Matt Hedden HFH 11/2/10

Over(e)view: Chain Complexes

$$\widehat{CF} = \bigoplus_{\mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}/2\langle \vec{x} \rangle$$

$$CF^+ = \mathbb{Z}[U, U^{-1}] / U\mathbb{Z}[U]$$

$$CF^- = \mathbb{Z}[U]$$

$$CF^\infty = \mathbb{Z}[U, U^{-1}]$$

∂ counts J-holomorphic disks connecting $\vec{x}$ to $\vec{y}$.

How method: (1) Enumerating $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ $\quad \vec{x} = \{x_1, \dots, x_g\}$

$x_i \in \alpha_i \cap \beta_{\sigma(i)} \quad \sigma \in S_g$.

(2) Determine $\pi_2(\vec{x}, \vec{y}) = \begin{cases} \emptyset & \text{if } 0 \neq \varepsilon(\vec{x}, \vec{y}) \in H_1(Y) \\ \mathbb{Z} \oplus H_2(Y) \end{cases}$

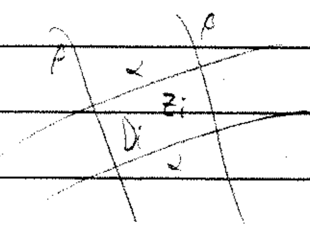
$[\Sigma]$ — Periodic domains i.e. null homologies for sum of α & β curves

(3) Encode elements $\phi \in \pi_2(\vec{x}, \vec{y})$ by their Domains

$$D(\phi) \in \mathbb{Z}\langle \Sigma \setminus \{\alpha \cup \beta\} \rangle$$

$$\| $$

$$\sum n_{z_i}(\phi) \cdot D_i$$



(4) Formula for $\mu(\phi) = \widehat{\chi}\big(D(\phi)\big) + n_{\vec{x}}\big(D(\phi)\big) + n_{\vec{y}}\big(D(\phi)\big)$

(5) ε-class splittings $\quad CF^\circ(Y) = \bigoplus CF^\circ(Y, \varepsilon_i)$

ε_i are $\varepsilon \sim$ classes

$$= \bigoplus_{s \in Spin^c(Y)} CF^\circ(Y, s)$$

Prop. $s_z(\vec{x}) - s_z(\vec{y}) = PD[\varepsilon(\vec{x}, \vec{y})]$

$s_{z_1}(\vec{x}) - s_{z_2}(\vec{y}) = PD[\gamma_{z_1} \cup -\gamma_{z_2}]$

(6) Formulas $\quad P$ = periodic domain

$$\mu(P) = \langle c_1(s_z(\vec{x})), [P] \rangle = \widehat{\chi}(P) + 2n_{\vec{x}}(P)$$

$$gr(\vec{x}) - gr(\vec{y}) = \mu(\phi) - 2n_z(\phi)$$ well-defined mod $\gcd_{\alpha \in H_2(Y)} \langle c_1(s_z(\vec{x})), \alpha \rangle$.

ϕ is any Whitney disk

Recall:

$$\partial^\infty \vec{x} = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) = \mathbb{Z} \oplus H_2(Y) \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) \cdot U^{n_z(\phi)} \cdot y$$

Q: $\langle \partial x, U^k y \rangle < \infty$?

This question forces a refinement of theory when $b_1(Y) > 0$ (i.e. $Y \neq \mathbb{Q}HS^3$)

A: Could be yes if there existed infinitely many $\phi \in \pi_2(\vec{x}, \vec{y})$ s.t.

(1) $n_z(\phi) = k$

(2) $\mu(\phi) = 1$.

The fix is slightly different for $\widehat{CF}$, CF^+ versus CF^-, CF^∞.

The reason is that CF^- + CF^∞ must allow disks w/ arbitrarily large $n_z(\phi)$.

So slightly stronger hypotheses will be necessary to get boundedness in CF^- and CF^∞.

Admissibility Hypotheses (Conditions on HD)

Note: If $Y = \mathbb{Q}HS^3$ (i.e. $b_1(Y) = 0$), then $\pi_2(\vec{x}, \vec{y}) = \mathbb{Z}\langle \Sigma \rangle$, but

$$\mu(\phi * [\Sigma]) = \mu(\phi) + 2.$$

So for these mflds, nothing further is needed.

Def. Call a HD weakly admissible for $s \in Spin^c(Y)$ if for every periodic domain P (i.e. nullhomology for α + β curves) with $n_z(P) = 0$,

and $\langle c_1(s), [P] \rangle = 0$,

then P has positive and negative coefficients.

Ex: $S^1 \times S^2$

$H_2(S^1 \times S^2) \cong \mathbb{Z}$

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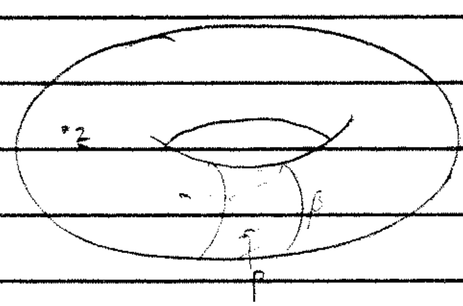
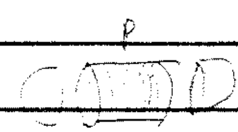


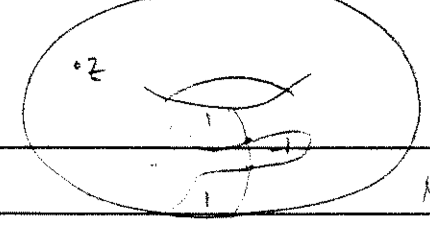


Consider the unique $s \in Spin^c(S^1 \times S^2)$ with $c_1(s) = 0$

But $P \geq 0$. So the HD is not weakly admissible.

has nonneg. coeff.

Ex: $S^1\times S^2$



α P

$[\alpha]=[\beta]$
$\Rightarrow [\alpha-\beta]=0$

Now, P has positive & negative multiplicities
$\Rightarrow$ HD is weakly admissible.

Note: If HD is weakly admissible for a torsion Spin^c-structure, (i.e. $c_1(s)$ is torsion in $H^2(Y)$)
$\Rightarrow$ HD is weakly admissible for every Spin^c-structure.

(Since every P is annihilated by $c_1(s)$ torsion)

Lemma. If a HD is weakly admissible for $s\in \text{Spin}^c(Y)$,
then given $\vec{x},\vec{y}\in\widehat{CF}(Y,s)$ or $CF^+(Y,s)$ (i.e. $s_z(\vec{x})=s_z(\vec{y})=s$),
$\exists$ only finitely many $\phi\in\pi_2(\vec{x},\vec{y})$ s.t. (1) $n_z(\phi)=k$ (or $n_z(\phi)\le k$)
and (2) $\mu(\phi)=1$
and (3) $D(\phi)\geq 0$.

Pf. Given ϕ s.t. (1), (2), (3) holds. Any other ϕ' satisfying (1), (2), (3) is of the form

$$\phi'=\phi * \underbrace{n[\Sigma]}_{n_z\neq 0} * \underbrace{P}_{n_z=0}$$

since $\pi_2(\vec{x},\vec{y})=\mathbb{Z}[\Sigma]\oplus\mathbb{Z}\langle\text{Periodic}\rangle$.

Now, $\mu(\phi')=\mu(\phi)+2n+\langle c_1(s),[P]\rangle$, $\mu(\phi')=\mu(\phi)=1$.

$$\Rightarrow -\frac{\langle c_1(s),[P]\rangle}{2}=n.$$

But since $n_z(\phi)=n_z(\phi')$, $n=0$.

Thus $\langle c_1(s),[P]\rangle=0$.

Admissibility tells us $P\not\geq 0$.

Assume there are infinitely many P s.t. $P\not\geq 0$, $\phi*P\geq 0$ & $\langle c_1(s),[P]\rangle=0$.

Case 1. (Simple.) When $H_2(Y)=\mathbb{Z}\langle P\rangle$.

$P\not\geq 0$, so considering $\phi*nP$ with $n\to\infty$, eventually gives us domains with negative coefficients.

Case 2. $H_2(Y)=\mathbb{Z}\oplus\mathbb{Z}=\mathbb{Z}\langle P_1\rangle\oplus\mathbb{Z}\langle P_2\rangle$.

Sps. $\phi=\phi*\underbrace{(n_1P_1+n_2P_2)}_{P}$

Sps. $\exists$ infinite number of such P for which $\phi_i=\phi*P^i\geq 0$.

If $n_1\to\infty$, then some coefficient of $D(\phi_i)\to-\infty$, unless $n_2\to+\infty$ and P_2 has cancelling coefficients.

This will eventually force a linear relation between the basis elements P_1 & P_2. $\to\Leftarrow$.

Case 3. $H_2(Y)$ higher dimensional. Similar to Case 2.

Def. A H.D. is <u>strongly admissible for $s \in Spin^c(Y)$</u> if $\forall$ P periodic domain with $n_z(P) = 0$ and $\langle c_1(s), [P] \rangle = 2n$, then P has some multiplicity $> n$.

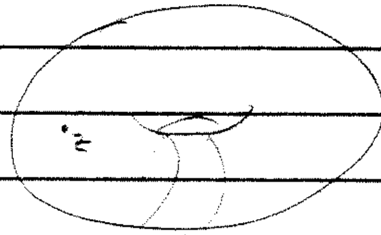
Lemma. If a H.D. is strongly admissible for $s \in Spin^c(Y)$, then $\exists$ only finitely many $\phi \in \pi_2(\vec{x}, \vec{y})$ s.t. (1) $\mu(\phi) = j$ (for any fixed j)
(2) $D(\phi) \geq 0$.

In particular, CF^- and CF^∞ have ∂ finite.

Lemma. Strong admissibility $\Rightarrow$ Weak admissibility for $s \in Spin^c(Y)$.

Henceforth, all diagrams for Y with $b_1(Y) > 0$ will need to be assumed to be weakly (strongly) admissible for $\widehat{CF}$ or CF^+ (CF^- or CF^∞).

Ex:
$S^1 \times S^2$

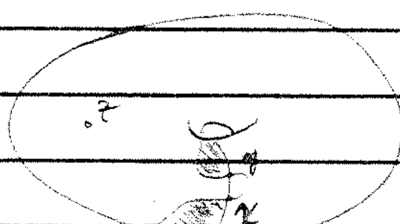


$\widehat{CF}^{\pm} = 0$

$\Rightarrow HF = 0$

$s_0 :=$ $Spin^c$-structure with $c_1(s) = 0$.

Ex:



$\widehat{CF}(S^1 \times S^2, s_0) = \mathbb{Z}/2\langle x \rangle \oplus \mathbb{Z}/2\langle y \rangle$

$CF^+(S^1 \times S^2, s_0) = \mathbb{Z}/2[U, U^{-1}] \big/ U\mathbb{Z}[U] \, (\langle x \rangle + \langle y \rangle)$

$\partial x = \mathcal{M}\Big($  $\Big) \cdot y + \mathcal{M}\Big($  $\Big) \cdot y + 0 \cdot x = 0 \pmod 2$

$\qquad\qquad\;\; \shortparallel \qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\;\; 1 \qquad\qquad\qquad\qquad\; 1$

Can't add any positive copies of this (would get neg. domains)

$^\wedge$ same mod 2 grading
$\Rightarrow$ no $\mu(\phi) = 1$ disk from x to x.

Can only subtract 1 copy of this for some reason.

$\partial y = 0$ since $gr(y) - gr(x) = \mu(\phi) - 2n_z(\phi) = -1 - 0$

So $\widehat{HF} = \widehat{CF} \cong (\mathbb{Z}/2)_* \oplus (\mathbb{Z}/2)_{*-1}$

$HF^+ = \widehat{CF}$.

<u>Digression</u>

Instead of doing all of this, we could just expand our coefficient groups to allow for the possibilities of infinite sums from the ∂ operator.

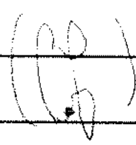
<u>Novikov coefficients</u>

$H_2(Y) = \mathbb{Z}, \quad \mathbb{Z}/2[\mathbb{Z}] = \mathbb{Z}/2[T, T^{-1}]$

$$\widehat{CF} \otimes_{\mathbb{Z}/2} \mathbb{Z}/2\left[H_2(Y)\right]$$

$$\mathbb{Z}/2\left[T, T^{-1}\right]$$

Mark a pt. in ∂ and count how many times ∂ of disks use the point.

$\partial x =$  $\cdot T \cdot y + y$

$$\partial x = (1+T)y$$

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Last Time: Admissibility (necessary on Y^3 with $b_1(Y^3) > 0$)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; rk(H_1(Y)) - rk(H_2(Y))$

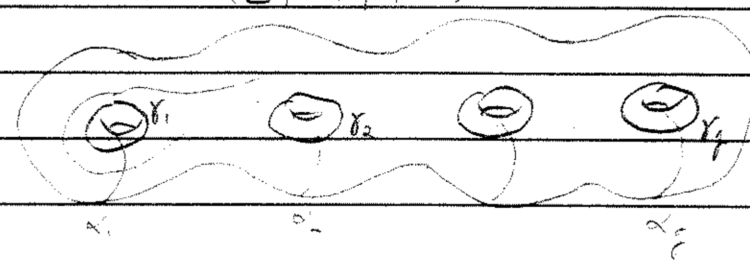
Lemma. Given (Y^3, s), $s \in Spin^c(Y)$, $\exists$ a H.D st

(1) $s = s_z(\vec{x})$ for some $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

(2) HD is (strongly) admissible for s.

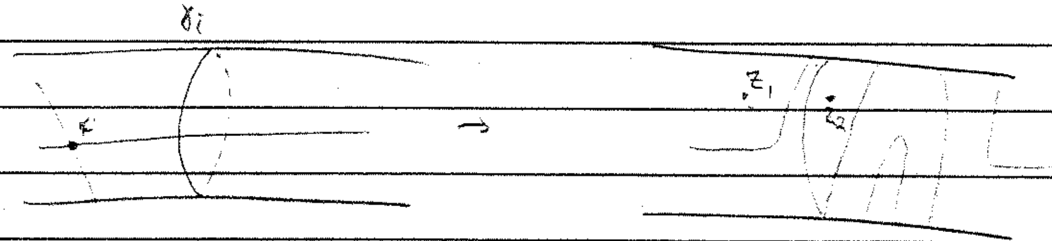
Pf. "Winding"

Start w/ an arbitrary HD. $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$



Choose γ_i dual to α_i s.t. $\{\alpha_i\} \cup \{\gamma_i\}$ generate $H_1(Y)$.

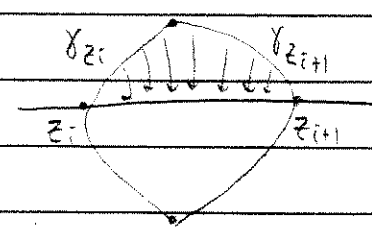
Examine the effect of winding α_i over γ_i N times



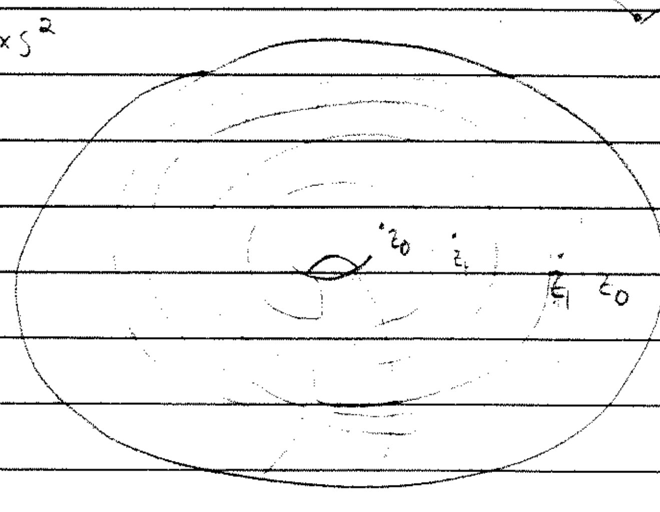
$$s_{z_i}(\vec{x}) - s_{z_{i+1}}(\vec{x}) = PD\Big[\underbrace{\gamma_{z_i} \cup -\gamma_{z_{i+1}}}_{\text{homologous to } [\gamma_i]}\Big]$$

Winding around every γ_i, we take

$$s_z(\vec{x}) \xrightarrow{\text{winding}} s_{z'}(\vec{x}) = s_z(\vec{x}) - \sum_{i=1}^{g} n_i \underset{\substack{\Uparrow \\ H^2(Y)}}{PD[\gamma_i]}$$



Ex: $S^1 \times S^2$

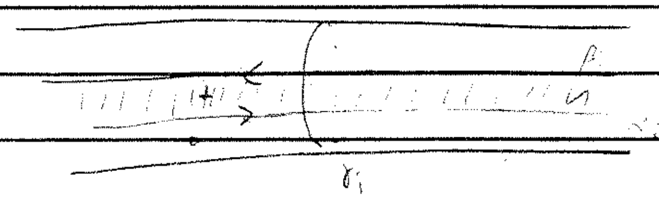


$PD(\gamma_{z_0} \cup -\gamma_{z_1}) = -PD\langle S^1 \rangle$

$PD(\gamma_{z_0} \cup -\gamma_{z_2}) = PD\langle S^1 \rangle$

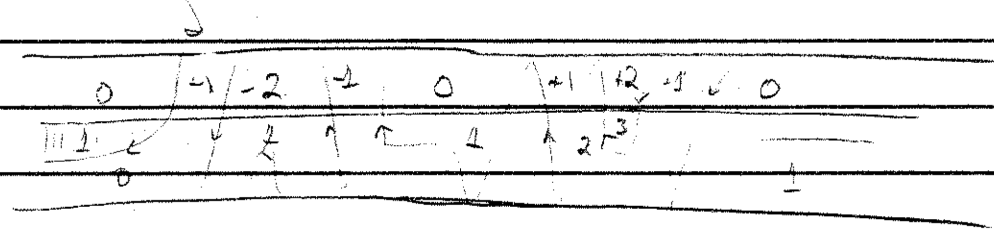
Note: Any HD has only a finite # of intersection points, so unless $Y^3 = \mathbb{Q}HS^3$, one cannot hope to represent every Spin^c-structure by pts. of intersection simultaneously.

(2) Achieving admissibility is ~~essentially~~ the same trick.



Sps. P is a periodic domain, & $\partial P = \alpha_i + \sum_{j \neq i} n_j \alpha_j + \sum m_j \beta_j$

Forced.



This shows that we can achieve weak admissibility, because winding is a local operation and we can take any periodic domain, and modify it so that it has arbitrarily high positive & negative coefficients. Strong admissibility is also achieved in this way, but requires keeping track of more stuff (See O.S.). $\square$

Adjunction Inequality

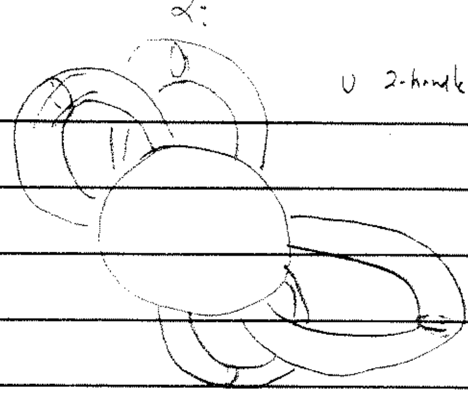
Thm. Suppose $HF^+(Y, s) \neq 0$. Then for any $\Sigma \hookrightarrow Y^3$, smoothly embedded, closed, oriented, and $g(\Sigma) > 0$, we have

$$|\langle c_1(s), [\Sigma] \rangle| \leq 2g(\Sigma) - 2 = -\chi(\Sigma) \qquad (*)$$

Pf. Since both sides are additive for disjoint union, it suffices to prove for connected surfaces.

Idea: Given any embedding $\Sigma \hookrightarrow Y$, $g(\Sigma) = g$, find an explicit HD where no spin^c-structure violating $(*)$ is realized.

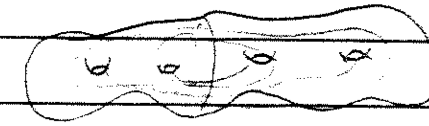
Given Σ,

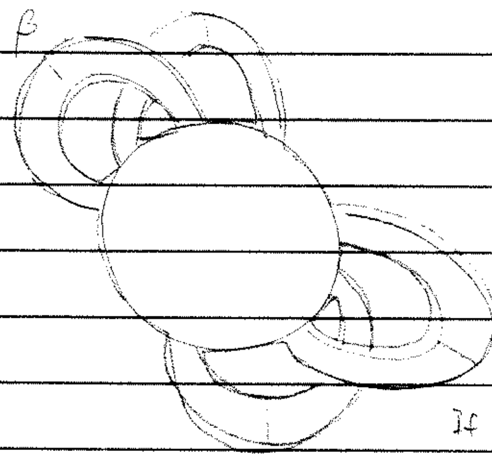


$n(\Sigma) =$ $\cong$

$\cong$

$\Sigma \times I$





Extend the Morse function on $n(\Sigma)$, giving rise to HD_1, & a Morse function f on all of Y.

Want to do this further, so that f

(1) Self-indexing

(2) has a unique index 0 and index 3 critical pt.

If one had multiple index 0 crit pts, then there must be "a unique flow to an index 1 critical pt." (Else, $H_0(Y) \not\cong \mathbb{Z}$).

But then these can be cancelled.

Can do the same to eliminate all but one index 3 crit. pt.

Only thing is be careful of is that we don't cancel a ind(3) crit pt. w/ ind(2) crit. pt. associated to β.

(Sps. m, n are two ind 3 crit. pts,

$\partial m = \beta$ Then m, n are cycles in $H^3(Y, \Sigma)$

$\partial n = 0$. $\Rightarrow H^3(Y, \Sigma) \geq \mathbb{Z} \oplus \mathbb{Z}$

$\cong$

$H_0(Y - \Sigma)$.)

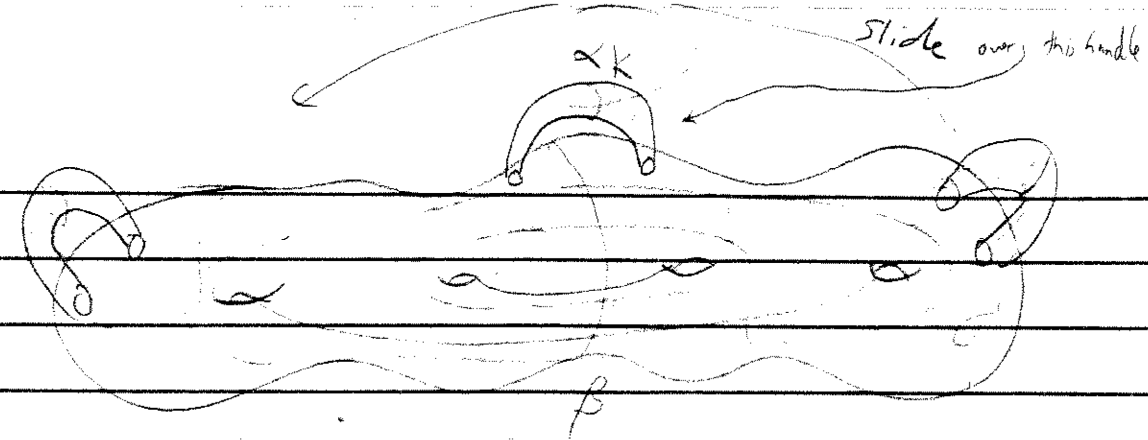
So we extend the Morse function f on $nb(\Sigma)$ to all Y s.t.

(1) f self-indexing

(2) f gives rise to a H.D. with our HD_1 as a sub-surface.

(3) $\exists$ 1-handle for the extension w/ feet on either side of β.

(4) By handle sliding, WMA all additional 1-handles have feet to the left of β (except one in (3)).



Claim: $\nexists\ \vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ s.t. $\langle c_1(s_z(x)), [\Sigma] \rangle < -2g(\Sigma) + 2$

Note: $[\Sigma] = [P]$, where P is the periodic domain on the "right" of β, and whose $\partial P = \beta \cup \alpha_k$.

& if $b_1(Y) = 1$ (i.e. $H_2(Y) = \mathbb{Z}\langle \Sigma \rangle$)

then HD is strongly admissible for any s s.t. $\langle c_1(s), [\Sigma] \rangle < 0$.

$\wedge$ weakly

Now, recall that $\langle c_1(s_z(x)), [P] \rangle = \hat{\chi}(P) + 2n_x(P) - \cancel{2n_z(P)}^{\,=0 \text{ b/c.}}$

$$= \underset{\uparrow \text{ in this case}}{\chi(P)} + 2n_x(P)$$

$$= -2g(\Sigma) + \underbrace{2n_x(P)}_{\geq (2)(\frac{1}{2}+\frac{1}{2})}$$ coming from the 2 intersections on β & α_k.

$$\geq -2g(\Sigma) + 2$$

Conditions on Domains relevant to determining $\# \widehat{\mathcal{M}(\phi)}$.

(1) If $\widehat{\mathcal{M}(\phi)} \neq \emptyset$, then $D(\phi) \geq 0$.

(2) $\mu(\phi) = 1$. If $\overline{D(\phi)} = \overline{D(\phi_1)} \sqcup \overline{D(\phi_2)}$,

then $\# \widehat{\mathcal{M}(\phi)} \neq 0$

$\Rightarrow$ Either $D(\phi_1)$ or $D(\phi_2)$ $\equiv 0$.

Pf. $\mu(\phi) = 1 \Rightarrow \mu(\phi_1) = n$ & $\mu(\phi_2) = 1 - n$.

But if $|n| > 1$, then $\mu(\phi_1)$ or $\mu(\phi_2)$ is negative

$\Rightarrow \mathcal{M}(\phi_1)$ or $\mathcal{M}(\phi_2)$ is empty

$\Rightarrow \mathcal{M}(\phi) = \emptyset$.

Thm

Thm $\mu(\phi) = 1$, $\overline{D(\phi)} = \overline{D(\phi_1)} \sqcup \overline{D(\phi_2)} \Rightarrow \mathcal{M}(\phi) = \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2)$.

If $n = 1$ or 0, then $\mathcal{M}(\phi) = \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2)$ where

$\mu(\phi_1) = 1$ $\mu(\phi_2) = 0$

or $\mu(\phi_1) = 0$ $\mu(\phi_2) = 1$

But if $\mu(\phi_i) = 0 = \dim(\mathcal{M}(\phi_i)) \Rightarrow \dim(\widehat{\mathcal{M}(\phi_i)}) = -1$

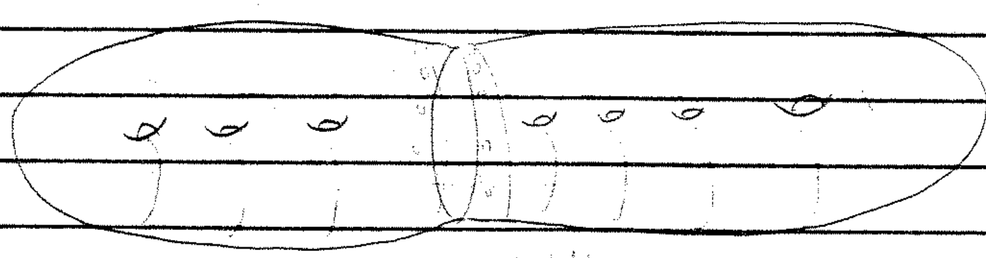
$\Rightarrow \widehat{\mathcal{M}(\phi_i)} = \emptyset$ unless $\mathbb{R}$-action isn't free, i.e. ϕ_i = constant.

(3) If $D(\phi) = \begin{cases} \text{bigon} \\ \text{rectangle} \end{cases}$ ~~with $\mu(\phi) = 1 \Rightarrow \# \widehat{\mathcal{M}(\phi)} = \pm 1$~~

Thm. If $D(\phi) = D(\phi_1) \sqcup D(\phi_2)$, then $\mathcal{M}(\phi) = \mathcal{M}(\phi_1) \sqcup \mathcal{M}(\phi_2)$

Pf. We saw last time, hypothesis implies

$\phi \in \pi_2(\vec{x}, \vec{y})$, then $\phi = \phi_1 * \phi_2$, with $\phi_1 \in \pi_2(\vec{x}, \vec{z})$, $\phi_2 \in \pi_2(\vec{z}, \vec{y})$



$D(\phi_1)$ supported on the left

$D(\phi_2)$ " " " right.

Suppose $\{u \in \mathcal{M}(\phi)\} \longleftrightarrow \left\{ \begin{array}{c} (F^2, \ell) \xrightarrow{\Phi} (\Sigma, j) \\ \downarrow \pi \\ (\mathbb{D}^2, i) \end{array} \right\}$

F^2 Riemann surf. w/ ∂.

Φ, π holomorphic

π g-fold branched covering

$$D(\phi) = \sum_{z_i \in D_i} \left(\text{mult. of } \Phi \text{ at } z_i \right) \cdot D_i$$

Signed number of pts. in preimage of z_i under Φ.

But Φ is orientation preserving (b/c holomorphic),

$$s_i = \# \{ \Phi^{-1}(z_i) \}.$$

The fact that $D(\phi)$ splits $\Rightarrow$

$$\begin{array}{ccccc} F^2 = & F^2_{\Phi_1} & \sqcup & F^2_{\Phi_2} & \xrightarrow{\Phi|_{\Phi_1}} \Sigma \\ & \downarrow \pi|_{F_{\Phi_1}} & & \downarrow \pi|_{F_{\Phi_2}} & \Phi|_{\Phi_2} \\ & \mathbb{D}^2 & & \mathbb{D}^2 & \end{array}$$

Which yields $\left\{ \begin{array}{c} F_{\Phi_1} \xrightarrow{\Phi_1} \Sigma \\ \downarrow \pi_1 \\ \mathbb{D}^2 \end{array} \right\} \sqcup \left\{ \begin{array}{c} F_{\Phi_2} \xrightarrow{\Phi_2} \Sigma \\ \downarrow \pi_2 \\ \mathbb{D}^2 \end{array} \right\}$

Conversely, given such a disjoint set of maps (i.e. something in $\mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2)$),

we can clearly reverse this.

$$\therefore \quad \mathcal{M}(\phi) = \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2). \qquad \square$$

Cor. $\widehat{HF}(Y_1 \# Y_2, s_1 \# s_2) \cong H_*\left(\widehat{CF}(Y_1, s_1) \otimes \widehat{CF}(Y_2, s_2), \partial\right)$

where $Y_1 \# Y_2$ denotes the connected sum of Y_1 & Y_2.

and $s_1 \# s_2$ denotes $[v_1 \sqcup_{S^2} v_2]$ where $[v_1] = s_1$, $[v_2] = s_2$.

(w/ $\mathbb{Z}_2$-coefficients, $\widehat{HF}(Y_1 \# Y_2, s_1 \# s_2) \cong \widehat{HF}(Y_1, s_1) \otimes_{\mathbb{Z}_2} \widehat{HF}(Y_2, s_2)$.)

Thm. $HF^-(Y_1 \# Y_2, s_1 \# s_2) \cong H_*\left(CF^-(Y_1, s_1) \otimes_{\mathbb{Z}[u]} CF^-(Y_2, s_2), \partial\right)$

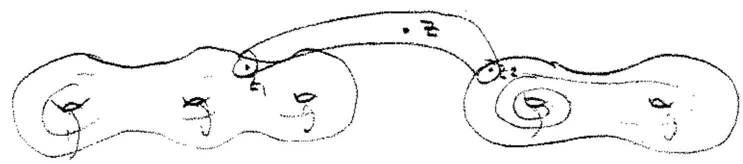
$HF^\infty(Y_1 \# Y_2, s_1 \# s_2) \cong H_*\left(CF^\infty(Y_1, s_1) \otimes_{\mathbb{Z}[u, u^{-1}]} CF^\infty(Y_2, s_2), \partial\right)$

Rmk. Using l.e.s. $\cdots \to HF^- \to HF^\infty \to HF^+ \to \cdots$

Pf. of Cor.

$Y_1 \# Y_2$ has a nice H.D.: $(\Sigma_1 \# \Sigma_2, \vec{\alpha}_1 \sqcup \vec{\alpha}_2, \vec{\beta}_1 \sqcup \vec{\beta}_2, z)$,

where $(\Sigma_i; \vec{\alpha}_i, \vec{\beta}_i, z)$ is a H.D. for Y_i, & $\Sigma_1 \# \Sigma_2$ is conn. sum at z_i.



(1) Exercise: Prove/see that this is indeed a H.D. for $Y_1 \# Y_2$. (2) Show $s_z(x_1 \times x_2) = s_{z_1}(x_1) \# s_{z_2}(x_2)$

Consider $\widehat{CF}(Y_1 \# Y_2, s_1 \# s_2) = \bigoplus_{\mathbb{T}_{\alpha_1 \cup \alpha_2} \cap \mathbb{T}_{\beta_1 \cup \beta_2}} \mathbb{Z}/2 \langle x_1 \times x_2 \rangle$

where $x_i \in \mathbb{T}_{\alpha_i} \cap \mathbb{T}_{\beta_i}$

So we have an isomorphism of chain groups $\widehat{CF}(Y_1 \# Y_2, s_1 \# s_2) \cong \widehat{CF}(Y_1, s_1) \otimes \widehat{CF}(Y_2, s_2)$.

$$\hat{\partial}(x_1 \otimes x_2) = \hat{\partial} x_1 \otimes x_2 + x_1 \otimes \hat{\partial} x_2 \quad (\text{No } -1 \text{ b/c } \mathbb{Z}/2)$$

$\hat{\partial}$ on $\widehat{CF}(Y_1 \# Y_2)$ counts J-holo disks in $\widehat{\mathcal{M}}(\phi)$ for some $\phi \in \pi_2(\vec{x}_1 \times \vec{x}_2, \vec{y}_1 \times \vec{y}_2)$

$$\partial(x_1 \times x_2) = \sum_{y_1 \times y_2} \sum_{\substack{\phi \in \pi_2 \\ n_z(\phi) = 0}} \# \widehat{\mathcal{M}}(\phi) \cdot (y_1 \times y_2)$$

$\Rightarrow D(\phi)$ has mult. 0 at z.

$\Rightarrow \overline{D(\phi)} = \overline{D(\phi_1)} \sqcup \overline{D(\phi_2)} \overset{\text{Thm.}}{\Rightarrow} \mathcal{M}(\phi) = \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2)$

$\overset{\text{Cor.}}{\Longrightarrow}$ But we saw then that either ϕ_1 or ϕ_2 was constant.

Thus $\mathcal{M}(\phi) = \mathcal{M}(\phi_1) \times pt. \sqcup pt. \times \mathcal{M}(\phi_2)$

$\phi_1 \in \pi_2(\vec{x}_1, \vec{y}_1)$ ↑ const. map to $\vec{x}_2$ ↑ const. map to $\vec{x}_1$ $\phi_2 \in \pi_2(\vec{x}_2, \vec{y}_2)$

$\vec{y}_1 \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$ $\vec{y}_2 \in \mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$

i.e. term in $\hat{\partial}\vec{x}_1 \otimes \vec{x}_2$ and in $\vec{x}_1 \otimes \partial \vec{x}_2$.

Cor (of Cor.)

$\widehat{HF}_*(Y)$ is independent of the HD upto stabilization.

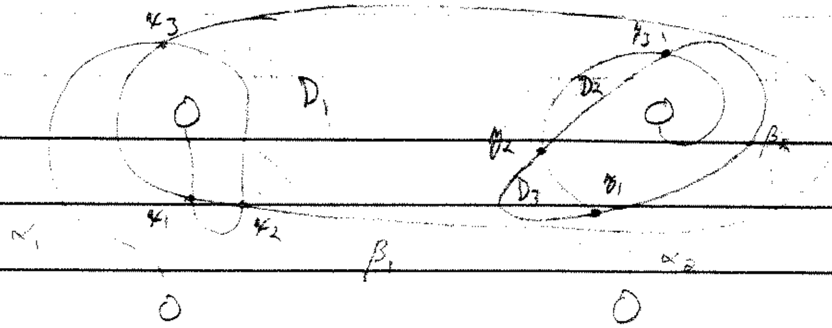
Pf Stabilization is the same as connect summing w/ a standard genus 1 H.D. of S^3. $(T^2, \alpha, \beta, *)$

The $(T^2, \alpha, \beta, *)$ has $\widehat{HF} \cong \mathbb{Z}/2$.

Tensoring w/ this does not change anything. $\square$

$Y = S^3$

$\widehat{HF}(Y)$ better be $\mathbb{Z}/2$!



$\phi_1 = D_1 + D_2$ connects

$x_3 y_3$ to $x_2 y_3$

$\phi_2 = D_1 + D_3$ connects

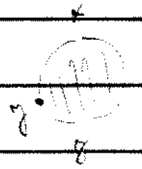
$x_3 y_1$ to $x_2 y_1$

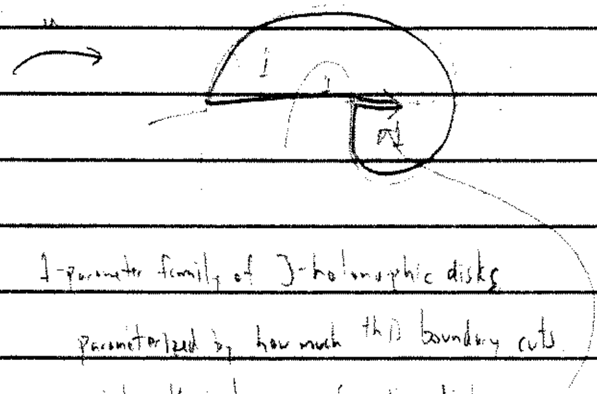
Depending on the almost complex structure, either ϕ_1 or ϕ_2 will have a holomorphic representative.

So this theory does <u>not</u> depend solely on the combinatorics of the diagram.

Cor. of Last Time. $\widehat{HF}$ is invariant under stabilization of HD's.

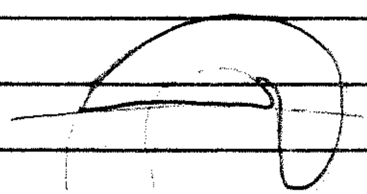
Corresponding invariance for HF^-, HF^∞, HF^+

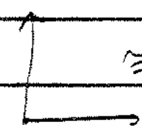




$$\mu(\phi) = 2\left(\hat{\chi}(\text{bigon})\right) + n_x(D(\phi)) + n_y(D(\phi)) + 1\,\hat{\chi}(\text{rectangle})$$
$$= 2\left(\tfrac{1}{2}\right) + \tfrac{1}{4} + 1(0) + \tfrac{3}{4}$$
$$= 2$$

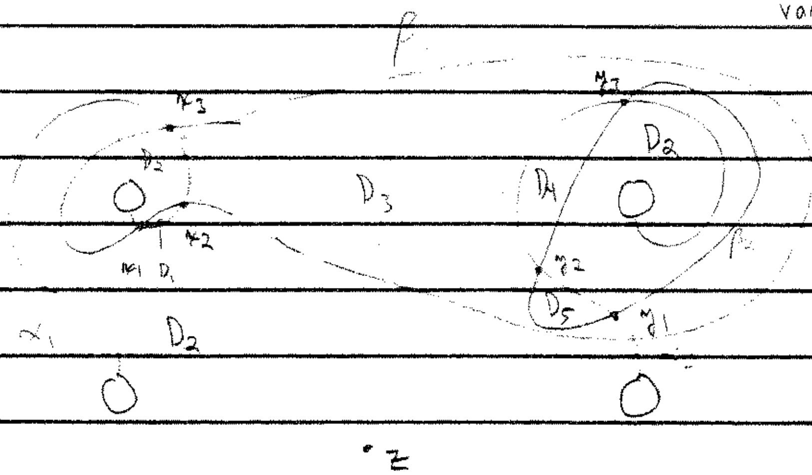
1-parameter family of J-holomorphic disks parameterized by how much the β boundary cuts into the interior of the disk





$\cong \mathbb{R}$

Real-line's worth of variation



$\alpha_1 \cap \beta_1$	$\alpha_1 \cap \beta_2$	$\alpha_2 \cap \beta_1$	$\alpha_2 \cap \beta_2$
$x_1\ x_2\ x_3$	$\emptyset$	Irrelevant	$y_1\ y_2\ y_3$

$\Rightarrow$ 9 generators $\{x_i y_j\}_{i,j=1,2,3}$

Also $H_2(S^3) \cong 0 \Rightarrow \pi_2(x, x) \cong \mathbb{Z} \oplus 0$

Let's work with $\widehat{HF}$ (i.e. requires $n_z(\phi) = 0$.)

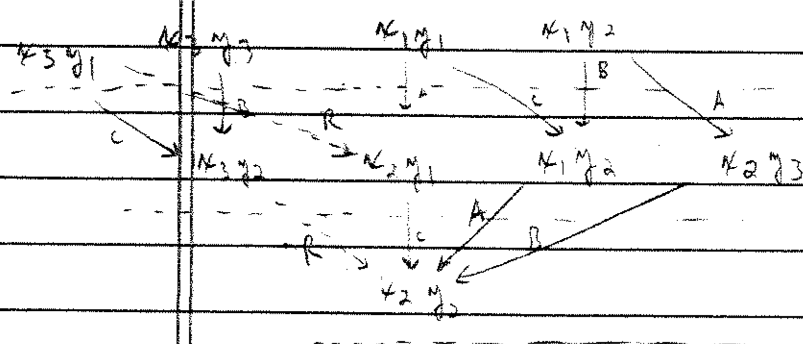
Note further: $\epsilon(\vec{x},\vec{y}) \in H_1(S^3) = 0 \Rightarrow \pi_2(\vec{x},\vec{y}) \neq \emptyset \ \forall \vec{x},\vec{y}$.

$\exists\ \phi_i \in \pi_2(x_1 y_i, x_2 y_i)$, $D(\phi_i) = 1 \cdot D_1$ $\forall i = 1,2,3$
$\mu(\phi_i) = 1$ & $\# \widehat{\mathcal{M}}(\phi_i) = 1 \pmod 2$ } A disks

Similarly $\exists\ \psi_i \in \pi_2(x_i y_3, x_i y_2)$ $\forall i = 1,2,3$
$\mu(\psi_i) = 1$ & $\# \widehat{\mathcal{M}}(\psi_i) = 1 \pmod 2$ } B disks

And $\exists\ \eta_i \in \pi_2(y_1 x_i, y_2 x_i)$ $\forall i = 1,2,3$
$\mu(\eta_i) = 1$ & $\# \widehat{\mathcal{M}}(\eta_i) = 1 \pmod 2$ } C disks

Recall, $gr(\vec{x},\vec{y}) = \mu(\phi) - 2\cancel{n_z(\phi)}^{0}$
$gr(x_1 y_i) = gr(x_2 y_i) + 1$
$gr(y_3 x_i, y_2 x_i) = 1$
$gr(y_1 x_i, y_2 x_i) = 1$

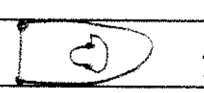


Can't get at the x_3's via these disks.

Further, since $\epsilon(\vec{x},\vec{y}) = 0\ \forall \vec{x},\vec{y}$, there are Whitney disks between any pair of pts. (w/ the right grading difference).

The solid arrows are when we know there is a holomorphic rep.

$R \in \pi_2(x_3 y_2, x_2 y_2)$, $D(R) = D_3 + D_4$

$D_3 =$  $\hat{\chi}(D_3) = \chi(\cancel{\text{annulus}}^{0}) - 4\left(\frac{1}{4}\right)$

$D_4 =$  $\hat{\chi}(D_4) = \frac{1}{2}$

so $\hat{\chi}(D(R)) = -\frac{1}{2}$ $\quad n_{x_3 y_2} = \frac{3}{4}$ $\quad n_{x_2 y_2} = \frac{3}{4}$ $\quad \Rightarrow \mu(R) = 1$.

In fact, $R_i \in \pi_2(x_3 y_i, x_2 y_i)$

$$\begin{tikzcd}[column sep=small] & x_1y_1 \arrow[dr, "\eta_1"] & \\ x_3y_1 \arrow[r, "R_1"] & x_2y_1 \arrow[u, "\phi_1^{-1}"'] & x_1y_2 \end{tikzcd}$$

$\Rightarrow \quad R_1 * \phi_1^{-1} * \eta_1 \in \pi_2(x_3y_1, x_1y_2)$

$D(R_1 * \phi_1^{-1} * \eta_1) = D(R_1) - D(\phi_1) + D(\eta_1)$

$= D_3 + D_4 - D_1 + D_5$

$\widehat{\mu}(R_1 * \phi_1^{-1} * \eta_1) = \emptyset$ by positivity

Similarly, $R_1 * \phi_1^{-1} * \eta_1 * \psi_1^{-1} * \phi_3$ connects x_3y_1 to x_2y_3

$D(R_1 * \phi_1^{-1} * \eta_1 * \psi_1^{-1} * \phi_3) = D_3 + D_5$

~~x_3y_1 x_3y_3 x_1y_1 x_1y_3~~

~~x_3y_2 x_2y_1 x_1y_2 x_2y_3~~

~~x_2y_2~~

If we know disk exists, solid arrow.

No arrow means considerations

of positivity etc. means

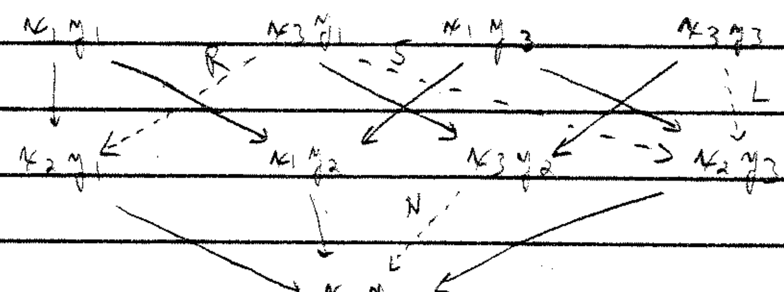
there is no disk.

Dotted arrow means we are not

sure yet.

(Homotopy theoretic considerations

allow it)

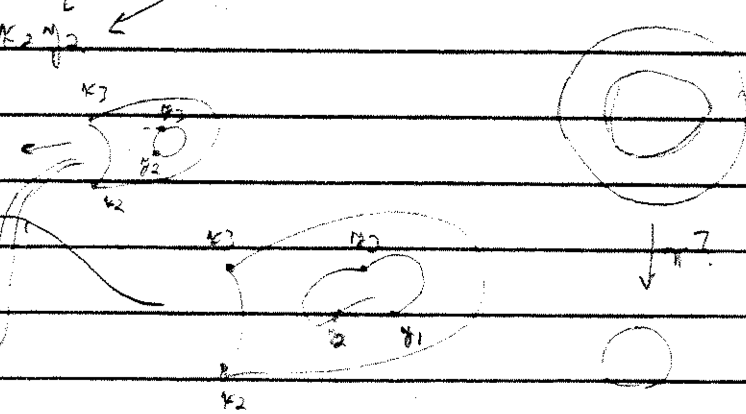


$D(R) = D_3 + D_4$

$D(S) = D_3 + D_5$

$D(L) = D_3 + D_4$

$D(N) = D_3 + D_4$



Lemma. Let $A = \{ z \in \mathbb{C} \mid \frac{1}{r} < |z| < r \}$

Then $\exists !\ A \xrightarrow{f} A$ which is holomorphic & maps

$$|z|_{\frac{1}{r}} \to |z|_r.$$

The map is $f(z) = \frac{1}{z}$.

The unique involution of A takes red $\leftrightarrow$ red, blue $\leftrightarrow$ blue

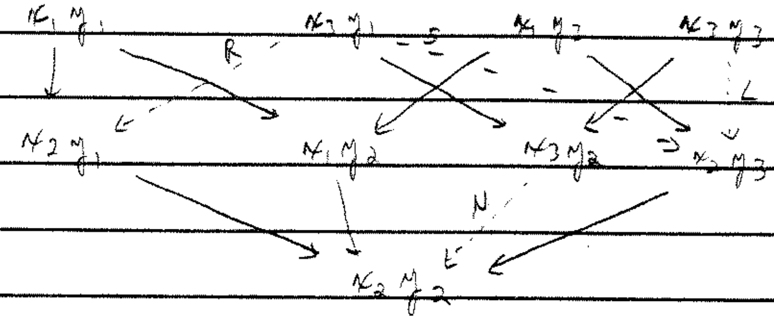
if and only if (the angle swept by outer red) = (the angle swept by inner red)

Claim: The answer will depend on the length of red cut along γ_1 for $D(R)$

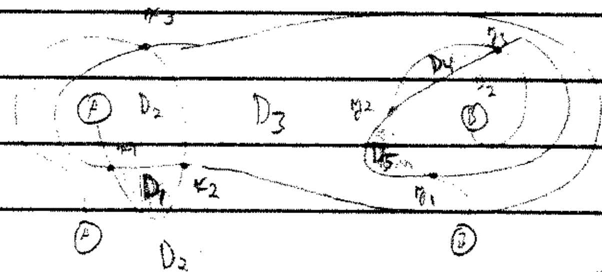
red " " γ_3 to γ_2 for $D(S)$, $D(N)$

blue " " γ_1 for $D(S)$.

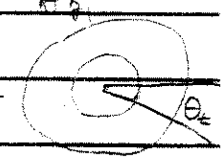
Matt Hedden HFH 11/18/10



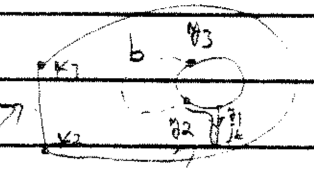
$H_1(S^3) \ni \varepsilon(\vec{x}, \vec{y}) = 0 \quad \forall x, y \qquad \pi_2(\vec{x}\,\vec{y}) = \mathbb{Z} \oplus \cancel{H_2(S^3)}$ $\rightarrow 0$



Where t is the length of the cut of this disk



$D_3 + D_4$

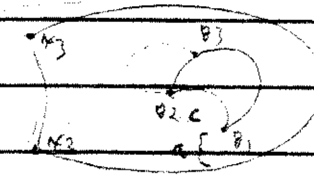


$D(R) = D_3 + D_4$

$D(L) = D_3 + D_4$

$D(S) = D_3 + D_5$

$D(N) = D_3 + D_4$



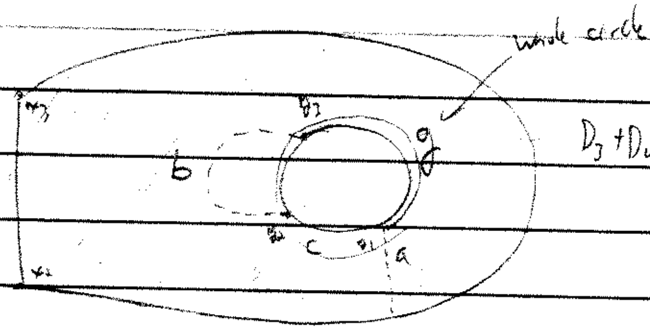
$D_3 + D_5$

Recall Lemma: $\exists!$ involution on the annulus.

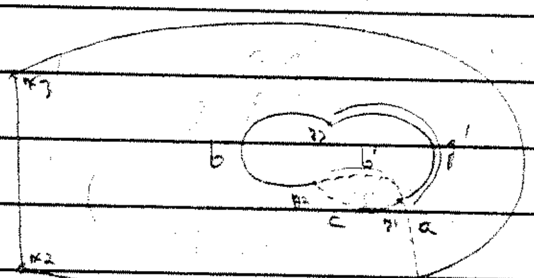
So the question is, when is the interior red angle equal to the exterior red angle?

If the length of this arc (along which we're cutting) is greater than twice the length of the red arc from x_2 to x_3, then there is some parameter for which $D(R)$ has a holomorphic rep.

Let a, b, and c be the lengths of the arcs labeled (w/ the fixed alm. cpx str. on Σ)



$D_3 + D_4$



$D_3 + D_5$

* *

Note: Even though Domains of R, L, N are the same, the ∂-pts. determine which arcs we can cut along!

Have holo rep. if

$D(R) = D_3 + D_4$

$D(L) = D_3 + D_4$ if $|b|$ is big enough

Yes or No together.

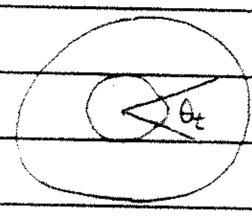
$D(S) = D_3 + D_5$ depending on b & c

$D(N) = D_3 + D_4$ if $|b|$ is big enough

WLOG, assume the outside red & blue arcs sweep out the same angle.

Let's assume $b+b' > g$!

L & N:



t is how far we cut along $\underline{b}$.

$\frac{g}{2b} < 1 \iff$ L & N admit holo. reps.

S: $\frac{b+b'}{g'+2c} < 1 \Rightarrow$ S has holo rep. OR ~~$\frac{g'}{b+b'+2a} < 1 \Rightarrow$ S has holo. rep.~~

cutting along c

R: $\frac{g}{2a} < 1 \iff$ R has holo. rep.

$\boxed{\text{Assume } |c| = |c'|,\ |b| = |b'|.}$

Then things simplify to

$g < 2b$ or $g > 2b$

$\Rightarrow$ If $b+b' > g$, then cutting along a doesn't help at all.

only cutting in along c.

$\Rightarrow$ Either S or (L & N). Length of a is independent, so R is yes or no independently.

$g < 2b$
$g > 2b$
$g < 2a$
$g > 2a$

$x_1 y_1$ $\quad$ $x_3 y_1$ $\quad$ $x_1 y_3$ $\quad$ $x_3 y_3$

$x_2 y_1$ $\quad$ $x_1 y_2$ $\quad$ $x_3 y_2$ $\quad$ $x_2 y_3$

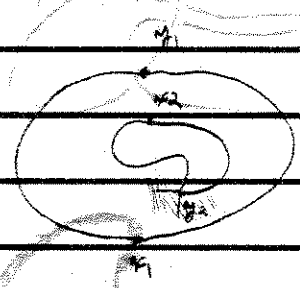
$x_2 y_1$

Exercise: Verify that these give rise to the same homology.

Exercise: Show that by isotoping α & β curves (i.e. changing the cpx structure on Σ) we can ensure that

$$\hat{\mu}(\phi) = \pm 1 \quad \text{for } \phi \in \pi_2(\vec{x}, \vec{y})$$

with $D(\phi)$ = Annulus with 1 obtuse corner.



i.e. $n_{x_1} = n_{x_2} = n_{y_1} = \frac{1}{4}$

$n_{y_2} = \frac{3}{4}$.