quad \text{homotopy}$$

where $(\mathbb{D})p$ is a branched $g:1$-fold cover, an honest cover on (a, b, x, y).

$\pi_2(\vec{x}, \vec{y}) \xleftrightarrow{1-1} \pi_2(x, y)$

$u: \mathbb{D}^2 \to \mathrm{Sym}^g(\Sigma)$ $\qquad \begin{array}{l} F \xrightarrow{\Phi} \Sigma \\ \downarrow p \\ \mathbb{D}\end{array}$

So, what is $\pi_2(x, y)$?

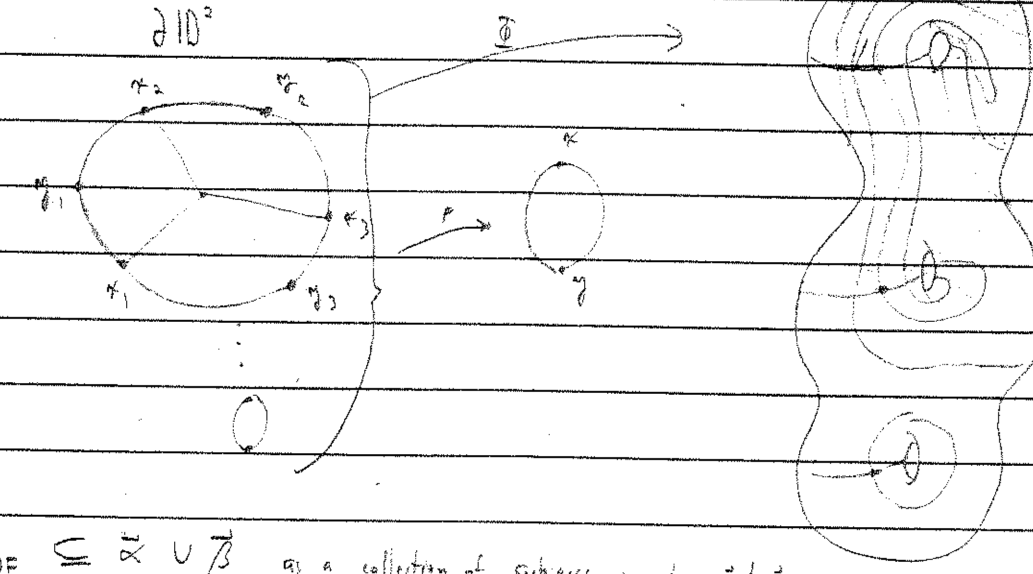
In particular, (1) Is $\pi_2(x, y) = \emptyset$?

Suppose $\pi_2(\vec{x}, \vec{y}) \neq \emptyset \Rightarrow \exists \begin{array}{l} F^2 \xrightarrow{\Phi} \Sigma \\ \downarrow p \\ \mathbb{D}^2\end{array}$

Consider $\Phi|_{\partial F} : \bigsqcup S^1 \to \vec{\alpha} \cup \vec{\beta} \subseteq \Sigma$

$\downarrow$ g-fold cover

$\partial \mathbb{D}^2$



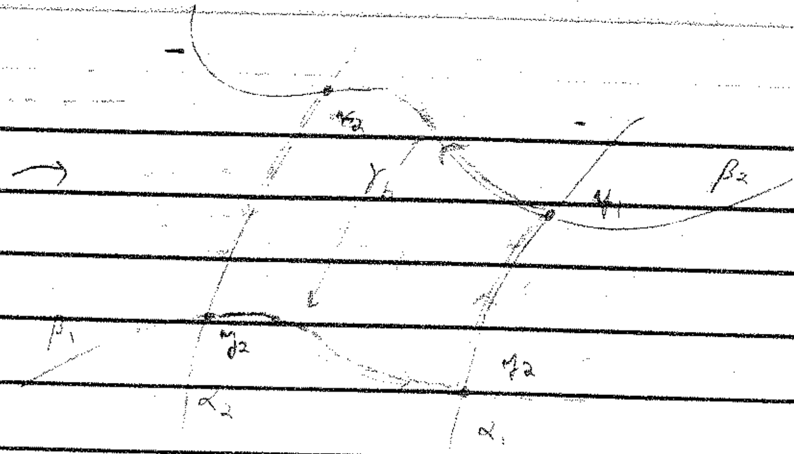
$\Phi|_{\partial F} \subseteq \vec{\alpha} \cup \vec{\beta}$ as a collection of subarcs connecting $\vec{x}$ & $\vec{y}$

$\underline{\Phi|_a} \subseteq \partial F = \left\{ g\text{-tuple of arcs in } \Sigma \text{ connecting } x_i \text{ to } y_{\sigma(i)} \right\}$ for some $\sigma \in S_g$
$\qquad \overset{\shortparallel}{\gamma_a}$

$\underline{\Phi|_b} \subseteq \partial F = \left\{ g\text{-tuple of arcs in } \Sigma \text{ connecting } y_j \text{ to } x_{\sigma'(j)} \right\}$ for some $\sigma' \in S_g$.
$\qquad \overset{\shortparallel}{\gamma_b}$

Ex:

$\partial F^2 \longrightarrow$



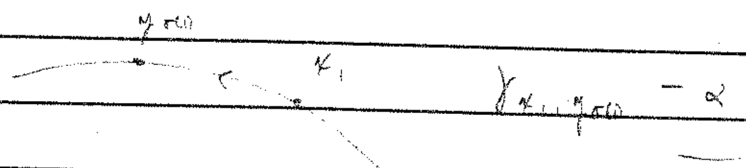
So if $\begin{array}{c} F^2 \to \Sigma \\ \downarrow \\ \mathbb{D}^2 \end{array}$ exists (i.e. $\pi_2(x, y) \neq \emptyset$)

Then $\partial F^2 \longrightarrow \gamma_a \cup \gamma_b$ collection of oriented arcs connecting $\vec{x}$ to $\vec{y}$ along α-curves and $\vec{y}$ to $\vec{x}$ along β-curves, resp.

Note: If $[\gamma_a \cup \gamma_b] \neq 0$ in $H_1(\Sigma)$, then $\gamma_a \cup \gamma_b \neq \partial \mathrm{Im}(F^2)$.

But, we have to account for all possible paths $\gamma_a \cup \gamma_b$.

Any other $\gamma_a' \cup \gamma_b'$ differs from $\gamma_a \cup \gamma_b$ by the addition/subtraction of curves in $\vec{\alpha} \cup \vec{\beta}$.



Ask: Is $[\gamma_a \cup \gamma_b] = 0$ in $H_1(\Sigma) / \mathrm{Span}[\vec{\alpha}] + \mathrm{Span}[\vec{\beta}]$

This is precisely the question of whether any path which could be ∂F^2 is null-homologous.

Def. $\varepsilon(\vec{x}, \vec{y}) := [\gamma_a \cup \gamma_b] \in H_1(\Sigma) / \mathrm{Span}[\vec{\alpha}] + \mathrm{Span}[\vec{\beta}]$

for any $\gamma_a \cup \gamma_b$ connecting $\vec{x}$ to $\vec{y}$ along α, $\vec{y}$ to $\vec{x}$ along β.

Prop. If $\pi_2(x, y) \neq \emptyset$, then $\varepsilon(\vec{x}, \vec{y}) = 0$.

Prop. If $\varepsilon(x, y) = 0$, then $\pi_2(x, y) \neq \emptyset$.

Pf $\varepsilon(x, y) = 0 \Rightarrow \exists\, \gamma_a \cup \gamma_b$ which is null-homologous in Σ.

$\Rightarrow \exists\, F^2 \xrightarrow{\Phi} \Sigma$ satisfying the boundary conditions.

It suffices to show that F^2 admits a branched covering $F^2 \xrightarrow{f} \mathbb{D}^2$.

This is true, but we will not prove it. Exercise

So $\pi_2(\vec{x},\vec{y}) \neq \emptyset \iff \varepsilon(\vec{x},\vec{y}) = 0$

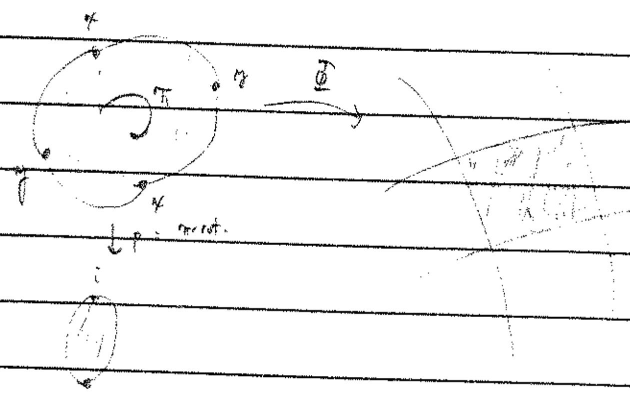
Nov, If $\pi_2(\vec{x},\vec{y}) \neq \emptyset$, how many elements does it have?

Modulo the Exercise, what we're asking is how many maps

$$\mathbb{D}^2 \xrightarrow{\Phi} \Sigma /_{\text{homotopy}} \quad \text{are there?}$$

$$\Phi : (F^2, a, b, x, y) \longrightarrow (\Sigma, \vec{\alpha}, \vec{\beta}, \vec{x}, \vec{y})$$

$$[F^2, a \cup b] \in H_2(\Sigma, \vec{\alpha} \cup \vec{\beta})$$



$$\cancel{H_2(\vec{\alpha} \cup \vec{\beta})} \longrightarrow \underset{\mathbb{Z}}{\underset{\cong}{H_2(\Sigma)}} \hookrightarrow H_2(\Sigma, \vec{\alpha} \cup \vec{\beta}) \xrightarrow{i} \underset{\mathbb{Z}^{2g}}{\underset{=}{H_1(\vec{\alpha} \cup \vec{\beta})}} \xrightarrow{j} \underset{\mathbb{Z}^{2g}}{\underset{\cong}{H_1(\Sigma)}} \to H_1(\Sigma, \vec{\alpha} \cup \vec{\beta})$$

What is $\operatorname{im}(i) = \ker(j)$?

$$0 \to \mathbb{Z} \to H_2(\Sigma, \vec{\alpha} \cup \vec{\beta}) \to \ker j$$

$$= \left\{ \sum n_i [\alpha_i] + \sum m_j [\beta_j] = 0 \in H_1(\Sigma) \right\}$$

For $g > 1$,

<u>Prop.</u> $\pi_2(\vec{x}, \vec{y}) = \mathbb{Z} \oplus \ker(\mathrm{Span}[\vec{\alpha}] + \mathrm{Span}[\vec{\beta}] \to H_1(\Sigma))$, if $\pi_2(\vec{x},\vec{y}) \neq \emptyset$.

<u>Recall:</u> For base space, $\pi_2(\vec{x},\vec{x}) = \{\text{constant}\}$. The trouble is that in this case ($g = 1$), our pull back construction is asking only for disks.

How to interpret $\ker\left(\mathrm{Span}[\vec{\alpha}] + \mathrm{Span}[\vec{\beta}] \xrightarrow{j} H_1(\Sigma)\right)$ & $H_1(\Sigma) /_{\mathrm{Span}[\vec{\alpha}] + \mathrm{Span}[\vec{\beta}]}$

<u>Prop.</u> $H_1(\Sigma) /_{\mathrm{Span}\,\vec{\alpha} + \mathrm{Span}\,\vec{\beta}} \longleftrightarrow H_1(Y; \mathbb{Z}) \cong H^2(Y; \mathbb{Z})$

$$\ker j \longleftrightarrow H_2(Y; \mathbb{Z})$$

Pf. Mayer-Vietoris from the Heegaard Diagram.

$$Y = (\Sigma \times I) \cup \{\alpha\text{-handlebody}\} \cup \{\beta\text{-handlebody}\}$$

$$Y = \underbrace{\{\alpha\text{-handlebody}\}}_{H_\alpha} \cup \underbrace{\{\beta\text{-handlebody}\}}_{H_\beta}$$

$$H_1(\Sigma) \xrightarrow{i} \underbrace{H_1(H_\alpha)}_{\mathrm{Span}\,\vec{\alpha}^*} \oplus \underbrace{H_1(H_\beta)}_{(\mathrm{Span}\,\vec{\beta}^*)} \to H_1(Y) \to 0$$

$$H_2(Y) \leftarrow \underbrace{H_2(H_\alpha)}_{\cong\, 0} \oplus \underbrace{H_2(H_\beta)}_{\cong\, 0} \leftarrow \underbrace{H_2(\Sigma)}_{\cong\, \mathbb{Z}} \xleftarrow{\ \cong\ } \underbrace{H_3(Y)}_{\cong\, \mathbb{Z}}$$

$$H_2(Y) \cong \ker i \cong \ker\left(\mathrm{Span}[\vec{\alpha}] + \mathrm{Span}[\vec{\beta}] \to H_1(\Sigma)\right)$$

$\ker i$ = collections of curves on Σ which are null-homotopic in both handlebodies