1. (21 points) Find the derivatives of each of the following. DO NOT SIMPLIFY.

(a) $f(x) = 3^{\sec x}\tan^{-1}(5x-1)$

Solution:

$$f'(x) = 3^{\sec x}\ln 3\,\sec x\tan x\,\tan^{-1}(5x-1) + \frac{3^{\sec x}}{1+(5x-1)^2}\,\frac{5}{1}$$

(b) $g(x) = \sqrt{\cos(e^x+1)}$

Solution:

$$g'(x) = \frac{1}{2\sqrt{\cos(e^x+1)}}\,(-\sin(e^x+1))\,e^x$$

(c) $y = x^{\tan x}$

Solution:

$$\begin{aligned}
\Longrightarrow \ln y &= \tan x\,\ln x \\
\Longrightarrow \frac{1}{y}\frac{dy}{dx} &= \left(\sec^2 x\right)\ln x + (\tan x)\frac{1}{x} \\
\Longrightarrow \frac{dy}{dx} &= y\left(\left(\sec^2 x\right)\ln x + \frac{\tan x}{x}\right) \\
&= x^{\tan x}\left(\left(\sec^2 x\right)\ln x + \frac{\tan x}{x}\right)
\end{aligned}$$

2. (12 points) Solve the initial value problem below.

$$\frac{dy}{dx} + xy = 2x, \qquad y(0) = 1$$

Solution:

$$\begin{aligned}
\Longrightarrow \frac{dy}{dx} &= 2x - xy \\
\frac{dy}{2-y} &= x\,dx \\
\int \frac{dy}{2-y} &= \int x\,dx \\
-\ln|2-y| &= \frac{x^2}{2} + C
\end{aligned}$$

$$y(0) = 1 \Longrightarrow C + 0 = -\ln 1 = 0 \Longrightarrow -\ln|2-y| = \frac{x^2}{2}$$

3. (14 points) Find each of the following limits.

(a) $\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{5x}$

Solution 1:

$$= \lim_{x\to\infty}\left[\left(1+\frac{1/3}{x}\right)^x\right]^5 = \left[\lim_{x\to\infty}\left(1+\frac{1/3}{x}\right)^x\right]^5 = \left[e^{1/3}\right]^5$$

Solution 2:

Let $f(x) = \left(1 + \dfrac{1}{3x}\right)^{5x}$. Then

$$\begin{aligned}
\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} 5x \ln\left(1 + \frac{1}{3x}\right) \\
&= 5 \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{3x}\right)}{\frac{1}{x}} \\
&= 5 \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{1}{3x}}\right)\left(\frac{-1}{x^2}\right)\left(\frac{1}{3}\right)}{\frac{-1}{x^2}} \\
&= \frac{5}{3}\left(\frac{1}{1+0}\right)
\end{aligned}$$

Thus $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{5/3}$

(b) $\displaystyle \lim_{x\to 0^+} \frac{\sqrt{x}}{\sin^{-1}\sqrt{x}}$

Solution:

$$= \lim_{x\to 0^+} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{\sqrt{1-x}}{1} = 1$$

4. (30 points) Find each of the following integrals.

(a) $\displaystyle \int \frac{x^2}{\sqrt{4-x^2}}\,dx$

Solution:

$$\begin{aligned}
x &= 2\sin\theta \\
dx &= 2\cos\theta\,d\theta \\
4 - x^2 &= 4 - 4\sin^2\theta = 4\cos^2\theta
\end{aligned}$$

Thus

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int \frac{4\sin^2\theta\, 2\cos\theta\,d\theta}{\sqrt{4\cos^2\theta}} \\
&= 4\int \sin^2\theta\,d\theta \\
&= 2\int (1-\cos 2\theta)d\theta \\
&= 2\theta - \sin 2\theta + C \\
&= 2\sin^{-1}\frac{x}{2} - 2\sin\theta\,\cos\theta + C \\
&= 2\sin^{-1}\frac{x}{2} - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C
\end{aligned}$$

(b) $\displaystyle \int e^{\sqrt{x}}\,dx$

Solution:

$$\begin{aligned}
w &= \sqrt{x} \\
\Longrightarrow dw &= \frac{dx}{2\sqrt{x}} = \frac{dx}{2w} \\
\Longrightarrow \int e^{\sqrt{x}}\,dx &= 2\int w\,e^w\,dw
\end{aligned}$$

Let $u = w,\ dv = e^w\,dw$. Then $du = dw$ and $v = e^w$. Thus

$$\begin{aligned}
2\int w\,e^w\,dw &= 2w\,e^w - 2\int e^w\,dw \\
&= 2w\,e^w - 2e^w + C \\
&= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C
\end{aligned}$$

4. (con't) Find each of the following integrals.

(c) $\displaystyle\int \frac{x+3}{x^3-x}\,dx$

Solution:

$$\frac{x+3}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\begin{aligned} \Longrightarrow \int \frac{x+3}{x^3-x}\,dx &= \int \left(\frac{-3}{x} + \frac{2}{x-1} + \frac{1}{x+1} \right) dx \\ &= -3\ln|x| + 2\ln|x-1| + \ln|x+1| + C \end{aligned}$$

5. (12 points) A reservoir shaped like a right circular cone (with the point down) is 16 ft across the top and 10 ft deep. If the tank is full of water weighing 62.4 lb/ft^3, how much work is required to pump the water to a point 4 ft above the top of the tank.

Solution:

$$\Delta V = \pi r^2 \Delta y = \frac{16\pi}{25} y^2 \Delta y$$

$$\Longrightarrow F(y) = (62.4)\Delta V = (62.4)\left(\frac{16\pi}{25}\right) y^2 \Delta y$$

$$\Longrightarrow W = \int_0^{10} (62.4)\left(\frac{16\pi}{25}\right) y^2(14-y)\,dy$$

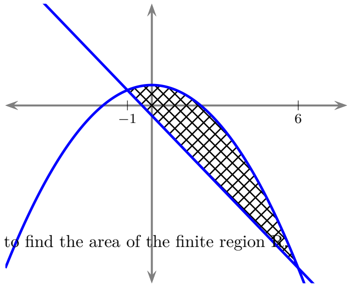
$$= (62.4)\left(\frac{16\pi}{25}\right)\left(\frac{14y^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^{10}$$

$$= (62.4)\left(\frac{16\pi}{25}\right)\left(\frac{14(10)^3}{3} - \frac{10^4}{4}\right)$$

$$= 86528\pi$$

6. (16 points) Consider the region R bounded by curves, $y = 4 - x^2$ and $5x + y = -2$.

(a) Sketch the region R.



(b) Set up but **DO NOT EVALUATE** the definite integral to find the area of the finite region R.

Solution:

$$\int_{-1}^{6} \left((4-x^2) - (-5x-2)\right)\,dx = \int_{-1}^{6} \left(-x^2 + 5x + 6\right)\,dx$$

(c) Set up but **DO NOT EVALUATE** the definite integral to find the volume of the solid generated by rotating the region R about the line $y = 9$.

Solution:

$$\begin{aligned} A(x) &= \left(\pi\,(9-(-5x-2))^2 - \pi\left(9-\left(4-x^2\right)\right)^2\right) \\ &= \pi\left((5x+11)^2 - \left(x^2+5\right)^2\right) \\ &= \pi\left(-x^4 + 15x^2 + 70x + 24\right) \Longrightarrow \end{aligned}$$

$$\text{Volume} = \int_{-1}^{6} A(x)\,dx = \pi \int_{-1}^{6} \left(-x^4 + 15x^2 + 70x + 24\right)\,dx$$

7. (27 points) Determine whether the series below converge or diverge. *Show your work and name the tests you are using.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^2+1}}$

Solution:

The series diverges by the nth-term test since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 \neq 0 \qquad (1)$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Solution:

i. Let $b_n = \frac{1}{n^p}$, $1 < p < 3$. Then the p-series $\sum b_n$ converges since $p > 1$.

ii. Let $a_n = \frac{\ln n}{n^3}$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^3}\,\frac{n^p}{1} \\ &= \lim_{n\to\infty} \frac{\ln n}{n^{3-p}} \\ &= 0, \qquad \text{since } 3-p>0\end{aligned}$$

iii. So by the LCT, $\sum a_n$ converges.

7. (con't) Determine whether the series below converge or diverge. *Show your work and name the tests you are using.*

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n\, n^3}{n!}$

Solution:

Let $a_n = \frac{2^n\, n^3}{n!}$

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{2^{n+1}\,(n+1)^3}{(n+1)!}\,\frac{n!}{2^n\, n^3} \\ &= \lim_{n\to\infty} \frac{2}{n+1}\left(\frac{n+1}{n}\right)^3 \\ &= 0\end{aligned}$$

It follows by the Ratio Test that $\sum a_n$ converges since $\rho = 0 < 1$.

8. (10 points) Find the (open) interval of convergence for the power series below. *Do not test for convergence at the endpoints.*

$$\sum_{n=1}^{\infty} \frac{x^n}{(-5)^n\,(n+1)}$$

Solution:

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\ &= \lim_{n\to\infty} \frac{|x|^{n+1}}{5^{n+1}\,(n+2)}\,\frac{5^n\,(n+1)}{|x|^n} \\ &= \lim_{n\to\infty} \frac{|x|}{5}\,\frac{n+1}{n+2} \\ &= \frac{|x|}{5}\end{aligned}$$

$\rho < 1 \Longrightarrow -5 < x < 5$

9. (10 points) Find the length of the curve $y = x^{3/2}$ from $x = 0$ and $x = 1$.

Solution:

$$\begin{aligned} \Longrightarrow \frac{dy}{dx} &= \frac{3\sqrt{x}}{2} \\ \Longrightarrow \mathrm{L} &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \frac{4}{9}\int_1^{13/4} \sqrt{u}\, du \\ &= \frac{8}{27}u^{3/2}\Big|_1^{13/4} \\ &= \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) \end{aligned}$$

10. (12 points) Find the Maclaurin series $\sum_{n=0}^{\infty} a_n x^n$ for each of the following functions. *Give an explicit formula for a_n in each part.* (You may freely use known formulas.)

(a) $\sin x^2$

Solution:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}$

(b) $(1-x)^{-2}$

Solution:

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \Longrightarrow$

$$\begin{aligned} \frac{1}{(1-x)^2} &= D_x\left(\frac{1}{1-x}\right) \\ &= 1 + 2x + 3x^2 + \cdots \\ &= \sum_{n=0}^{\infty}(n+1)x^n \end{aligned}$$

11. (12 points) Find the Taylor polynomial of order 3 generated by $f(x) = \sqrt{x+1}$ at $a = 3$.

Solution:

$$\begin{aligned} f'(x) = \frac{1}{2}(x+1)^{-1/2} &\Longrightarrow f'(3) = \frac{1}{4} \\ f''(x) = \frac{-1}{4}(x+1)^{-3/2} &\Longrightarrow f''(3) = \frac{-1}{32} \\ f'''(x) = \frac{3}{8}(x+1)^{-5/2} &\Longrightarrow f'''(3) = \frac{3}{256} \end{aligned}$$

$$\begin{aligned} \Longrightarrow P_3(x) &= f(3) + f'(3)\frac{(x-3)}{1!} + f''(3)\frac{(x-3)^2}{2!} + f'''(3)\frac{(x-3)^3}{3!} \\ &= 2 + \frac{x-3}{4} - \frac{(x-3)^2}{64} + \frac{(x-3)^3}{512} \end{aligned}$$

12. (8 points) Determine whether the following improper integral converges or diverges. *Justify your claim.*

$$\int_1^9 \frac{dx}{(x-1)^{2/3}} \qquad \text{converges}$$

Solution:

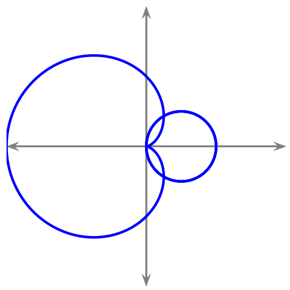
$$\begin{aligned} &= \lim_{a\to 1^+}\int_a^9 \frac{dx}{(x-1)^{2/3}} \\ &= 3\lim_{a\to 1^+}(x-1)^{1/3}\Big|_a^9 \\ &= 3\lim_{a\to 1^+}\left((9-1)^{1/3} - (a-1)^{1/3}\right) \\ &= 6 \end{aligned}$$

13. (16 points) Consider the functions below.

$$\begin{aligned} r_1 &= \cos\theta \\ r_2 &= 1-\cos\theta \end{aligned}$$

(a) Sketch the graphs of these functions in polar coordinates.

(b) Find all points of intersection of the graphs in (a).



Solution:

Set $\cos\theta = 1-\cos\theta$ and solve. Thus $\cos\theta = \frac{1}{2}$. It follows that $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$. The intersection points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$ and $(0, 0)$.

(c) Set up but **DO NOT EVALUATE** the definite integral for the area which is inside r_2 and outside of r_1.

Solution:

Let A_1 be the area of the cardioid and let A_2 be the area lens shaped common region in quadrant I. Then the indicated area is $A_1 - 2A_2$ where

$$\begin{aligned} A_1 &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\,d\theta \\ A_2 &= \frac{1}{2}\left(\int_0^{\pi/3} (1-\cos\theta)^2\,d\theta - \int_{\pi/3}^{\pi/2} \cos^2\theta\,d\theta\right) \end{aligned}$$