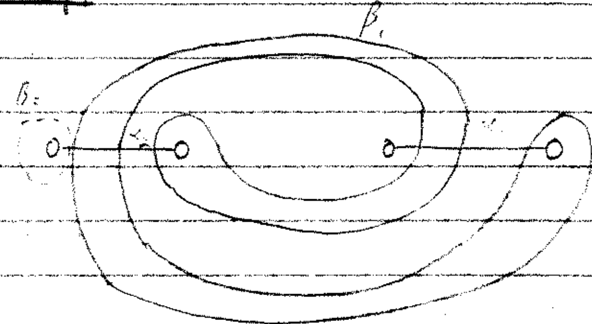
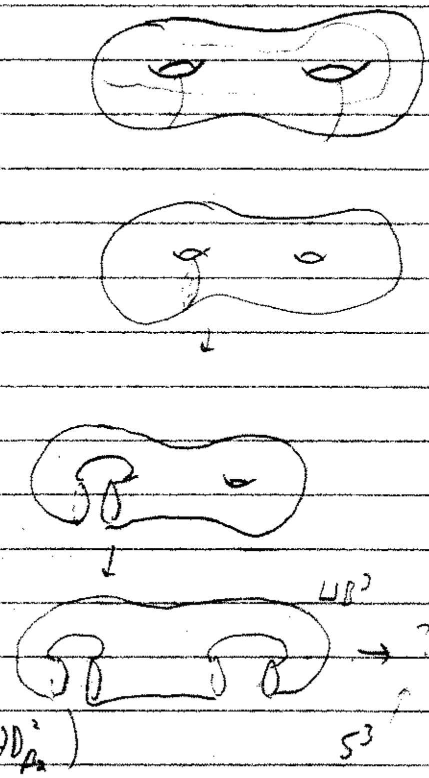
Matt Hedden - HFH - 1/13/10

Knot Floer Homology





$(\Sigma, \alpha_1, \alpha_2, \beta_1) \rightsquigarrow Y$ with $\partial Y = T^2$

Claim: $Y \cong S^3 \setminus \text{nbhd}(\text{trefoil})$

Pf.

- We can see S^3 by doing handle-slides of β_1 around β_2 to see a stabilized H.D.

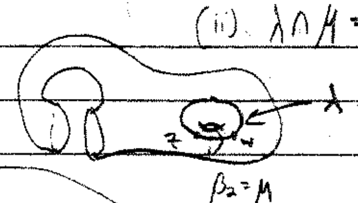
  After isotopy, it is clear that $Y \cup S^1 \times D^2_{\beta_2} \cong S^3$.

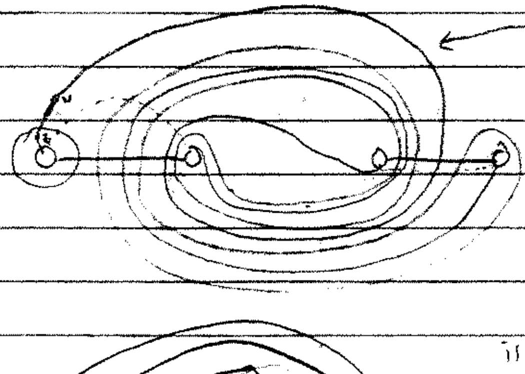
- β_2 is the meridian of $S^1 \times D^2_{\beta_2}$ (i.e. $\beta_2 \sim pt. \times \partial D^2_{\beta_2}$)

  And β_2 is meridian of knot (μ).

  To find a longitude for K, we need a curve λ s.t. (i) $\lambda \subseteq \partial Y$

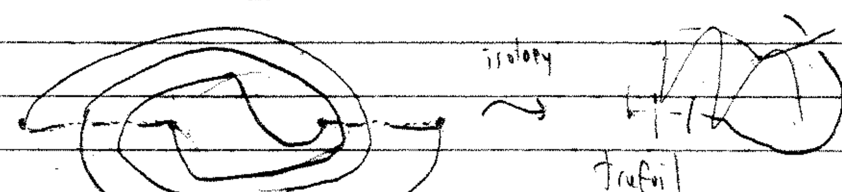
  (ii) $\lambda \cap \mu = 1$



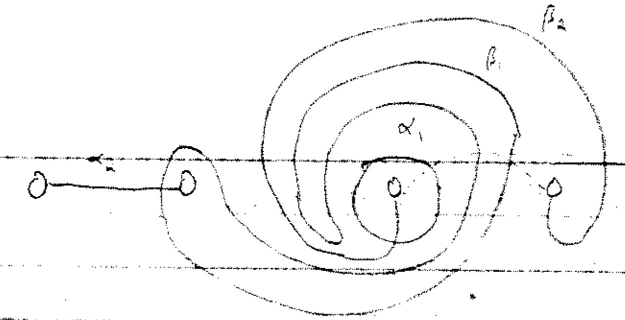


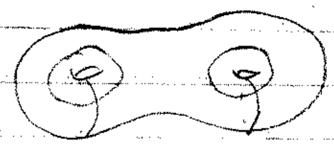
Goal: Find arc on H.D. that avoids β_1 & β_2.

Projection of the knot



Trefoil





While this is a H.D. for S^3, the curves don't bound disks in the handlebodies we can picture - only abstractly.

So going through the same process, we could come up with a knot (ignoring β_2), and a projection for the knot, but it will be a projection onto the abstract Heegaard surface, not what we think of as a projection of the knot (in S^3).

Def- A H.D. adapted to $K \subset Y$ is the following: $(\Sigma_g, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\}, z, w)$ s.t.

(1) $(\Sigma_g, \vec{\alpha}, \vec{\beta})$ is a H.D. for Y.

(2) $t_a' \cup t_b \cong K$, where t_a is an arc on Σ connecting z to w in the complement of $\vec{\beta}$

t_b is an arc on Σ connecting w to z in the complement of $\vec{\alpha}$.

t_a' is a slight push-off of t_a into the handlebody specified by $\vec{\beta}$.

We will call this a "doubly pointed H.D. adapted to K."

Prop. • Any $K \subseteq Y$ has an adapted Heegaard diagram.

• Any two such diagrams can be connected by

pointed Heegaard moves:

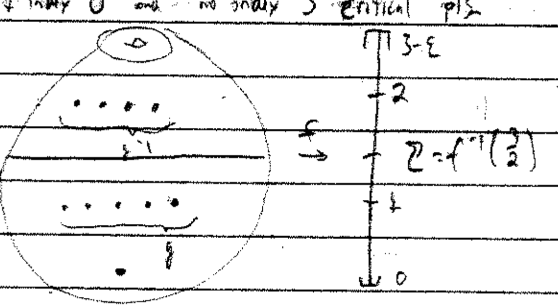
(1) Isotopies not crossing the base points.

(2) Handleslides not crossing the base points.

(3) (de-) stabilizations

"Pf" of Existence

First, find a H.D. for $Y - n(K)$. To do this, consider a self-indexing Morse function in $Y - n(K)$ with ~~1~~ index 0 and no index 3 critical pts.



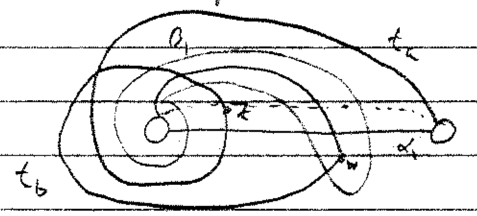
$\vec{\alpha} = \underset{\text{of ind. 1}}{\text{ascending mflds}} \cap \Sigma$

$\beta = \underset{\text{of ind. 2}}{\text{descending mflds}} \cap \Sigma.$

Once we have H.D. for $Y - n(K)$, we find a curve on $\left(\Sigma \text{ surgered along } \vec{\beta} \right) \cong T^2$
which is a meridian of K.
It suffices to find curve μ that is disjoint from $\vec{\beta}$ and for which $\left(\Sigma, \{\vec{\alpha}\}, \{\vec{\beta} \cup \mu\} \right)$ specifies Y.

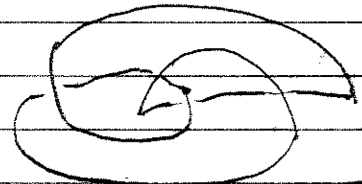
Now, place a basepoint close to μ on either side.

The moves between diagrams is achieved by considering ~~seq~~ one-parameter sequences of Morse functions (keeping track of a meridian of the torus boundary component the whole time)



$t_a \to t_a'$

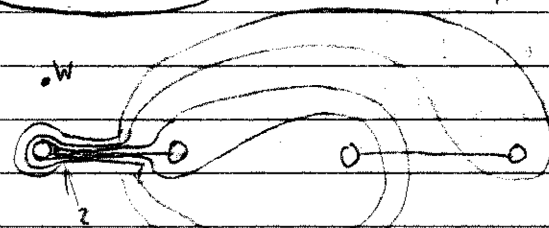
t_b



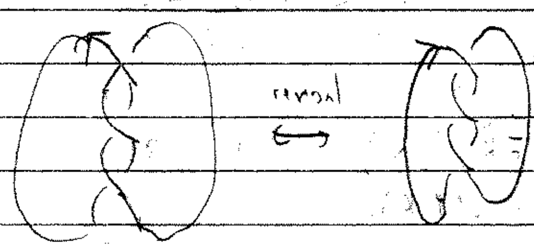
Trefoil

Ron asks how Matt knows where z & w go

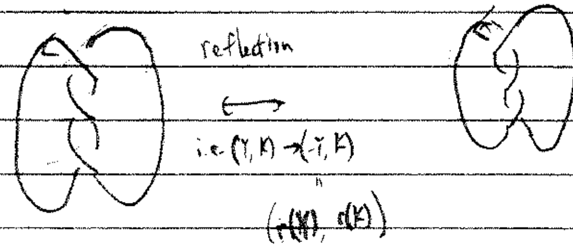
A: Follow z and w through from this diagram through handleslide, isotopy + destabilization.



Remark: Switching roles of z + w reverses the orientation of the knot
(because of our construction)



Prop.
Reflection can be achieved by
$$(\Sigma, \vec{\alpha}, \vec{\beta}, z, w) \to (\Sigma, \vec{\beta}, \vec{\alpha}, z, w)$$
$$\shortparallel$$
$$(\Sigma, \vec{\alpha}, \vec{\beta}, z, w) \to (-\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$$



Recall: $z \rightsquigarrow V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subseteq \mathrm{Sym}^g(\Sigma)$

∂-operator counted intersections of Whitney disks w/ this hypersurface:

$$n_z(\phi) \equiv \mathrm{Im}(\phi) \cap V_z.$$

So, $w \rightsquigarrow V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$

$$n_w(\phi)$$

Now, recall: $\left(\widehat{CF}(Y), \hat{\partial}\right)$ $\qquad \widehat{CF} = \bigoplus_{\mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}/2\langle \vec{x} \rangle$

$$\hat{\partial}\vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1 \\ n_z(\phi) = 0}} \# \widehat{\mathcal{M}}(\phi) \cdot \vec{y}$$

$$\widehat{CFK}(Y, K) = \bigoplus_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}/2\langle \vec{x} \rangle$$

$$\hat{\partial}^K \vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1 \\ n_z(\phi) = n_w(\phi) = 0}} \# \widehat{\mathcal{M}}(\phi) \cdot \vec{y}$$

Notational Change (without Motivation)

$(\Sigma, \vec{\alpha}, \vec{\beta}, z) \leftarrow$ H.D. for Y, $\widehat{CF}(Y)$ uses V_z.

or (of Y!)

$(\Sigma, \vec{\alpha}, \vec{\beta}, z, w) \leftarrow$ H.D. for (Y, K) $\quad \widehat{CF}(Y)$ from a knot diagram uses V_w.

Now, $\widehat{HFK}(Y, K) = H_*\left(\widehat{CFK}(Y, K), \hat{\partial}^K\right)$

The Knot Floer Homology groups of (Y, K).

Exercise: Compute $\widehat{HFK}(S^3, \text{trefoil})$

HFK 1/20/10

Last Time: Algebra Background: • Defined Filtration function of Chain complex

$\Updownarrow$

- Filtered Chain Complex.
- Filtered Chain map
- Filtered Chain Homotopy Equivalence

- Defined, Given $(\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$ 2-ptd. HD adapted to K,

$$A : \widehat{CF}(Y,s) \to \mathbb{Z} \quad \text{a relative } \mathbb{Z}\text{-filtration}$$

associated to K by

$$A(\vec{x}) - A(\vec{y}) = n_z(\phi) - n_w(\phi) \quad \text{for any } \phi \in \pi_2(\vec{x}, \vec{y})$$

extended in a natural way to $\widehat{CF}(Y,s)$ by

$$A\left(\sum_{i=1}^{N} \vec{x}_i\right) = \max_{i \in 1,\dots,N} \left(A(\vec{x}_i)\right).$$

Exercise: Given $\phi_1, \phi_2 \in \pi_2(\vec{x}, \vec{y}) \leftrightarrow \mathbb{Z} \oplus H_2(Y)$

$\mathbb{Z}$: from $\pi_2(\mathrm{Sym}^g(\Sigma))$, represented by Domain $= \Sigma$; $\mathbb{Z}\langle S \rangle$

$H_2(Y)$: Periodic domains, i.e. null-homologies for $\ker(\mathrm{Span}\,\vec{\alpha} + \mathrm{Span}\,\vec{\beta} \to H_1(\Sigma))$

$$\phi_1 = \phi_2 * nS * \left(\sum n_i P_i\right)$$

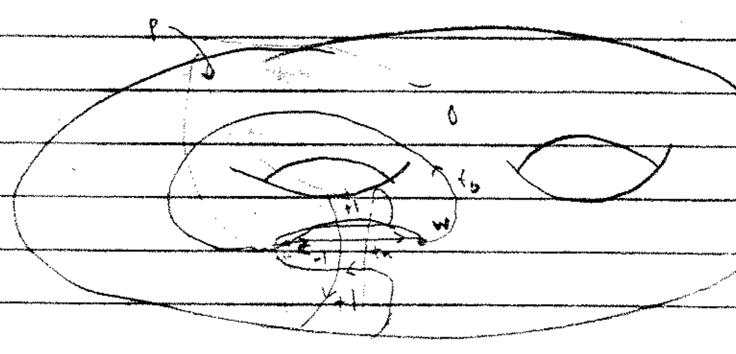
$$D(\phi_1) = D(\phi_2) + n \underset{\Sigma}{\underset{\shortparallel}{D(S)}} + \sum_i n_i P_i$$

$$n_z(\phi_1) - n_w(\phi_1) = n_z(\phi_2) + n\,\cancel{n_z(D(S))} + \sum n_i n_z(P_i)$$
$$- n_w(\phi_2) - n\,\cancel{n_w(D(S))} - \sum n_i n_w(P_i)$$

$$n_z(D(S)) = n_w(D(S))$$

Show that $n_z(P_i) - n_w(P_i) = \#_{\text{alg.}} \underset{H_1(Y)}{\underset{\psi}{[K]}} \cap \underset{H_2(Y)}{\underset{\psi}{[P_i]}}$

So, if $[K] = 0 \Rightarrow A$ is well defined.



$P \sim [P] \in H_2(S^1 \times S^2) \cong \mathbb{Z}$

$[P \cup D_\alpha \cup D_\beta]$ is a closed 2-chain in Y.

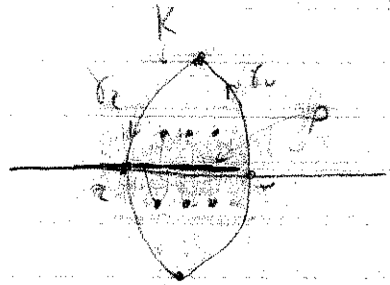
$z, w \rightsquigarrow K \rightsquigarrow [K]$

$\quad\quad\quad\quad \shortparallel$

$V_\alpha \supseteq t_\alpha' \cup t_b' \subseteq V_\beta$

$\# K \cap [P \cup D_\alpha \cup D_\beta]$

$$\# ((t_\alpha' \cup t_\beta'') \cap P) = n_z(P) - n_w(P)$$



Thm. A 2 ptd. HD for K defines a filtration of $\widehat{CF}(Y, s)$ for each $s \in Spin^c(Y)$, provided that $[K] = 0$, i.e., null homologous.

($[K] \cap W = 0 \quad \forall W \in H_2(Y)$

i.e. $[K]$ is torsion, $n[K] = 0$ for some $n \in \mathbb{Z}$)

Knots which aren't null-homologous, but which satisfy $n[K] = 0$ are called rationally null homologous.

Rmk. The associated filtration of $\widehat{CF}(Y, s)$ (Σ, A) is finite length.

i.e., $0 \subseteq \mathcal{F}(i) \subseteq \mathcal{F}(i+1) \subseteq \cdots \subseteq \mathcal{F}(i+N) = \widehat{CF}(Y, s)$.

This is because $\exists$ only finitely many $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Consider the spectral sequence associated to $\mathcal{F}(Y, K)$ (the whole filtration) — Pages in Spectral Sequence

$$\widehat{CFK}(Y, K, i) = \left(\frac{\mathcal{F}(Y, K, i)}{\mathcal{F}(Y, K, i-1)}, \partial \right) \qquad (E_1, \partial_1)$$

induced differential on quotients

$$H_*\left(\left(\frac{\mathcal{F}(Y, K, i)}{\mathcal{F}(Y, K, i-1)}, \partial^{\text{induced}} \right) \right) = \widehat{HFK}(Y, K, i) \qquad (E_2, \partial_2)$$

associated homology

Knot Floer homology group in Alexander grading

Recall, $\left(CFK(Y,K), \partial\right)$ is the same as $\widehat{CF}(Y)$, but where ∂ doesn't count J-holo disks with $n_z(\phi) = 0$

Prop. $\widehat{HFK}(Y,K) = \bigoplus_{i\in\mathbb{Z}} \widehat{HFK}(Y,K,i)$

Pf. First, observe $\bigoplus_{i\in\mathbb{Z}} \dfrac{\mathcal{F}(Y,K,i)}{\mathcal{F}(Y,K,i-1)} \cong \widehat{CF}(Y)$

Because for every $\vec{x} \in \widehat{CF}(Y)$, $\vec{x} \in \dfrac{\mathcal{F}(Y,K,i)}{\mathcal{F}(Y,K,i-1)}$, where $i = A(\vec{x})$.

We need to see that the differentials coincide.

$\left(\dfrac{\mathcal{F}(Y,K,i)}{\mathcal{F}(Y,K,i-1)}, \partial^{\text{induced}}\right)$

$\partial^{\text{ind.}}\vec{x}$ = same as $\partial\vec{x} \in \widehat{CF}(Y)$, but if $r \in \partial\vec{x}$, $r \in \mathcal{F}(Y,K,i-1)$, we set $r = 0 \in \partial^{\text{ind.}}\vec{x}$.

Suppose $\langle \partial\vec{x}, r \rangle \neq 0$

$\Rightarrow \exists$ J-holo disk connecting $\vec{x}$ to r

$\langle \partial^{\text{ind}}\vec{x}, r \rangle = 0 \iff r \in \mathcal{F}(Y,K,i-1)$

$\iff A(\vec{x}) - A(r) \geq 1$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad \underbrace{n_z(\phi) - n_w(\phi)}_{=0 \text{ by Def. of } \partial \text{ (in } \wedge \text{ complex)}}$

$\langle \partial^{\text{ind}}\vec{x}, r \rangle \neq 0 \iff r \in \mathcal{F}(Y,K,i)/\mathcal{F}(Y,K,i-1)$, $r \notin \mathcal{F}(Y,K,i-1)$

$\iff A(\vec{x}) - A(r) = 0$

$\iff n_z(\phi) = 0.$

Observe: $\widehat{HFK}(Y,K)$ can thus be thought of as a bigraded homology theory associated to $K \subseteq Y$.

$$\widehat{HFK}(Y,K) = \bigoplus_{\substack{i\in\mathbb{Z} \\ *\in\mathbb{Z}}} \widehat{HFK}_*(Y,K,i)$$

$A(\vec{x}) - A(\vec{y}) = n_z(\phi) - n_w(\phi)$.

Maslov grading

Gr $\qquad Gr(\vec{x}) - Gr(\vec{y}) = \mu(\phi) - 2n_w(\phi)$

Actually, $* \in \mathbb{Z}/d\,v(c_1(s))$ Chern class of spin^c structure

Given a 2-ptd. HD, how do we compute $A(\vec{x}) - A(\vec{y})$?

- Given $\vec{x}, \vec{y} \longrightarrow$ compute $\varepsilon(\vec{x}, \vec{y}) \in H_1(Y)$
- If $\varepsilon(\vec{x}, \vec{y}) = 0$, then the collection of curves $\gamma_{\vec{x}} \cup \gamma_{\vec{y}}$ connecting $\vec{y} \to \vec{x}$ " β ; $\vec{x} \to \vec{y}$ along α

  (supplemented by closed α's + β's)

  $= \partial D$

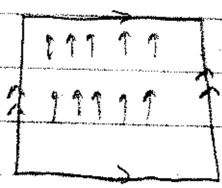
  $\nwarrow$ 2-chain w/ $\partial \subseteq \Sigma$.

- $n_z(D) - n_z(W)$

  multiplicities of D at z + w resp.

The Alexander grading A, interpreted as a function to "relative Spin^c structures" on $Y - n(K)$

Prop. Given a torus T^2, $\exists!$ nonvanishing v.f. on T^2, up to hotopy. $\mathcal{V}$



$(0,p) \subseteq T_p \mathbb{R}^2$, invariant under $x \mapsto x+1$

$y \mapsto y+1$.

Def. For a 3mfld. M w/ $\partial M^3 = \coprod_{i=1}^{N} T_i^2$, a relative Spin^c structure on M, often denoted $\underline{\text{Spin}}^c(M)$, is a homology class of nowhere vanishing vector fields in M s.t. restricting to $\mathcal{V}$ on each T_i^2.

underline for relative

i.e. We want a v.f. v s.t. $v|_{T_i} = \mathcal{V}$ +

$v \sim v'$ if v is homotopic to v' through n.v.v.f. on M that restrict to $\mathcal{V}$, after we remove some interior balls.

(Take any n.v.v.f. on Y, remove nbhd. of K, and the v.f.$|_{\partial n(K)}$ should lie in $T\partial n(K)$..)

Next time: Last semester: $\vec{x} \longmapsto S_w(\vec{x}) \in \text{Spin}^c(Y)$. $\quad S_w(\vec{x}) - S_w(\vec{y}) = PD[\varepsilon(\vec{x}, \vec{y})] \in H^2(Y)$

Now: $\vec{x} \longmapsto S_{z,w}(\vec{x}) \in \underline{\text{Spin}}^c(Y)$. $\quad S_{z,w}(\vec{x}) - S_{z,w}(\vec{y}) = PD[\varepsilon(\vec{x}, \vec{y})]$

Lefschetz-Poincaré Duality

$H^2(Y - n(K), \partial)$

Relative Cohomology

$\cong$

$H^2(Y) \oplus \mathbb{Z}$.

1/25/11 HFH Matt Hedden

What we have: If $[K] = 0 \in H_1(Y)$,
then $A(\vec{x}) - A(\vec{y}) = n_z(\phi) - n_w(\phi)$, $\phi \in \pi_2(\vec{x}, \vec{y})$
$$A\left(\sum_{i=1}^{N} x_i\right) = \max_i A(x_i)$$
is a well-defined (relative) $\mathbb{Z}$-filtration of $\widehat{CF}(Y, s)$, $\forall s \in Spin^c(Y)$.

We had $\{\varepsilon\text{-classes}\} \leftrightarrow \{Spin^c(Y) \text{ classes}\}$, ε-classes $\in H_1(Y)$

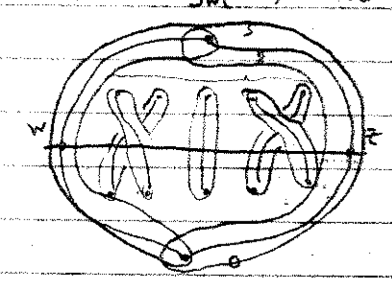
Promised: $\{\varepsilon\text{-classes}\} \leftrightarrow \{Spin^c(Y,K) \text{ classes}\}$

$\hat{H}_1(Y-K) \cong H_1(Y) \oplus \mathbb{Z}$

i-splitting $\bigoplus_{i \in \mathbb{Z}} \widehat{HFK}(Y, K, i)$ (Alexander grading)

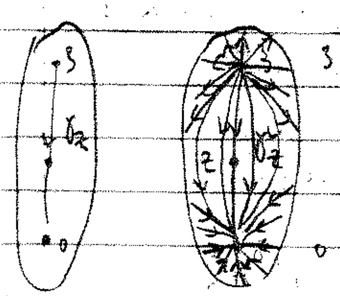
Given $(\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$, we want $s_{z,w}(\cdot) : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y, K)$.

To define $s_{z,w}(\cdot)$, we perform a similar operation to what we did to construct $s_w(\cdot) : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y)$



Outside nbhd$(\gamma_{x_1} \cup \cdots \cup \gamma_{x_g} \cup \gamma_z \cup \gamma_w)$, we take $v_{z,w}(\vec{x})$ to be $-\nabla f$, where f is self-ind. Morse function specifying $(\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$

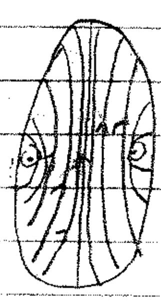
For parity reasons, we can extend $v_{z,w}(\vec{x})$ over nbhd. of $\gamma_{x_1} \cup \cdots \cup \gamma_{x_g}$ s.t. it is non-vanishing.



Rotating this picture about obvious (vertical) axis, we obtain integral curves for $-\nabla f$ in nbhd(γ_z).

Replace $-\nabla f$ in nbhd(γ_z) by

(Note that v.f.'s agree on boundary)



This modified v.f. in this nbhd. acquires a component in the direction $\perp$ to the plane of the page. Upon rotation, meridian of K bounds closed integral curve for modified v.f.

Finally, remove the knot + consider $V_{z,w}(\vec{x})$, the restriction of our modified v.f. to the complement.

$V_{z,w}(\vec{x})\big|_{T^2}$ can be taken in $T_*(T^2)$.

Recall, $s, s' \in Spin^c(Y) \longrightarrow s - s' \in H^2(Y)$

Prop. $\underline{s}, \underline{s}' \in \underline{Spin}^c(Y,K)$, then $\underline{s} - \underline{s}' \in H^2(Y \setminus K, \partial)$.

Pf. Idea: $[(Y \setminus K, \partial), (S^2, pt.)] \in H^2(Y \setminus K, \partial)$.

n.v.v.f. w/ fixed ∂ condition gives rise to a homotopy class

Prop. Associated to $\vec{x}, \vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\underline{s}_{z,w}(\vec{x}) - \underline{s}_{z,w}(\vec{y}) = PD[\varepsilon(\vec{x},\vec{y})]$, where $\varepsilon(\vec{x},\vec{y}) \in H_1(Y \setminus K)$.

Pf. $[\underline{s}_{z,w}(\vec{x}) - \underline{s}_{z,w}(\vec{y})]$ is realized by a 2-chain $\in C^2(Y \setminus K, \partial)$, supported in nbhd $(\gamma_{\vec{x}} \cup -\gamma_{\vec{y}})$

$\Rightarrow [\underline{s}_{z,w}(\vec{x}) - \underline{s}_{z,w}(\vec{y})] = n\, PD[\underbrace{\gamma_{\vec{x}} \cup \gamma_{\vec{y}}}_{H_1(Y \setminus K)}]$.

Thom Isomorphism thm., tubular nbhd. thm. etc. $\longrightarrow n = 1$.

Note: $[\gamma_{\vec{x}} \cup -\gamma_{\vec{y}}] = [\varepsilon(\vec{x},\vec{y})]$

$\gamma_{\vec{x}}$ arcs along α conn. $\vec{x}$ to $\vec{y}$

$\cup -\gamma_{\vec{y}}$ arcs along β conn. $\vec{y}$ to $\vec{x}$.

Indeed, gradient flow takes $(\gamma_{\vec{x}} \cup -\gamma_{\vec{y}})'$ to $(\gamma_\alpha \cup -\gamma_\beta)$, where $(\gamma_{\vec{x}} \cup -\gamma_{\vec{y}})'$ is a small perturbation of $(\gamma_{\vec{x}} \cup -\gamma_{\vec{y}})$.

Prop. $\underbrace{(A(\vec{x}) - A(\vec{y}))}_{\mathbb{Z}}[\mu] = [\varepsilon(\vec{x},\vec{y})]$, if $\varepsilon(\vec{x},\vec{y}) = 0 \in H_1(Y)$,

$H_1(Y \setminus K) \cong H_1(Y) \oplus \mathbb{Z}\langle \mu \rangle$, where $\mu \in Y \setminus K$ is a meridian of K.

$\| \; (n_z(\phi) - n_w(\phi))[\mu]$

Exercise: If $[K] = 0$, then $H_1(Y \setminus K) \cong H_1(Y) \oplus \mathbb{Z}\langle[\mu]\rangle$.

Pf. (of Prop.) Recall, Pf. of $n_z(P) - n_w(P) = lk(\partial P, K) = \# P \cap K$

claim: " " " for a periodic domain P.

$n_z(\phi) - n_w(\phi) = lk(\partial D(\phi), K) = \# \underset{\nwarrow \text{2-chain}}{D(\phi)} \cap \underset{\nwarrow \text{1-cycle}}{K}$.

By construction of $D(\phi)$, the only place K intersects $D(\phi)$ is on H.S. at z & w resp. $\Rightarrow \# D(\phi) \cap K = n_z(D(\phi)) - n_w(D(\phi))$
(keeping track of orientations) $= n_z(\phi) - n_w(\phi)$.

or $\widehat{CF}(Y, s)$

Filtration can now be viewed as a filtration by relative $Spin^c$-structures, each of which equal s when viewed in $Spin^c(Y)$.

Note: Natural map

$$\underline{Spin^c(Y, K)} \to Spin^c(Y)$$
$$[v] \mapsto [\text{extension of } \vec{v} \text{ over nbd. of } K]$$

(In $\widehat{HF}$ land, $s_{z,w}(\vec{x}) \longmapsto s_z(\vec{x})$
$\widehat{Spin}^c(Y, K) \to Spin^c(Y)$

To get $A(\vec{x})$ in $\mathbb{Z}$ (more invariantly), we can do one of 2 things:

Equivalent
(1) Try to extract # from $s_{z,w}(\vec{x})$.

(2) Use fact that $\widehat{HFK}$ has symmetries to pin down A.

(1)

Recall, for $s \in Spin^c(Y)$, $c_1(s) \in H^2(Y; \mathbb{Z})$

From this, we could get #'s by

Given $[F] \in H_2(Y)$, $HF^+_*(Y, [F], i) = \bigoplus_{\{s \mid \langle c_1(s), [F] \rangle = 2i\}} HF^+_*(Y, s)$, $i \in \mathbb{Z}$

Morally picking a homology class $\alpha \in H_2(Y)$ endows HF° with extra grading.

In $\widehat{HFK}$ context, we can do the same:

$c_1(\underline{s}) \in H^2(Y - K, \partial)$

$c_1(\underline{s}) := [v] - [-v]$, where $[\vec{v}] = \underline{s}$ (n.v.v.f.)
$\in H^2(Y - K, \partial)$

or equivalently, = relative Chern class of $v^\perp$ (orthogonal 2-plane field perpendic. to v)

or equiv. , = obstruction to extending nonzero section of $v^\perp$ to interior

1/27/10 Matt Hedden HFH

Relative $\mathbb{Z}$-filtration to absolute $\mathbb{Z}$-filtration

$H^2(Y-K,\partial)$ $\quad$ $H_2(Y-K,\partial)$

$\cdot\,\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightsquigarrow \underline{s}_{z,w}(\vec{x}) \in \underline{\mathrm{Spin}}^c(Y,K) \rightsquigarrow \frac{1}{2}\langle c_1(\underline{s}_{z,w}(\vec{x})) - PD[\mu], [F,\partial]\rangle \in \mathbb{Z}$

μ = meridian of knot, F = Seifert surface for K.

$c_1(\underline{s}) := [v]-[-v]$ $\qquad$ $c_1(\underline{s}_{z,w}(\vec{x})) = c_1(v^\perp)$

$v^\perp$ = perp. 2-plane field to v, n.v.f. representing $\underline{s}_{z,w}(\vec{x})$

So given $\alpha \in H_2(Y-K,\partial) \rightsquigarrow A_\alpha(\vec{x}) \in \mathbb{Z}$ an absolute filtration function.

Ex: If $H_*(Y^3) \cong H_*(S^3)$, then $H_2(Y-K,\partial) \cong \mathbb{Z}\langle F\rangle$, where $\partial F = K$ (oriented).

Another way

Use symmetry of $\widehat{HFK}$

Prop. $Y = \mathbb{Z}HS^3$, $\widehat{HFK}(Y,K,\underline{s}) \cong \widehat{HFK}(Y,K,J\underline{s} \mp PD[\mu])$

for $\underline{s} \in \underline{\mathrm{Spin}}^c(Y,K)$, $J\underline{s} := [-v]$, where $[v]=\underline{s}$.

If $Y = \mathbb{Z}HS^3$, then $H_1(Y-K) \cong H^2(Y-K,\partial) \cong \mathbb{Z}$

$\updownarrow$ 1-1

$\underline{\mathrm{Spin}}^c(Y)$

Picking $\underline{s}_0$, then $\forall\ \underline{s} \in \underline{\mathrm{Spin}}^c(Y,K)$, $\underline{s}_0 - \underline{s} \in H^2(Y-K,\partial)$.

Claim: $\exists!\ \underline{s}_0$ s.t. $J\underline{s}_0 = \underline{s}_0$.

If $\underline{s}_{z,w}(\vec{x}) = \underline{s}_0$, set $A(\vec{x}) = 0$.

Alternatively, the symmetry prop. is a reflection of the symmetry of the Alexander poly.

We could instead define A to be the unique choice of A s.t.

(1) $A(x) - A(y) = n_z(\phi) - n_w(\phi)$?

(2) $\sum_{i\in\mathbb{Z}} \chi(\widehat{HFK}(Y,K,i)) \cdot T^i = \Delta_K(T)$

$[K]=0$, 0-surgery (using nullhomology)

$(Y,K) \longrightarrow (Y_0(K))$

An equivalent def. of $\underline{\mathrm{Spin}}^c(Y,K)$ for K s.t. $[K]=0$ is

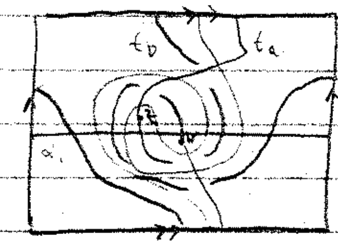
$\underline{\mathrm{Spin}}^c(Y,K) := \mathrm{Spin}^c(Y_0(K)) \xrightarrow{1-1} H^2(Y_0(K)) \cong H^2(Y) \oplus \mathbb{Z}\langle\hat{F}\rangle$

$\hat{F}$: Seifert surface, capped w/ meridian disk of $S^1\times D^2$ used for surgery

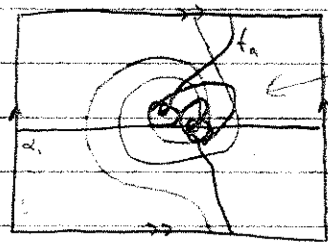
Given $(\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$, want $s_{z,w}(\vec{x}) \in Spin^c(Y_0(K))$

<u>Ex:</u> Figure Eight Knot.

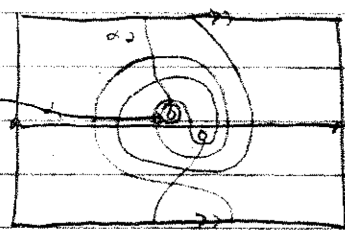
<u>Exercise:</u> Verify that this specifies the Figure 8 Knot



Problem: this is not a Heegaard diagram for 0-surgery.



Attach a handle using these two basepoints, and use new α-curve running along the arc t_a.

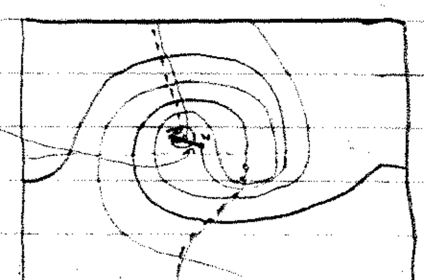


μ meridian for the knot

<u>Claim</u> (1) After stabilization, along t_a, HD continues to represent S^3, & μ (final β-curve) is the meridian of some knot K.

(2) K is the same knot specified by the original 2-pted. H.D.

(in this case the Figure 8)

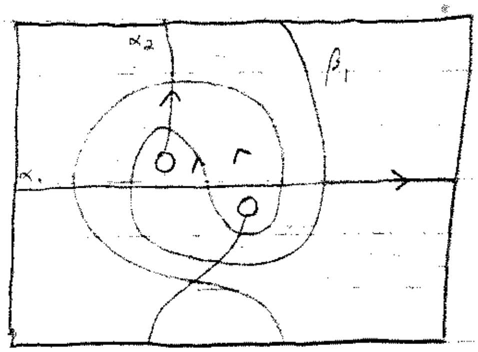


Now t_a

<u>Claim:</u> Now t_a is isotopic to the old t_a:

Now, we can do surgery, replacing μ w/ 0-framed longitude

(Need a curve intersecting μ once & H_1(resulting 3-manifld) $\cong \mathbb{Z}$.

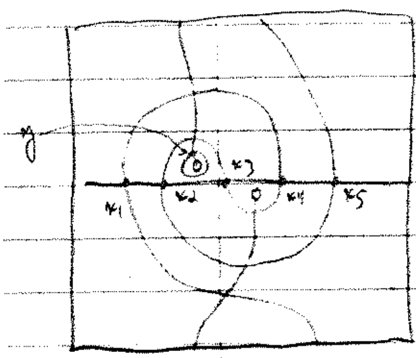


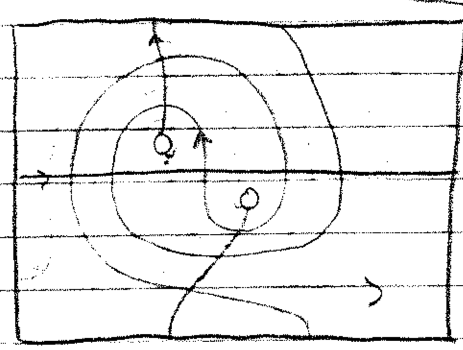
(Accidentally)
using negative Right hand Rule -

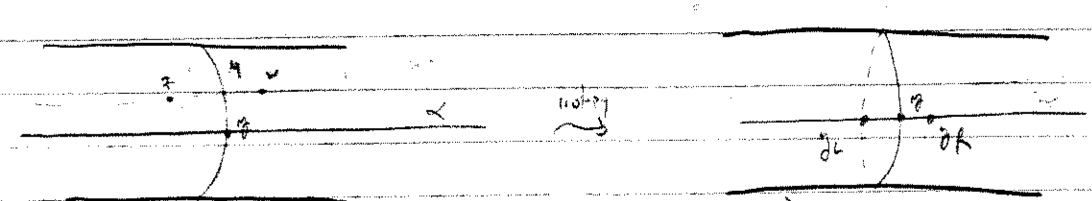
$$|H_1(Y)| = \det \begin{bmatrix} \#\alpha_1 \cap \beta_1 & \#\alpha_1 \cap \beta_2 \\ \#\alpha_2 \cap \beta_1 & \#\alpha_2 \cap \beta_2 \end{bmatrix} = \det \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = 1$$

$C_2 \xrightarrow{\partial^{21}} C_1$ in Morse complex or CW-complex from combinatorial data

Wrong order, so we have to wrap β_2 around meridian some number of times, changing to get det $= 0$.







$HD = (\Sigma_g, \vec{\alpha}, \vec{\beta}, z, w)$ $\quad \vec{x}$

$\rightsquigarrow HD' = (\Sigma_{g+1}, \vec{\alpha} \cup \alpha_{g+1}, \vec{\beta} \cup \mu, z, w) \quad \rightsquigarrow \vec{x} \cup \eta$

$\rightsquigarrow HD'' = (\Sigma_{g+1}, \vec{\alpha} \cup \alpha_{g+1}, \vec{\beta} \cup \lambda, z, w) \quad \rightsquigarrow \vec{x} \cup \eta_L$

0-framed longitude

Given $\vec{x} \in \mathbb{T}_\alpha \cup \mathbb{T}_\beta$ for $HD = (\Sigma_g, \alpha, \beta, z, w)$ repr. K,

We can do a sequence of moves $\rightsquigarrow$

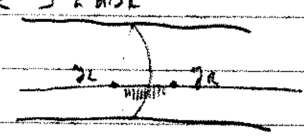
$$\vec{x}_{close} = \vec{x} \cup m^L \in \mathbb{T}_{\alpha'} \cup \mathbb{T}_{\beta'} \text{ fr. HD specifying } Y_0(K).$$

$$s_{z,w}(\vec{x}) := s_w(\vec{x}_{close})$$

~~Equivalently~~ The correspondence of the relative spin^c-structures (as v.f.'s) is: starting w/ a n.v.v.f. tangent to ∂-torus (running longitudinally), isotope the v.f. to be meridional on ∂-torus (w/ oriented meridian). Now this extends in the 0-surgered mnfld.

Note: $s_z(\vec{x} \cup m^L) - s_z(\vec{x} \cup m^R) = \varepsilon(\vec{x} \cup m^L, \vec{x} \cup m^R) = 0$

since $\exists$ a ADK



Given $(\Sigma, \alpha, \beta, z, w)$,

$$\langle c_1(s_w(\vec{x}_{close})), [\hat{F}] \rangle = \hat{\chi}(P) + 2n_{\vec{x}_{close}}(P) \cancel{- 2n_w(P)} \quad (\star)$$

↑ euler measure of a periodic domain P repr. the homology class of capped off Seifert surface

Exercise: Compute $(\star)$ for $\vec{x}_i \cup m^L$ $i = 1, \dots, 5$ in Figure 8 Example.