The Gopakumar-Vafa conjecture \(\text{GV}\) predicts that the Gromov-Witten invariants \(GW_{A,g}\) of a Calabi-Yau 3-fold can be expressed in terms of some other invariants \(n_{A,h}\), called BPS numbers, by a transform between their generating functions:

\[
\sum_{A \neq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A \neq 0} n_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{2 \sin \frac{kt}{2}}{2} \right)^{2h-2} q^{kA}
\] (0.1)

The content of the conjecture is that the BPS numbers \(n_{A,h}\) are integers. (Gopakumar and Vafa also conjectured that for each \(A\), the coefficients of (0.1) satisfy \(n_{A,h} = 0\) for large \(h\); we do not address this finiteness statement here.)

The GV formula (0.1) can be viewed as a statement about the structure of the space of solutions to the \(J\)-holomorphic map equation. For a generic almost complex structure \(J\), each \(J\)-holomorphic map is the composition \(f = \varphi \circ \rho\) of a multiple-cover \(\rho\) and an embedding \(\varphi\). The embeddings are well-behaved: they have no nontrivial automorphisms, and the moduli space of \(J\)-holomorphic embeddings is a smooth manifold. But multiply-covered maps cause severe analytical problems with transversality. In the symplectic construction of the GW invariants, these problems are avoided by lifting to a cover of the moduli space and turning on a lift-dependent perturbation \(\nu\) of the equation; this destroys the multiple-cover structure and only shows that the numbers \(GW_{A,g}\) are rational. But it also suggests an interpretation of the GV formula: the righthand side of (0.1) might be a sum over embeddings, with the sum over \(k\) counting the contributions of the multiple covers of each embedding.

This viewpoint is very similar to C. Taubes’ work [1] relating Gromov invariants to the Seiberg-Witten invariants on 4-manifolds, and our approach has been fundamentally influenced by Taubes. It is also similar to the 4-dimensional situation described by Lee and Parker in [LP]. In both cases, the set of \(J\)-holomorphic embeddings in each homology class is isolated and compact for generic \(J\) — a simplifying circumstance that does not appear to be true in the context of formula (0.1). This seems to suggest that the BPS numbers may only be \textit{virtual} counts of simple \(J\)-holomorphic maps.

In this paper we take this later approach: we show that the GW invariant of any symplectic Calabi-Yau 6-manifold \(X\) can be expressed as the sum of local contributions coming from \(\varepsilon\)-tubular neighborhoods \((C, \varepsilon, J)\) of finitely many smooth, embedded \(J\)-holomorphic curves (finite in each genus and degree). For generic \(\varepsilon\) each such neighborhood – called a cluster – has a well-defined invariant \(GW(C, \varepsilon, J)\), depending only on \(\varepsilon\) and \(J\) in the neighborhood, that counts the contribution to the GW invariant of \(X\) of all curves, including multiple covers, whose images lie in the neighborhood. These clusters come in types, depending on the associated invariant.

We then compute the cluster invariants by an isotopy argument, deforming \(J\) on \(X\) to make each cluster equal to a certain standard “elementary cluster” whose local invariant can be explicitly computed. The

---

The research of EI is supported in part by the NSF grant DMS-0905738 and that of THP by the NSF grant DMS-1011793.
count of clusters of each type is an integer, but changes during the isotopy according to several types of wall-crossing formulas. Compensating for this behavior in the spirit of Taubes’ work, and extending the arguments of [IP1] provides an integral virtual count of simple $J$-holomorphic curves that immediately implies the GV conjecture.

There is an extensive literature revolving around this GV conjecture. J. Bryan and R. Pandharipande have a series of papers about it, including two ([BP1] and [BP2]) relevant to our approach. For algebraic 3-folds, several BPS-type integer invariants have been defined using holomorphic bundles, including the Pandharipande-Thomas [PT] and Donaldson-Thomas invariants, with conjectural GV-type correspondences GW/DT/PT between them. For toric 3-folds, Maulik, Oblomkov, Okounkov and Pandharipande proved the GW/DT correspondence by calculating both sides explicitly in a computational tour de force [MOOP]. Pandharipande and Pixton [PP] established the GW/PT correspondence for CY complete intersections in products of projective spaces. Other instances have been observed when a change of variables in the GW series produces integer invariants, including a formula for Fano classes ($A \in \text{H}^2(X)$ with $c_1(A) > 0$) in symplectic 6-manifolds proved by A. Zinger’s [Z], and the computation of Klemm and Pandharipande for Calabi-Yau 4-folds [PK]. In Section 8 we combine our result on the Calabi-Yau classes first with Zinger’s to obtain a GV-type formula for all symplectic 6-manifolds, and then with Klemm and Pandharipande’s to obtain a GV-type formula for semipositive symplectic manifolds.

1. Curves in symplectic Calabi-Yau 6-manifolds

The Gromov-Witten invariants of a closed symplectic manifold $(X, \omega)$ are constructed in two steps. One first forms the universal moduli space and its stabilization-evaluation map $se$

$$
\overline{M}(X) \xrightarrow{se} \overline{M}_{g,n} \times X^n
$$

over the space $\mathcal{J}$ of $\omega$-tame almost complex structures. This moduli space consists of equivalence classes (up to reparametrizations of the domain) of pairs $(f, J)$ where $f : C \to (X, J)$ is a stable pseudo-holomorphic map whose domain $C$ is a nodal Riemann surface; it has components $\overline{M}_{A, g}$ labelled by the genus $g$ of $C$ and the homology class $A = f_*[C] \in H_2(X)$ with $\omega(A) \geq 0$. The restriction of $\pi$ to each component is proper, and the fibers $\overline{M}_{A, g}(X)$ carry a $d$-dimensional virtual fundamental class

$$
[\overline{M}^j_{A, g}(X)]^{virt} \in \text{H}_d(\overline{M}_{A, g}(X); \mathbb{Q})
$$

where

$$
d = 2c_1(X)A + (\dim X - 6)(1 - g)
$$

(cf. [IP3]). These are assembled in a formal power series

$$
GW(X) = \sum_{\omega(A) > 0} \sum_{g=0}^{\infty} [\overline{M}^j_{A, g}(X)]^{virt} t^{2g-2} q^A
$$

with coefficients in the “rational Novikov ring” $\Lambda$ (where $t q^A = q^A t$ and $q^{A+B} = q^A q^B$). Note that the first sum, over all positive $A \in H_2(X)$ omits any trivial contributions (i.e. from the class $A = 0$). For consistency, the term “$J$-holomorphic map” will always mean a non-trivial map, and a “$J$-holomorphic curve” in $X$ is the image of such a map. When $f$ is an embedding, we identify it with its image curve $C$ in $X$.

Gromov-Witten invariants are especially simple if $c_1(X) = 0$ and $\dim X = 6$; such spaces are often called symplectic Calabi-Yau 6-manifolds. In this case both terms in (1.2) vanish, so the virtual fundamental class has dimension $d = 0$ for all $g$ and $A$. It is frequently convenient to define the energy of a
pair \((A,g)\) by

\[ E(A,g) = \max\{\omega(A),g\} \geq 0 \]

and to restrict attention to the subset \(\overline{M}^E(X)\) “below energy \(E\),” meaning the union of all components with \(E(A,g) \leq E\), and the corresponding fibers \(\overline{M}^{J,E}(X)\) of \((\mathbb{M},\mathbb{E})\).

Using the terminology of [MS], a pseudo-holomorphic map \(f\) is called simple if it does not factor though a multiple cover, or equivalently, if it is somewhere injective. A map \(f \in \overline{M}^J(X)\) is regular if the linearization \(D_{f,J}\), given by \((\mathbb{M},\mathbb{E})\), is onto. Since the index \((\mathbb{M},\mathbb{E})\) is 0, each regular map \(f\) has a well-defined sign \((f,J) = \pm 1\), given by the mod 2 spectral flow from \(D_{f,J}\) to any complex operator. Finally, by a Baire subset of the parameter space we mean a countable intersection of open and dense sets.

**Theorem 1.1.** Let \((X,\omega)\) be a symplectic Calabi-Yau 6-manifold. Then for each \(E > 0\) there is a Baire subset \(\mathcal{J}_E^\infty\) of \(\mathcal{J}\) such that for each \(J \in \mathcal{J}_E^\infty\)

(a) All simple \(J\)-holomorphic maps below energy \(E\) are regular and embeddings.

(b) The projection \(\pi\) in \((\mathbb{M},\mathbb{E})\) is a local diffeomorphism around any regular point.

Theorem 1.1 is proved in the Appendix using standard techniques. For our purposes, it is best to replace \(\mathcal{J}_E^\infty\) with a set \(\mathcal{J}_E^{\text{iso}}\) that emphasizes slightly different properties.

**Definition 1.2.** Denote by \(\mathcal{J}_E^{\text{iso}}\) the set of all \(J \in \mathcal{J}\) such that the moduli space \((\overline{M}^{J,E})^{\text{simple}},\text{with the Gromov topology, consists of isolated points, each an embedding into} X\).

**Corollary 1.3.** \(\mathcal{J}_E^{\text{iso}}\) is nonempty and dense. For each \(J \in \mathcal{J}_E^{\text{iso}},\) every \(J\)-holomorphic map \(f: \Sigma \rightarrow X\) below energy \(E\) is a composition \(\varphi \circ \rho\) of a map \(\rho: \Sigma \rightarrow C\) of complex curves and a \(J\)-holomorphic embedding \(\varphi: C \rightarrow X\) (the decomposition is unique up to reparametrizations of \(C\)).

**Proof.** For each \(J \in \mathcal{J}_E^{\text{iso}}\) each simple \(J\)-holomorphic map \(f\) is an embedding and regular by part (a) of Theorem 1.1. By (b), each such point \((f,J)\) is an isolated point of the fiber \(M^{J,E}\) of \(\pi\). The last statement is a consequence of the general results of §6.1 in [MS]. \(\square\)

Unfortunately, the images in \(X\) of the pseudo-holomorphic maps that appear in Theorem 1.1 may accumulate. To focus on the images, consider the “underlying curve” map

\[ c: \overline{M}(X) \rightarrow \mathcal{J} \times \text{Subsets}(X) \]

that associates to each \((f,J)\) the pair \((J,f(C))\) where \(f(C)\) the image of \(f\), regarded as an element of the metric space \(\text{Subsets}(X)\) of all compact subsets of \(X\) with the Hausdorff distance (defined using a fixed background metric on \(X\)). We denote by \(C^J(X)\) the image of \(\overline{M}^J(X)\) and by \(C^E(X)\) the image of \(\overline{M}^E(X)\). With this notation, the underlying curve map becomes

\[
\begin{align*}
\overline{M}^E(X) & \xrightarrow{\pi} \mathcal{J} \\
\mathcal{J} & \xrightarrow{c} C^E(X)
\end{align*}
\]

(1.4)

Gromov compactness implies that \(c\) is continuous and proper. Viewed differently, convergence in \(C^E(X)\) defines a topology on \(\overline{M}^E(X)\) that we will call the “rough topology”.

Notice that the map \(c\) identifies a curve with all its multiple covers. The following partial converse is an immediate consequence of Gromov convergence and the definition of \(\mathcal{J}_E^{\text{iso}}\).

**Lemma 1.4.** For \(J \in \mathcal{J}_E^{\text{iso}}\), if \(f\) is a limit point of a sequence \(\{f_n\}\) in \(\overline{M}_E^{J,E}(X)\) in the rough topology, then \(f\) is a composition \(\varphi \circ \rho\) as in Corollary 1.3 with either \(\deg \rho > 1\), or else \(\deg \rho = 1\) but \(\text{genus}(\Sigma) > \text{genus}(C)\). \(\square\)
Each class $A$ in $H_2(X; \mathbb{Z})$ in the positive cone $\omega(A) > 0$ can be written as $A = d[\beta]$ where $\beta$ is a primitive homology class with $\omega(\beta) > 0$ and $d$ is a uniquely determined positive integer. Let $\Omega(d)$ be the number of prime factors of $d$, counted with multiplicity. The integer $d$ is called the degree of $A$. For any map $f$ from a genus $g$ curve representing a degree $d$ class, we define its level to be

$$\ell(f) = \Omega(d) + g \quad (1.5)$$

The components of the moduli space are filtered by the degree and genus, and therefore by their level; the level filtration will be used frequently in later sections. For each $E$, the sets

$$\overline{\mathcal{M}}^E_m(X) = \bigcup \left\{ \mathcal{M}^E_{A,g}(X) \mid A = d\beta \text{ where } \beta \text{ is primitive and } \Omega(d) + g \leq m \right\} \quad (1.6)$$

filter $\overline{\mathcal{M}}^E(X)$, and their images under $\mathcal{M}^E_m = c(\mathcal{M}^E_m(X))$.

Note that for each $J \in J^E_{\text{isol}}$, the map $c$, when applied to multiply-covered maps, decreases the level but respects the filtration; for such parameter the fiber $\mathcal{C}^E_m \subset \mathcal{C}(X)$ can also be described as the collection of embedded $J$-holomorphic curves with level at most $m$. With this notation, $m = 0$ corresponds to genus zero curves representing primitive classes and $\mathcal{C}^E_m$ is the collection of embedded $J$-holomorphic curves in $X$ with $g + \Omega \leq m$, and energy at most $E$.

**Lemma 1.5.** For any fixed $J \in J^E_{\text{isol}}$

(a) $\mathcal{C}^E_m \subseteq \mathcal{C}^E_{m+1}$ is a filtration of $\mathcal{C}^E$ with $\mathcal{C}^E_m = \mathcal{C}^E$ for $m$ large.

(b) $\mathcal{C}^E_m$ and $\mathcal{C}^E = \bigcup \mathcal{C}^E_m$ are compact countable subsets of the metric space $\mathcal{C}(X)$.

(c) For any neighborhood $U$ of $\mathcal{C}^E_{m-1}$, the set $\mathcal{C}^E_m \setminus U$ is a finite collection of embedded $J$-holomorphic curves.

In particular, there are finitely many genus zero $J$-holomorphic curves with energy less than $E$ representing primitive classes.

**Proof.** Condition (a) is immediate by definition and the fact that only finitely many homology classes are represented by $J$-holomorphic maps below energy $E$.

Next, each set $\mathcal{C}^E_m$ is compact because it is the image of the compact set $\mathcal{M}^E_m$ under the continuous map $\mathcal{M}^E_m \to \mathcal{M}^E$. Then $\mathcal{C}^E_m \setminus U$ is a closed subset of the compact metric space $\mathcal{C}^E_m$, so any infinite sequence $\{C_i\}$ has an accumulation point $C_0$. Because only finitely many homology classes are represented by $J$-holomorphic maps below energy $E$ we may assume, after passing to a subsequence, that they all have the same genus and same homology class $[C_i] = k\beta$ for the same primitive class $\beta$ and the same $k$ with $\Omega(k) + g \leq m$. By Lemma 1.4 the limit is a multiple cover of a curve of a strictly lower level, but that’s impossible because $U$ is open. Thus $\mathcal{C}^E_m \setminus U$ is finite. Finally, taking $U_k$ to be the $1/k$ tubular neighborhood of $C_{m-1}$, we conclude that $\mathcal{C}^E_m \setminus \mathcal{C}^E_{m-1} = \bigcup_k (\mathcal{C}^E_m \setminus U_k)$ is countable, and hence $\mathcal{C}^E_m$ and $\mathcal{C}^E$ is countable.

\[ \square \]

2. Clusters in symplectic manifolds

A decomposition of the moduli space $\overline{\mathcal{M}}^{S,E}(X)$ over a subset $S \subset J$ is a way of writing it as a finite disjoint union $\bigcup_i \mathcal{O}_i$ of subsets $\mathcal{O}_i$ that are both open and closed in the rough topology. Given such a decomposition, the isomorphism

$$\tilde{H}_*(\overline{\mathcal{M}}^{S,E}(X); \Lambda) \cong \bigoplus_i \tilde{H}_*(\mathcal{O}_i; \Lambda) \quad (2.1)$$

gives a decomposition of the GW invariant over $S$, namely

$$GW^E(X) = \sum_i GW^E(\mathcal{O}_i) \quad (2.2)$$
where $GW(O_i)$ is the corresponding component of $(2.3)$ under the isomorphism $(2.4)$.

A simple way to obtain such sets $O$ is to choose an open set $U$ in $C(X) \subseteq \operatorname{Subsets}(X)$ whose boundary does not intersect $C^{J,E}$.

**Lemma 2.1.** Each subset $U$ of $C(X)$ with $\partial U \cap C^{E,J} = \emptyset$ has a well-defined invariant $GW^E(U,J)$. The collection of $J$ for which $\partial U \cap C^{E,J} = \emptyset$ is open in the $C^0$ topology on $J$, and the invariant $GW^E(U,J)$ is locally constant as a function of $J$.

**Proof.** The assumption implies that the intersections of both $U$ and its complement $U^c$ with $C^{E,J}$ are open subsets of the image $C^{E,J}(c)$. Since $c$ is continuous, $O = c^{-1}(U)$ and $O^c = c^{-1}(U^c)$ are open and closed subsets of $\overline{M}^{E,J}(X)$. Define $GW^E(U,J) = c_*GW^E(O,J)$ by formula (2.2) associated to the decomposition $O \sqcup O^c$.

Next note that condition $\partial U \cap C^{E,J} = \emptyset$ is an open condition on $J$: if not, there would be a sequence of $J_n \to J$ in $C^0$ and a sequence of $J_n$-holomorphic curves $C_n$ in $\overline{U}$ with energy bounded by $E$ (which by our convention includes a bound on the genus). But then, by Gromov compactness a subsequence would converge to a $J$-holomorphic curve $C$ with $C \in \partial U \cap C^{E,J}$ because $\partial U$ and $C^{E,J}$ are both closed, contradiction.

Therefore there exits a $C^0$ ball $V$ around $J$ in $J$ such that $\partial U \cap C^{E,V} = \emptyset$. This gives rise to the same decomposition $O \sqcup O^c$ but now of the moduli space $\overline{M}^{E,J}(X)$ over the entire ball $V$, therefore the contribution of $U$ is defined for each $J \in V$ and it is constant on $V$. □

The geometric content of the invariant $GW^E(U,J)$ is most clearly seen by choosing $U$ to be a ball of small radius in the Hausdorff distance centered at a $J$-holomorphic curve. The set $O_1$ is then a cluster of holomorphic curves in $X$, which we will require to have the following properties:

**Definition 2.2.** A cluster $(C,\varepsilon,J)$ in $X$ below energy $E$ (an “$E$-cluster”) consists of an almost complex structure $J \in J$, an embedded $J$-holomorphic curve $C$ and a radius $\varepsilon > 0$ with the following conditions:

(a) All non-constant $J$-holomorphic maps in the ball $B(C,\varepsilon)$ with energy $\leq E$ represent $k[C]$ for some $k \geq 1$, and have genus $g \geq g(C)$;

(b) $C$ is the only $J$-holomorphic map in its degree and genus in the ball $B(C,\varepsilon)$.

(c) There are no $J$-holomorphic curves with energy $\leq E$ at precisely $\varepsilon$ Hausdorff distance from $C$.

The curve $C$ is called the core of the cluster and the cluster is regular if the embedding $C \hookrightarrow X$ is a regular $J$-holomorphic map.

The next lemma shows that small balls in the Hausdorff metric are often clusters. In fact, conditions (a) and (b) of Definition 2.2 are automatic for small $\varepsilon$ when $C$ is regular. Condition (c) is the important one: it implies that $O = B(C,\varepsilon,J)$ has a well-defined contribution

\[
GW^E(O) \in \tilde{H}_0(O;\Lambda).
\] \hspace{1cm} (2.3)

**Lemma 2.3** (Cluster existence). For each $J \in \mathcal{J}^{E}_{\text{isol}}(X)$ and each simple $J$-holomorphic curve $C$, the set $S$ of $\varepsilon > 0$ for which the ball $B(C,\varepsilon)$ is an $E$-cluster is open and dense in a non-empty interval $[0,\varepsilon_C]$, and the contribution (2.3) is locally constant on $S$.

**Proof.** By definition, for any $J \in \mathcal{J}^{E}_{\text{isol}}(X)$, all simple $J$-holomorphic curves are embedded and isolated in their degree and genus. Since $C^{J,E}$ is compact, and an embedded curve $C$ can appear as an accumulation point only of curves representing $k[C]$ and having genus at least that of $C$, Lemma 1.4 implies that conditions (a) and (b) of Definition 2.2 hold for all $\varepsilon \leq \varepsilon_C$.

Finally, by Lemma 1.5, the image of $C^{J,E}_m$ under the distance function $B(C,\varepsilon_C) \to [0,\varepsilon_C]$ is countable and compact, so its complement – which is the set of $\varepsilon$ that satisfy condition (c) of Definition 2.2 – is open and dense.

**Theorem 2.4** (Cluster decompositions). Given $E$ and $J \in \mathcal{J}^{E}_{\text{isol}}(X)$, each subset $U$ of $C(X)$ such that $\partial U \cap C^{J,E} = \emptyset$ has a finite $E$-cluster decomposition $\{O_i = (C_i, J_i, \varepsilon_i)\}$, and hence

\[
GW^E(U) = \sum_i GW^E(O_i).
\] \hspace{1cm} (2.4)
Theorem 3.2. Structures. Similar examples exist over any higher genus curve, although they involve non-integrable almost complex structures.

Proof. Fix a curve \( Y \) and then point out that the GW series for these curves has been calculated by J. Bryan and R. Pandharipande [BP2].

Definition 3.1. In a symplectic Calabi-Yau 6-manifold \( X \), a cluster is called elementary if

(a) The core curve \( C \) is balanced, meaning its normal bundle splits as \( N_{C} = L \oplus L \) and the normal component \( D_{\alpha} \) of the linearization also splits as \( D^* \pm D^* \).

(b) The only non-trivial J-holomorphic maps into \( B(C) \) are multiple covers of the embedding \( C \hookrightarrow X \).

(c) For each such cover \( \rho \), the pullback operator \( \rho^* D^* \) is injective (\( C \) is super-rigid).

Property (c) implies that the core \( C \) of an elementary cluster is a regular J-holomorphic map, and one can show it also implies (b) for sufficiently small \( \varepsilon > 0 \) by the rescaling argument of [BP2].

When \( C \) is a rational curve, the unit disk bundle in \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) is an elementary cluster. Similar examples exist over any higher genus curve, although they involve non-integrable almost complex structures.

Theorem 3.2. For any smooth complex curve \( C \), there exits an elementary cluster whose core is \( C \).

Proof. Fix a curve \( C \) of genus \( g \), a theta characteristic \( L \), and a Kähler structure \((J,g,\omega)\) on the total space \( Y \) of \( L \oplus L \to C \) compatible with its holomorphic structure. This means that \( L^2 \) is identified with the canonical bundle \( K_C \) of \( C \) and that the canonical bundle \( K_Y = \pi^*(L^{-2}K_C) \) of \( Y \) is trivial. Thus \( Y \) is a Calabi-Yau 6-manifold. We will perturb \( J \) to obtain an almost complex structure on the unit disk bundle \( D \subset Y \) that makes \( D \) an elementary cluster.

First consider the total space \( Z \) of \( p : L \to C \). Its canonical bundle \( K_Z = T^{2,0}Z \) is the pullback \( p^*(L^*K_C) = p^*(L) \), which has a canonical section \( \beta \) whose divisor is the zero section. Pullback \( \beta \), regarded as a 2-form, by the bundle projections \( p_1, p_2 : Y \to Z \) onto the first and second copy of \( L \) and set \( \alpha = p_1^*\beta + p_2^*\beta \). Then \( \alpha \) is a closed \((2,0)\)-form on \( Z \) whose divisor is zero section of \( L \oplus L \), which is
the core curve $C$ of the disk bundle $D \subset Y$. Following Junho Lee, define a bundle map $K_{\alpha} : TY \to TY$ by $g(u, K_{\alpha}v) = \overline{\alpha}(u, v)$ for all $u, v \in TY$ and set

$$J_\alpha = (I + JK_{\alpha})^{-1} J(I + JK_{\alpha}).$$

Then $J_\alpha$ is an $\omega$-tame almost complex structure after replacing $\alpha$ by $t\alpha$ for small $t > 0$ (cf. [L] and Section 1 of [LP]). In this context, Lee proved in [L] that the image of every non-trivial $J_\alpha$-holomorphic map into $Y$ lies in the zero set $C$ of $\alpha$ or is everywhere tangent to $ker K_{\alpha}$. It is straightforward to check that at each point $p \in Y$ not on the zero section, $ker K_{\alpha}$ is vertical. Consequently, any map whose image is tangent to $ker K_{\alpha}$ lies in a fiber of $Y \to C$. We conclude that every $J_\alpha$-holomorphic map into $D$ that represents $k[C]$ for some $k \neq 0$ is a map into the core curve $C$ of the disk bundle $D$.

Along the zero section of $Z$, $\nabla K_{\alpha}$ decomposes under the splitting $L \oplus L$ as $\nabla K_{\beta} \oplus \nabla K_{\beta}$ where $\nabla K_{\beta}$ is a 1-form with values in endomorphisms of $L$. Correspondingly, the normal component $D^N$ of the linearization $D$ (cf. (3.1)) splits as $D^N_{\beta} \oplus D^N_{\beta}$. As in Section 4 of [LP], the normal component of $D_{\beta}$ is the sum $\overline{\partial} + R_{\beta}$ of the $\overline{\partial}$-operator on $L$ and a bundle map $R_{\beta} : L \to T^{0,1}C \otimes L$ that satisfies $JR_{\beta} = R_{\beta} J$. The injectivity condition (c) of Definition 3.1 is then exactly the statement of Theorem 4.2 of [LP].

By Lemma 3.1, an elementary cluster has a well-defined invariant $GW(C)$, independent of $\varepsilon$. As usual, there is an associated disconnected invariant

$$Z(C) = \exp(GW(C))$$

obtained by exponentiating in the Novikov ring. It turns out that $Z(C)$ is the more easily calculated.

**Theorem 3.3.** The disconnected GW invariant of an genus $g$ elementary cluster is given by

$$Z^{\text{elem}}(C) = 1 + \sum_{d \geq 1} \sum_{\rho \vdash d} \prod_{\square \in \rho} \left(2 \sin \frac{h(\square)t}{2}\right)^{2g-2} q^{dC}$$

(3.1)

where the second sum is over all partitions $\rho$ of $d$, the product is over the squares in the Ferrer’s diagram, and $h(\square)$ is the associated hhooklength.

**Proof.** Because for an elementary curve the linearization on covers is injective, the contribution to the GW invariant of its multiple covers can be calculated using the Euler class of Taubes obstruction bundle (this is the 6-dimensional version of the setup in [LP]).

Consider the moduli space $\underline{M}_{d,\chi}(C)$ of degree $d$ holomorphic maps $\rho$ to $C$ whose domain is possibly disconnected and has Euler characteristic $\chi$, and where $\rho$ is non-trivial on each connected component. This carries a virtual fundamental cycle $[\underline{M}_{d,\chi}(C)]^{vir}$ of even complex dimension $b = d(2-2g) - \chi$. The operators $D_N$ and $D'$ of Definition 3.1 induces a families of real operators over $\underline{M}(C)$ whose fibers at $\rho$ are the pullback operators $\rho^* D_N$ and $\rho^* D'$. By Definition 3.1, the corresponding index bundles satisfy $\text{Ind } D_N = \text{Ind } D' \oplus \text{Ind } D'$. *A priori*, these are real virtual bundles, but Definition 3.1 insures that $-\text{Ind } D'$ is an actual vector bundle of rank $b$.

With this notation, the elementary contribution is equal to the integral

$$Z_{d,\chi}(C, J_\alpha) = \int_{[\underline{M}_{d,\chi}(C)]^{vir}} e(Ob)$$

of the Euler class of the Taubes obstruction bundle $Ob = -\text{Ind}_R D_N$. Writing $Ob' = -\text{Ind}_R D'$, we have

$$e(Ob) = e(Ob' \oplus Ob') = e(Ob') \cup e(Ob') = p_b(Ob') = (-1)^{b/2} c_b(Ob' \otimes_R C).$$

This Chern class factors through $K$-theory (general Euler classes do not). Because $D' = \overline{\partial}_L + R$ is a 0'th order deformation of the complex operator $\overline{\partial}_L$, the complexification of the index bundles of $D'$ and $\overline{\partial}_L$ are equal in $K$-theory, so

$$c_b(Ob' \otimes_R C) = c_b(-\text{Ind } \overline{\partial}_L \otimes_R C) = c_b(-\text{Ind } \overline{\partial}_L \oplus (-\text{Ind } \overline{\partial}_L)^*)$$
Now consider the \( T = \mathbb{C}^\ast \times \mathbb{C}^\ast \)-action on the total space \( Y \) of the holomorphic bundle \( N_C = L \oplus L \). With the antidiagonal \( \mathbb{C}^\ast \)-action, \( Y \) is an equivariant local Calabi-Yau 3-fold. Bryan-Pandharipande defined a “residue” generating function \( Z_T^X(Y) \) whose coefficients

\[
Z_{d,\chi}^T(Y) = \int_{\left[ M_{d,\chi}(C) \right]_{\vir}} e(\operatorname{Ind} L \oplus (-\operatorname{Ind} L))
\]

are equivariant integrals (defined by localization). They also calculated the reduction of \( Z_{d,\chi}^T(Y) \) to the antidiagonal action to be the ordinary integral

\[
Z_{d,\chi}^T(Y) = \left(-1\right)^{b/2} \int_{\left[ M_{d,\chi}(C) \right]_{\vir}} \left[ M \circ d \right]_{\chi}(Y) \vir \left(-\operatorname{Ind} L \oplus (-\operatorname{Ind} L)\right)
\]

and also equal to the coefficient of the series appearing on the right hand side of (3.1) (cf §2 and §7 of [BP2]). Combining the last five displayed equations completes the proof.

The series (3.1) depends only on the genus \( g \) of \( C \). Taking log and separating the \( d = 1 \) term, we can write

\[
GW_{g}^{\elem}(q^C) = \log(Z_{g}^{\elem}(C)) = q^C \left( 2 \sin \frac{t}{2} \right)^{2g-2} + \sum_{d \geq 2} \sum_{h \geq g} GW_{d,h}(g) q^{dc} t^{2h-2}.
\]

In particular, the core \( C \) of any elementary cluster has sign \( C > 0 \).

Now apply the “BPS transform”, which takes an arbitrary element of the Novikov ring to another by

\[
\sum_{A,g} N_{A,g} t^{2g-2} q^A \mapsto \sum_{A,g} n_{A,g} \sum_{k=1}^{\infty} \frac{1}{k} \left( 2 \sin \frac{kt}{2} \right)^{2g-2} q^{kA}
\]

This transform is well-defined and invertible (see [BP1]). Thus we can write

\[
GW_{g}^{\elem}(q^C) = \sum_{d \geq 0} n_{d,h}(g) \sum_{k=1}^{\infty} \frac{1}{k} \left( 2 \sin \frac{kt}{2} \right)^{2h-2} q^{kdC}.
\]

for uniquely determined coefficients \( n_{d,h}(g) \) that are, \textit{a priori} rational numbers. These coefficients have been been explicitly calculated in low degree and genus (e.g. for degree \( d \leq 2 \) and genus \( g \leq 1 \)). A combinatorics argument, given in the Appendix, handles the case \( g \geq 2 \), yielding a basic fact:

\textbf{Theorem 3.4.} The local BPS conjecture is true for elementary clusters.

\qed

\[4. \text{The structure of Moduli space}\]

Consider next the universal moduli space of simple maps (with smooth, connected domains)

\[\mathcal{M}_{\text{simple}} \xrightarrow{\pi} \mathcal{J}\]

with the projection \( \pi(f,J) = J \). Note that \( \mathcal{M}_{\text{simple}} \) is an open subset of the universal moduli space \( \mathcal{M}(X) \) of (1.1), and simple maps have trivial automorphism groups (cf. [MS]). It contains as an open subset \( \mathcal{M}_{\text{emb}} \) of maps that are embeddings.

We say \( x_0 \in C \) is a \textit{injective regular point} of \( f : C \to X \) if there exists a neighborhood \( U \) of \( x_0 \) such that (i) \( f^{-1}(f(x)) = x \) for all \( x \in U \) and (ii) \( df(x) \neq 0 \) for all \( x \in U \). These conditions imply that the image \( f(U) \) is an embedded submanifold of \( X \) and \( f^{-1}(f(U)) = U \). The Micallef-White Theorem [MW] implies that the set of regular points of a simple \( J \)-holomorphic map is open and dense in \( C \).
All of the analysis in this section is localized around a point \( p = (f, C, J) \) in the moduli space consisting of a simple \( J \)-holomorphic map \( f : C \to X \) whose domain is a smooth, connected complex curve \( C = (\Sigma, j) \), and \( J \) is an almost complex structure. After fixing a Sobolev norm that is stronger than the \( C^1 \) norm, the space \( \text{Map}(\Sigma, X) \) of maps, the space \( Cx(\Sigma) \) of complex structures on \( \Sigma \), and \( J \) are separable Banach manifolds (cf. \cite{Pal}, \cite{MS}).

All of the analysis in this section is localized around a point \( p = (f, C, J) \) in the moduli space consisting of a simple \( J \)-holomorphic map \( f : C \to X \) whose domain is a smooth, connected complex curve \( C = (\Sigma, j) \), and \( J \) is an almost complex structure. After fixing a Sobolev norm that is stronger than the \( C^1 \) norm, the space \( \text{Map}(\Sigma, X) \) of maps, the space \( Cx(\Sigma) \) of complex structures on \( \Sigma \), and \( J \) are separable Banach manifolds (cf. \cite{Pal}, \cite{MS}).

### 4.1. Slice and linearization

The moduli space \( \mathcal{M}_{g,n}(X) \) is naturally a subset of the quotient of \( \text{Map}(\Sigma, X) \times Cx(\Sigma) \times J \) by the action of the diffeomorphism group of \( \Sigma \). In practice, one chooses a slice for the diffeomorphism action at \( p \) and regards the moduli space as a subset of the slice. If \( C \) has no automorphisms, a slice can be defined by fixing a trivialized universal deformation of \( C \), that is, an open set \( U \subset \mathbb{C}^N \) of dimension \( N = 3g - 3 + n \) and a complex structure on \( U \times \Sigma \to U \) such that the fibers are complex, the central fiber \( \{0\} \times \Sigma \) is biholomorphic to \( C \), and every small deformation of \( C \) is equivalent, under a diffeomorphism, to exactly one fiber of \( U \times \Sigma \). Then

\[
\text{Slice}_p \subset \text{Map}(C, X) \times U \times J
\]

is a slice. If \( \text{Aut}(C) \) is non-trivial, fix injective regular points \( x_1, \ldots, x_k \) in \( C \) so that the pointed curve \( \tilde{C} = (C, x_1, \ldots, x_k) \) has \( \text{Aut}(\tilde{C}) = 1 \); one can then fix a trivialized universal deformation of \( \tilde{U} \times \Sigma \to \tilde{U} \).

At each image point \( y_i = f(x_i) \), choose a local transverse 4-disk \( V_i \subset X \) through \( y_i \) and transverse to \( f(C) \). Standard results (cf. Section 3.4 of \cite{MS}) show that the space \( \text{Map}_p(C, X) \) of maps satisfying \( f(x_i) \in V_i \) for all \( i \) is locally a manifold near \( p \), and hence

\[
\text{Slice}_p \subset \text{Map}(\tilde{C}, X) \times \tilde{U} \times J
\]

is again a slice. Thus defined, each point in the slice is a pair \((f, J)\) where \( f : C \to X \) is a map whose domain has no non-trivial automorphisms.

The slice comes with a projection

\[
\pi : \text{Slice}_p \to J
\]

defined by \( \pi(f, J) = J \) and a vector bundles \( F \to \text{Slice}_p \) whose fiber over \((f, J)\) is \( \Omega^{0,1}(f^*TX) \). Around each \( p = (f, J) \), the moduli space, considered as a subset of \( \text{Slice}_p \), is the zero set of the section \( F \) of \( \Omega^{0,1} \) defined by

\[
F(f, J) = \overline{\partial}_J f. \tag{4.2}
\]

Using the natural isomorphism \( T_0 \tilde{U} = H^{0,1}(T \tilde{C}) \), the tangent space to the slice can be written as

\[
T_p \text{Slice}_p = \mathcal{E}_p \oplus T_J J \tag{4.3}
\]

where \( \mathcal{E} \) is the bundle over the slice whose fiber over \((f, J)\) is

\[
\mathcal{E}_{(f, J)} = \left\{ \xi = (\zeta, j) \in \Omega^0(f^*TX) \oplus H^{0,1}(T \tilde{C}) \mid \zeta(x_i) \in TV_i \right\}.
\]

The linearization of the \( J \)-holomorphic map equation \([\text{eq} 4.2]\) at \( p \) is the operator \( \mathcal{L}_p : \mathcal{E}_p \oplus T_J J \to \mathcal{F}_p \) given by

\[
\mathcal{L}_p(\xi, K) = D_p(\xi, k) + \frac{1}{2} K \circ f_\ast \circ j \tag{4.4}
\]

where \( D_p : \mathcal{E}_p \to \mathcal{F}_p \), which is the linearization under variations that fix \( J \), applied to \( \xi = (\zeta, k) \) is

\[
D_p(\xi)(w) = \frac{1}{2} \left[ \nabla_w \zeta + J \nabla_{jw} \zeta + (\nabla_\zeta J)(f_\ast(jw)) + Jf_\ast k(w) \right]. \tag{4.5}
\]

Both \( \mathcal{L}_p \) and \( D_p \) depend on \( J \) only through the 1-jet of its restriction to the image of \( f \). Also note that, even if the domain \( C \) is an unstable curve, \( D_p \) restricts to an isomorphism \( \mathcal{E}_p^C \to \mathcal{F}_p^C \) between the subbundles

\[
\mathcal{E}_p^C = \{ (\zeta, k) \in \mathcal{E}_p \mid \zeta \in \Omega(f^*TC) \} \quad \text{and} \quad \mathcal{F}_p^C = \{ \eta \in \mathcal{F}_p \mid \eta \in \Omega^{0,1}(f^*TC) \}.
\]
For notational simplicity, we will henceforth write $C$ as $C$.

4.2. **The universal moduli space.** The following theorem describes the structure of $\mathcal{M}_{\text{simple}}$ and two types of natural submanifolds. First, the “wall” $W \subset M$ is the subset defined by

$$W = \bigcup_{r \geq 1} W^r, \quad W^r = \{(f, J) \in M_{\text{simple}} \mid \dim \ker D_p = r\}. \quad (4.6)$$

Second, each embedded complex curve $\iota_C : C \hookrightarrow X$ determines a set $\mathcal{J}_C \subset \mathcal{J}$ consisting of all $J \in \mathcal{J}$ for which $\iota_C$ is $J$-holomorphic. The universal moduli space $\pi : M_{\text{simple}} \to \mathcal{J}_C$ over $\mathcal{J}_C$ has a canonical section $J \mapsto (\iota_C, J)$ whose image is a submanifold $M_C = \{\iota_C\} \times \mathcal{J}_C$.

**Theorem 4.1.** (a) $M_{\text{simple}} \to \mathcal{J}$ is a manifold and its tangent space at $p = (f, J)$ is $\ker \mathcal{L}_p$.

(b) $\pi$ is a local diffeomorphism on $M \setminus W$.

(c) $W^r$ is a codimension $r^2$ submanifold of $M_{\text{simple}}$.

(d) For each embedded curve $C$, $M_C = \{\iota_C\} \times \mathcal{J}_C$ is transverse to each $W^r$.

**Proof.** Parts (a) and (b) are well-known; we present their proofs for later use. By the Implicit Function Theorem, $M_{\text{simple}}$ is a manifold at those points $p$ where $\mathcal{L}_p$ is onto. Equation (4.4) shows that this fails only at maps $(f, J)$ where there is a non-zero $c \in \ker D_p^* \setminus 0$ with

$$0 = \int_C \langle c, Kf, j \rangle.$$

for every variation $K$ in $J$. Since $f$ is simple, it has an injective regular point $x \in C$. One can find a bump function $\beta \geq 0$ supported near $x$ and a variation $K$ in $J$ so that $Kf, j = \beta c$, thereby concluding that $c \equiv 0$ (Proposition 3.4.1 of [MS]).

(b) The projection $\pi^*(\xi, K) = K$ of a non-zero element in $\ker \mathcal{L}_p$ is zero if and only if $0 \neq \xi \in \ker D_p$. But $D_p$ has index 0, so is onto at each point where $\ker D_p = 0$. Hence for each $K$ we can use (4.4) to obtain $\xi$ with $\mathcal{L}_p(\xi, K) = 0$; then $(\xi, K)$ is tangent to $M$ and $\pi^*(\xi, K) = K$. Thus $\pi^*$ is an isomorphism at each point not on $W$.

(c) Let $\text{Fred} \to M$ be the vector bundle whose fiber over $(f, J)$ is the space of index 0 Fredholm maps of type (??). Fred is the union of strata $\mathcal{F}^r = \{A \in \text{Fred} \mid \dim \ker A = r\}$ where each $\mathcal{F}^r$ a submanifold of codimension $r^2$ whose normal bundle at $A$ is naturally identified with $\text{Hom}(\ker A, \text{coker} A)$ (cf. [K]). Hence it suffices to show that the section of Fred defined by $\Psi(f, J) = D_f, J$ is transverse to each stratum $\mathcal{F}^r$. More specifically, at a point $(f, J) \in M_{\text{simple}}$ with $\Psi(f, J) \in \mathcal{F}^r$, we must show that for any non-zero elements $\kappa \in \ker D_p^*$ and $\kappa \in \ker D_p$ there is a variation in $(f, J)$ so that

$$\int_{\Sigma} \langle c, (\delta D_p)\kappa \rangle \neq 0.$$ 

Consider variations $(\dot{f}, \dot{J}) = (0, L)$ where $L$ vanishes along the image $f(C)$. Such variations are in $\ker \mathcal{L}_p \setminus 0$ so are tangent to $M$. Equation (4.5) then shows that $(\delta D_p)\kappa = (\nabla_x K) f, j$, and this depends only on the normal component $\kappa^N$ of $\kappa$.

Now $c$ and $\kappa$ are solutions of elliptic equations which, as remarked after (4.6), are not everywhere tangent to $C$. By the unique continuation property, $\kappa^N$ and $c^N$ are non-zero almost everywhere, so we can fix an injective regular point $x \in C$ where $\kappa^N(x)$ and $c^N(x)$ are not zero. This point has a neighborhood $U$ such that $V = f(U)$ is a submanifold of $X$; let $N_V$ denote its normal bundle and let $\beta \geq 0$ be a smooth cutoff function supported on the ball $B(f(x), \varepsilon) \subset X$ with $\beta_{\varepsilon}(f(x)) = 1$. Next, choose a section $\varphi$ of $\Omega^{0,1}(V, \text{End}(TX))$ along $V$ with $\varphi|_{TV} = 0$ and with $\varphi(\kappa)(f, v) = -c(jv)$ for all $v \in T_x C$. Integrating in the directions normal to $V$ then yields an $\text{End}(TX)$-valued $(0, 1)$-form $K$ supported near $f(x)$, vanishing along $V$ and satisfying $(\nabla_x \Phi) f, j = (\varphi(\kappa)) f, j = c$ at $x$. For this $K$, with $\varepsilon$ small, the integral (4.7) is

$$\int_{\Sigma} \langle c, (\nabla_x K) f, j \rangle \geq \frac{1}{2} \int_U \beta \varepsilon |c|^2 > 0. \quad (4.7)$$

Finally, (d) holds because this inequality is achieved using a perturbation that vanishes along $C$. □
4.3. Wall crossings. We next study the local geometry of the moduli space around a point \( p = (f, J) \) on the wall \( W^1 \) in the universal moduli space \( M_{\text{emb}} \) and in its subset \( M_C = \{ f_C \} \times J_C \). We assume that \( p \) corresponds to a \( J \)-holomorphic embedding \( f : C \to X \) of a smooth curve \( C \) which we identify with its image in \( X \).

First observe that there is a subset \( A \subset W^1 \) where the wall fails to be transverse to \( \ker \pi_{\ast} \). While \( A \) is intrinsic, it can be described locally near \( p_0 \in W^1 \cap M_{\text{emb}} \), by choosing smoothly varying non-zero elements \( \kappa \in \ker D_p \) and \( c \in \ker D_p^\ast \) along \( W^1 \), extending to a neighborhood \( O \) of \( p_0 \) in \( M_{\text{emb}} \), and defining a function \( \psi : O \to \mathbb{R} \) by

\[
\psi(p) = \int_C \langle c, D_p \kappa \rangle.
\]

Then the zero set of \( \psi \) is exactly \( W^1 \cap O \) and

\[
A \cap O = \{ p \in W^1 \cap O \mid (d\psi)_p \kappa = 0 \}.
\]

**Lemma 4.2.** The set \( A \subset W^1 \) is a codimension 1 submanifold transverse to \( M_C \).

**Proof.** Locally, on each set \( O \) as above, \( A = \varphi^{-1}(0) \) where \( \varphi : W^1 \to \mathbb{R} \) is defined by \( \varphi(p) = (d\psi)_p(\kappa) = \langle c, (\delta_p D_p) \kappa \rangle \). The lemma follows provided we can show that for each \( p \in A \) there is a \( v \in T_p(W^1 \cap M_C) \) with \( (d\varphi)_p(v) \neq 0 \). Notice that each \( p = (f, J) \in A \) is a critical point of \( \psi \). Therefore the hessian \( H\psi \) is well-defined and symmetric at \( p \), and we must show that \( (H\psi)_p(\kappa, v) \neq 0 \). As in (4.7), we take \( v = (0, K) \) where \( K \) is supported on ball \( B \) around the image of a somewhere injective point of \( f \), and we now assume both \( K = 0 \) and \( \nabla K = 0 \) on \( B \). Then \( v \in T_p(W^1 \cap M_C) \) and again formula (4.5) shows that \( \delta_p D(\kappa) = (\nabla_\kappa K) f_s j \) and

\[
(H\psi)_p(\kappa, v) = \int_{f^{-1}(B)} \langle c, (\nabla_\kappa \nabla_\kappa K) f_s j \rangle.
\]

The argument used for (4.7) produces a \( K \) with \( K = \nabla K = 0 \) on \( B \) that makes this integral non-zero. \( \Box \)

We next apply the Kuranishi method at a point \( p \in W^1 \) that is not in \( A \). Fix three vectors in the tangent space to the slice at \( p = (f, J) \): a generator \( (\kappa, 0) \) of \( \ker D_p \cong \mathbb{R} \), and vectors \((0, K_1)\) and \((0, K_2)\) satisfying

\[
K_1|_{T_C} = 0 \quad \pi_{\text{cok}}(K_2 f_s j) \neq 0
\]

where \( \pi_{\text{cok}} \) is the projection to the cokernel of \( D_p \). Fix a path \( \gamma(t) = J_t \) in \( J_C \) with \( \gamma(0) = J \) and tangent vector \( \dot{\gamma}(0) = K_1 \), and a 2-dimensional surface \( S \subset \mathcal{J} \) containing \( \gamma \) whose tangent space \( T_{J_t} S \) is \( \text{span}(K_1, K_2) \). Because \( J_t \) is constant along \( f(C) \), \( \gamma(t) \) lifts to a path \( \tilde{\gamma}(t) = (f, J_t) \) in \( \mathcal{M}^S \). In this context, a 3-dimensional subset \( B \subset \text{Slice}_p \) with coordinates \((x, y, z)\) is a called a lift of \( S \) if

(i) \( \tilde{\gamma}(t) \subset B \) for \( |t| \) small.

(ii) \( \pi : B \to S \) is given by \( \pi(x, y, z) = (y, z) \), and \( \tilde{\gamma}(t) = (0, t, 0) \).

(iii) \( T_p B \) is spanned by \( \ker D_p \) and \( T_{J_t} S \), and

\[
\frac{\partial}{\partial x} = (\kappa, 0) \quad \frac{\partial}{\partial y} = (0, K_1) \quad \frac{\partial}{\partial z} = (0, K_2)
\]

**Theorem 4.3** (Kuranishi model). At \( p \in \mathcal{M} \cap (W^1 \setminus A) \), there is a coordinate 3-ball \( B' \subset \text{Slice}_p \) satisfying (i)-(iii) above such that \( \mathcal{M}^S = \pi^{-1}(S) \) is locally the variety

\[
\mathcal{M}^S = \{(x, y, z) \in B' \mid z = x(ax + by)\}
\]

with \( a \neq 0 \), and with \( b \neq 0 \) if \( K_1 \) is transverse to \( T_p W^1 \).
Proof. Under the identification (4.3), $M^S(X)$ is locally the zero set of the section (4.2) of $F$. Since $p \in \mathcal{W}^1$, we have $\dim \ker D_p = 1$ and index $D_p = 0$, so there are $L^2$-orthogonal decompositions

$$E_p = E_0 \oplus E_1, \quad F_p = F_0 \oplus F_1$$

Locally identify the slice $\text{Slice}_p$ with its tangent space (4.3) at $p$ and locally trivialize the bundle $F$ near $p$. With these identifications, (4.2) has an expansion

$$F(\xi) = \mathcal{L}_p(\xi) + Q(\xi)$$

where $Q(\xi)$ vanishes to second order at $\xi = 0$. In coordinates, using (4.3), $\xi$ can be written as $\xi = (x, y, z, \alpha)$ where $(x, y, z) \in B \subset E_0 \times S$ and $\alpha \in E_1$. Substituting, using (4.4), and noting that $D_p\kappa = 0$ and $K_1|_{TC} = 0$ gives

$$F(x, y, z, \alpha) = D_p\alpha + \frac{\xi}{2} \cdot K_2f, j + Q(x, y, z, \alpha).$$

Under the decomposition $F_p = F_0 \oplus F_1$, write $K_2f, j = 2(c_p, K_j')$ and $Q = (-q, Q')$, and note that $c_p \neq 0$ by (4.10). Define a map $\eta : E_0 \times S \times E_1 \to E_0 \times S \times E_1$ by $\eta((x, y, z, \alpha)) = (x, y, z, \eta_{x,y,z}(\alpha))$ where

$$\eta_{x,y,z}(\alpha) = \alpha + D_p^*G[zK_2' + Q'(x, y, z, \alpha)]$$

where $G$ is the Green operator of the Laplacian $D_pD_p^*$ on $F_1$. After completing $E$ in a Sobolev norm $\| \cdot \|_k$ with $k \geq 3$ and completing $F$ in the $\| \cdot \|_{k-1}$ norm, $D_p$ and $D_p^*G$ are bounded operators and, by the Inverse Function Theorem, $\eta$ is a local diffeomorphism near 0 in the $\| \cdot \|_k$ space and hence, by the Sobolev embeddings, on $C^\infty$. The calculation

$$D_p(\eta_{x,y,z}(\alpha)) = D_p\alpha + D_pD_p^*G[zK_2' + Q'(x, y, z, \alpha)]$$

$$= D_p\alpha + [z(\frac{\partial}{\partial z}K_2f, j - c_p) + (Q + q)(x, y, z, \alpha)]
= F(x, y, z, \alpha) - zc_p + q(x, y, z, \alpha)$$

shows that the restriction of $\eta^{-1}$ to $\xi = (x, y, z) \in \mathcal{E}_0 \times S$ gives new coordinates — still called $(x, y, z)$ — on a possibly smaller ball $B'$ in which $M^S$ is the variety

$$M^S = \{(x, y, z) \in B' \mid z = q'(x, y, z)\}.$$

in a trivialization $F_0 = \mathbb{R}$ that identifies $c_p$ with 1. Here $q'$ is real-valued, smooth and vanishes to second order at the origin, so can be written as $ax^2 + bxy + cy^2 + axz + byz + \gamma z^2 + r(x, y, z)$ where $r$ vanishes to order 3 at the origin. The diffeomorphism $z \mapsto z(1 - ax - by - \gamma z)$ of $B'$ preserves $\pi$ and the level set $\{z = 0\}$, giving new coordinates in which

$$M^S = \{z = ax^2 + bxy + cy^2 + r_2(x, y, z)\} \quad (4.12)$$
with \(\pi(x, y, z) = (y, z)\), again with \(r_2\) vanishing to order 3 at the origin. Intrinsically, for \(v = x(\kappa, 0) + y(0, K_1)\), the quadratic part of \((4.12)\) is the quadratic form associated with the Hessian

\[
(HF)_p(v, w) = \int_C \langle c_p, (\delta_v L)_p(w) \rangle.
\]

at the point \(p \in W^1\). In particular, for \(w = (\kappa, 0)\), the last term of \((4.4)\) vanishes, so

\[
\int_C \langle c_p, (\delta_v L)_p(\kappa, 0) \rangle = \int_C \langle c_p, (\delta_v D)_p \kappa \rangle = (d\psi)_p(v)
\]

from \((4.3)\). Consequently,

- \(a = d\psi(\kappa, 0) \neq 0\) by \((4.9)\) and the hypothesis \(p \notin A\).
- \(b = d\psi(0, K_1) \neq 0\) because \((0, K)\) is transverse to \(W^1\).
- \(c = 0\) since the path \(\tilde{\gamma}(t) = (0, t, 0)\) described after \((4.10)\) lies in \(\mathcal{M}^S\), so \((4.12)\) becomes \(0 = ct^2 + O(t^3)\) for all small \(t\).

Therefore the quadratic form \(ax^2 + bxy\) is non-degenerate, so a final change in the coordinate \(z\) replaces the equation in \((4.12)\) by the one in \((4.11)\).

We will need several statements about generic paths in \(\mathcal{J}\). These are consequences of the Sard-Smale Theorem, applied in the following manner. Suppose that \(\pi: \mathcal{M} \to \mathcal{J}\) is a smooth, index \(\iota\) Fredholm map of separable Banach manifolds and \(\mathcal{N} \subset \mathcal{M}\) is a submanifold. Fix points \(p, q \in \mathcal{J}\) and let \(\mathcal{P} = \mathcal{P}_\mathcal{J}\) be the space of \(C^k\) paths \([0, 1] \to \mathcal{J}\) from \(p\) to \(q\). Pulling back by the evaluation map \(\mathcal{P} \times [0, 1] \to \mathcal{J}\) and projecting to the first factor gives a space

\[
\mathcal{\tilde{M}} = \left\{ (\gamma, t, f) \mid \gamma \in \mathcal{P}, f \in \mathcal{M}^\gamma(t) \right\} \to \mathcal{P}.
\]

(4.13)

General results (cf. [Pal]) show that \(\mathcal{P}\) and \(\mathcal{\tilde{M}}\) are separable Banach manifolds, \((4.13)\) is a Fredholm map, and \(\mathcal{\tilde{N}}\) is a submanifold of \(\mathcal{\tilde{M}}\). By the Sard-Smale Theorem \([S]\) there is a Baire set \(\mathcal{P}^*\) of \(\mathcal{P}\) so that the fiber \(\mathcal{M}^\gamma\) of \((4.13)\) over each \(\gamma \in \mathcal{P}^*\) is a submanifold of \(\mathcal{M}\) transverse to \(\mathcal{N}\) that is either empty or of dimension \(1 + \iota\). The intersection with \(\mathcal{N}\) is therefore a manifold of dimension \(\iota + 1 - \text{codim} \mathcal{N}\).

We will apply this reasoning twice: first for paths in \(\mathcal{J}\), then for paths in \(\mathcal{J}_C\).

**Proposition 4.4.** Any path in \(\mathcal{J}\) with endpoints in \(\mathcal{J}_E^*\) can be deformed, keeping its endpoints, to a path \(\gamma\) such that \(\mathcal{M}^\gamma_{\text{simple}} \to \gamma\) is a 1-dimensional manifold, consisting of embeddings, and intersecting the wall transversely in isolated points, all in \(W^1 \setminus A\). Moreover, any path \(\gamma\) with these properties is in \(\mathcal{J}_{\text{isol}}\).

**Proof.** By Theorem 4.1, \(\mathcal{M}_{\text{simple}}\) is a manifold and \(W^r\) is a codimension \(r^2\) submanifold. Similarly, \(\mathcal{M}_{k, \text{simple}}\) is a manifold for any number \(k\) of marked points. Lemma 4.2 shows that \(\mathcal{A}\) is a codimension 1 submanifold of \(W^1\), and Lemma 4.3 implies that \(D = ev^{-1}(\Delta)\) and \(S = \Phi^{-1}(0)\) are codimension 6 submanifolds of \(\mathcal{M}_{2, \text{simple}}\) and \(\mathcal{M}_{1, \text{simple}}\) respectively. Applying the Sard-Smale Theorem with \(\mathcal{N} = W^r, A, D\) and \(S\) gives a Baire subset of \(\mathcal{P}\) over which the fibers are manifolds of the expected dimension. The intersection is a single Baire subset \(\mathcal{P}^* \subset \mathcal{P}\) so that for each \(\gamma \in \mathcal{P}^*\), \(\mathcal{M}^\gamma_{\text{simple}} \to \gamma\) is a 1-manifold consisting only of embedded curves whose only interaction with \(W\) are transverse intersections at points of \(W^1 \setminus A\). Note that the only critical points of \(\pi: \mathcal{M}^\gamma_{\text{simple}} \to [0, 1]\) are these wall crossing points. The proof is completed by observing that the local model \((4.14)\) below shows that the wall crossing points are non degenerate critical points of \(\pi\) and therefore (a) they are isolated points of \(\mathcal{M}^\gamma_{\text{simple}}\) and (b) all points of the fiber \(\mathcal{M}^\gamma_{\text{simple}}\) are isolated for each \(t\).

**Corollary 4.5.** \(\mathcal{J}_{\text{isol}}^E\) is a dense and path-connected subspace of \(\mathcal{J}\).

**Proof.** Density was shown in Corollary 4.3. Path-connectedness follows from Lemma 4.4 and the fact that \(\mathcal{J}\) is path-connected.
We conclude this section by proving a version of Proposition 4.4 for paths in the subspace $\mathcal{J}_C$ of $\mathcal{J}$. The proof is somewhat different because the restriction of $\mathcal{M}_{\text{simple}}$ to $\mathcal{J}_C$ is not a manifold (see the last sentence of this section).

**Proposition 4.6.** Any path in $\mathcal{J}_C$ with endpoints in $\mathcal{J}_C^*$ can be deformed, keeping its endpoints, to a path $\gamma$ in $\mathcal{J}_C \cap \mathcal{J}_{\text{isol}}$ whose lift $\tilde{\gamma}$ intersects $W$ transversally at finitely many points, all in $\mathcal{W}^1 \setminus \mathcal{A}$. 

**Proof.** First consider the subset $\mathcal{M}_C = \{f_C\} \times \mathcal{J}_C$. This is a submanifold of $\mathcal{M}_{\text{emb}}$ that is transverse to each $\mathcal{W}^r$ and to $\mathcal{A}$ by Theorem 4.1 and Lemma 4.2. By Sard-Smale, there is a Baire set of paths $\gamma$ in $\mathcal{J}_C$ for which the lift $\tilde{\gamma} = (f_C, \gamma)$ to $\mathcal{M}_{\text{simple}}$ intersects the wall transversally at finitely many points, all along $\mathcal{W}^1 \setminus \mathcal{A}$. The local model (4.13) below then implies that the core curve $\tilde{\gamma}(t)$ is an isolated point of $\mathcal{M}^{\gamma(t)}$ (and is clearly embedded).

It remains to find another Baire set of paths $\gamma$ for which the points of $\mathcal{M}^{\gamma(t)} \setminus \mathcal{M}_C$ are embedded and isolated for each $t$. Denote by $\mathcal{M}^{\gamma(t)}_{\text{simple}} \to \mathcal{J}_C$ the collection of simple $J$-holomorphic curves which have at least one point $x$ off $C$ in each component of their domain. The results of Theorem 4.1, Lemma A.3 and Lemma 4.2 all extend to the moduli space $\mathcal{M}^{\gamma(t),E}_{\text{simple}}$ by using variations supported around $x$ but vanishing along $C$; such variations are tangent to $\mathcal{J}_C$.

With that, the proof of Proposition 4.4 extends to give a Baire subset $\mathcal{P}^*$ of the space $\mathcal{P} \mathcal{J}_C$ of paths in $\mathcal{J}_C$ so that for each $\gamma \in \mathcal{P}^*$, the points of $\mathcal{M}^{\gamma(t),E}_{\text{simple}}$ are embedded and isolated for all $t$.

**Corollary 4.7** (Local models). Suppose that $\gamma(t)$ is a path in $\mathcal{J}$ such that $\pi: \mathcal{M}^{\gamma(t)}_{\text{simple}} \to \gamma$ intersects a wall over $\gamma(0)$ at a point $(f_0, J_0)$ of $\mathcal{W}^1 \setminus \mathcal{A}$. Then the local model (4.13) at $(f_0, J_0)$ is of two types:

1. If $\gamma$ satisfies the conclusions of Proposition 4.4, then $\mathcal{M}^{\gamma} \to \mathcal{J}$ is locally
   \[ V = \{(x, t) \in B \mid t = ax^2\} \quad \text{with } a \neq 0 \text{ and } \pi(x, t) = t = \gamma(t). \]  

2. If $\gamma \subset \mathcal{J}_C$ satisfies the conclusions of Proposition 4.4, then $\mathcal{M}^{\gamma} \to \mathcal{J}$ is locally
   \[ V = \{(x, t, 0) \in B \mid 0 = x(ax + bt)\} \quad \text{with } a, b \neq 0, \text{ and } \pi(x, t, 0) = t = \gamma(t). \]

**Proof.** To get (4.14), take $L = \gamma(0)$ and apply Theorem 4.3 restricted to the plane $\{y = 0\}$ (so $K$ is not used) and the last line of the proof of Theorem 4.3 requires only $a \neq 0$). Similarly, to get (4.15), take $K = \gamma(0)$, noting that $(0, K)$ is transverse to the wall, and restrict to the plane $\{z = 0\}$.

The distinction between the local models (4.14) and (4.15) is crucial. By Proposition 4.4, Model (1) applies where a generic path in $\mathcal{J}_{\text{isol}}$ crosses a wall. This is precisely the local model for the creation (if $a > 0$) or annihilation (if $a < 0$) of a pair of curves in the moduli space. Similarly, Proposition 4.6 shows that Model (2) applies where a generic path in the subspace $\mathcal{J}_C \subset \mathcal{J}$ crosses a wall. In Case (2), the local model is not a manifold: it is two lines crossing at the origin. It can be smoothed using the $z$ parameter (taken to be 0 in (4.15)), as will be done in Lemma 5.4.

## 5. The cluster isotopy theorem

For notational simplicity, given two clusters $\mathcal{O} = (C, \varepsilon, J)$ and $\mathcal{O}' = (C', \varepsilon', J')$ whose central curves $C$ and $C'$ have the same genus and homology class, write

\[ GW^E(\mathcal{O}) \approx GW^E(\mathcal{O}') \]

to mean that the difference is a finite sum of terms of the form $\pm GW^E(C_i', \varepsilon_i, J_i')$ of strictly higher level (4.3), compared to that of $C$. With this notation, for example, the conclusion of the cluster refinement Lemma 2.5 simply says that for generic $0 < \varepsilon' < \varepsilon$

\[ GW^E(C, \varepsilon, J) \approx GW^E(C, \varepsilon', J). \]  

(5.1)

We now use the results of Section 4 and an isotopy argument to prove that the GW series of any cluster is equivalent, in the above sense, to the series (3.2) of an elementary curve.
Cluster Isotopy Theorem 5.1. For any regular cluster $\mathcal{O} = (C, J_0, \varepsilon_0)$ centered at a genus $g$ embedded $J_0$-holomorphic curve $C$,

$$GW^E(\mathcal{O}) \approx \text{sign}(C, J_0) GW_{g\text{emb},E}^E(q^C).$$  \hspace{1cm} (5.2)

Proof. The proof of Theorem 4.2 shows that there exist $J_1 \in J_C$ and $\varepsilon_1$ so that $\mathcal{O}_{\text{elem}} = (C, \varepsilon_1, J_1)$ is an elementary cluster. Choose a path $\gamma(t) = J_t$ in $J_C$ from $J_0$ to $J_1$ (the proof of Theorem A.2 of \cite{IP2} shows that $J_C$ is connected). By Lemma 4.6 we can assume, after a deformation, that $\gamma$ is a path in $J_C \cap J_{\text{isol}}$ and there is a finite set $\text{Sing} = \{t_i\}$, not containing 0 or 1, such that $\tilde{\gamma} = (f_C, \gamma(t_i))$.

- lies in $\mathcal{M}^\gamma \setminus W$ for all $t \notin \text{Sing}$, and
- lies in a 2-dimensional surface $S_i \subset \mathcal{M}$ given by Theorem 4.3 for $t \in [t_i - \delta, t_i + \delta]$.

Choose $\delta > 0$ small enough so that intervals $[t_i - \delta, t_i + \delta]$ do not overlap, and let their endpoints be $0 < \tau_1 < \cdots < \tau_{2k} < 1$. For each $i$, fix a cluster $\mathcal{O}_i = (C, \varepsilon_i, J_{\tau_i})$. Then $\tilde{\gamma}$ can be regarded as the composition of paths $\tilde{\gamma}_i : [\tau_i, \tau_{i+1}] \to \mathcal{M}^\gamma$ of two types:

(i) Paths in $\mathcal{M}_{\text{emb}} \setminus W$. For these, Lemma 5.2 below shows that $GW^E(\mathcal{O}_i) \approx GW^E(\mathcal{O}_{i+1})$.

(ii) Paths in $\mathcal{M}_{\text{emb}} \cap S_i$, crossing transversally the wall at a single point of $A \cap W^I$.

For these, Lemma 5.4 below shows that $GW^E(\mathcal{O}_i) \approx -GW^E(\mathcal{O}_{i+1})$.

Altogether, we conclude that

$$GW^E(\mathcal{O}) \approx (-1)^\sigma GW^E(\mathcal{O}_{\text{elem}})$$

where $\sigma$ is the number of transverse wall-crossings, which is exactly the spectral flow of the operator $D_\gamma$ along the path $\tilde{\gamma}$. The path ends at an elementary curve, which has positive sign by the calculation in §3. Thus $(-1)^\sigma$ is exactly the sign of the initial curve $(C, J_0)$.

The first isotopy lemma used above can be stated in a way that allows the central curve to move through embeddings.

Lemma 5.2 (First Isotopy Lemma). Fix $E > 0$. Then for any path $(C_t, J_t)$ in $\mathcal{M}_{\text{emb}} \setminus W$ with $J_t$ in $J_{\text{isol}}$ and any $\varepsilon_0, \varepsilon_1$ such that $(C_0, \varepsilon_0, J_0)$ and $(C_1, \varepsilon_1, J_1)$ are clusters,

$$GW^E(C_0, \varepsilon_0, J_0) \approx GW^E(C_1, \varepsilon_1, J_1).$$  \hspace{1cm} (5.3)

Proof. It follows from Theorem 4.1, and the compactness of $[0, 1]$ that there is a $\delta > 0$ such that, for each $t \in [0, 1]$, $C_t$ is the only $J_t$-holomorphic curve in its degree and genus in the ball $B(C_t, \delta)$ (in Hausdorff distance). By Lemma 2.3 we can choose, for each $0 \leq t \leq 1$, an $0 < \varepsilon_t < \delta$ such that $(C_t, \varepsilon_t, J_t)$ is a cluster. Then, by Lemma 2.4 $(C_{\varepsilon_t}, \varepsilon_t, J_t)$ has a well-defined invariant $GW^E$ for all $s$ in an open interval around $t$. These open intervals cover $[0, 1]$; take a finite subcover $\{I_k\}$. Then $GW^E(C_{\varepsilon}, \varepsilon_k, J_s)$ is constant for $s$ in each $I_k$ and $C_s$ is the only $J_s$-holomorphic curve in its genus and homology class in that ball. Theorem 2.4 shows that on the intersection of two consecutive intervals the corresponding $GW^E$ invariants differ by the contributions of higher-level clusters. The lemma follows.

From the previous section we know that each path in $J$ with endpoints in $J^E$ can be deformed, keeping its endpoints, to a path $\gamma$ in $J_{\text{isol}}$ such that the projection

$$\pi : \mathcal{M}_{\text{emb}}^\gamma \to \gamma$$  \hspace{1cm} (5.4)

has finitely many non-degenerate critical points, none an endpoint, each locally modeled by $\gamma(t) = ax^2 + bx + c$. If $a > 0$ in the local model, then $\gamma$ can be parameterized so that $\gamma(t) = t$ and $\pi^{-1}(t) = \{x \mid t = ax^2\}$ is empty for $t < 0$ and is two distinct curves $C_t^\pm$ for $0 < t < \delta$ (and vice versa if $a < 0$). A second isotopy lemma relates the GW invariants of clusters centered on these curves $C_t^\pm$.

Lemma 5.3 (Wall-crossing in $J$). Fix $E > 0$, a path $\gamma$ in $J_{\text{isol}}$ and a non-degenerate critical point $(C_0, J_0)$ of $(5.3)$ for $J_0 = \gamma(0)$. Then there exits a $\delta > 0$ and a neighborhood $U$ of $(C_0, J_0)$ in $\mathcal{M}$ such that if $|t| < \delta$ and $\mathcal{M}_{\text{emb}}^{(t)} \cap U = \{C_t^\pm\}$, then for any two clusters $\mathcal{O}^+ = (C_t^+, \varepsilon, J_t)$, $\mathcal{O}^- = (C_{t}^-, \varepsilon', J_t)$

$$GW^E(\mathcal{O}^+) \approx -GW^E(\mathcal{O}^-).$$
Proof. The local model (4.14) at $(C_0, J_0)$ implies that there is a $\varepsilon_1 > 0$ and a ball $U = B(C_0, \varepsilon_1)$ in $\mathcal{C}(X)$ that contains $C_t^\pm$ and no other $J_t$ holomorphic curves in the degree and genus of $C_0$ for all $|t| < \varepsilon_1$. Because $J_t \in \mathcal{J}^{E}_{isol}$, Lemma 2.3 ensures that $\varepsilon_1$ can be chosen so that $(C_0, \varepsilon_1, J_0)$ is a cluster. As $J$ varies, the associated invariant $GW^E(U, J_s)$ is, by Lemma 2.1, well-defined and independent of $s$ for small $s$.

As above, the local model shows that $U \cap C^{J_s, E}$ is $\{C_t^\pm\}$ for $s = t$, and is empty for $s = -t$. Taking $s = t$ and applying Theorem 2.4, $GW$ completed by noting that the invariants $GW$ theorem with the core curve remains if we fix the complex structure on the core curve $\varepsilon$ as similarly perturbed. By Lemma 2.3 the $GW$ invariants of the corresponding clusters $\{\gamma\}$ with projection $(\gamma')$ transversally at $t = 0$. In this picture, for each $0 < t < \delta$, there are four curves to consider: the incoming core curve $(C, J_{-1})$, the outgoing core curve $(C, J_1)$, and a second pair of curves $(C', J_{-1})$ and $(C', J_t)$.

Lemma 5.4 (Wall-crossing in $\mathcal{J}_C$). Fix $E > 0$ and a path $\gamma'$ in $\mathcal{J}^{E}_C \cap \mathcal{J}^E_{isol}$ so that $\gamma(t)$ crosses the wall transversally at $t = 0$ at a point $(C_0, J_0)$ in $\mathcal{W}^1 \setminus A$. Then there exits a $\delta > 0$ so that each incoming cluster $\mathcal{O}_{-\delta} = (C_{-\delta}, \varepsilon, J_{-\delta})$ and each outgoing cluster $\mathcal{O}_{\delta} = (C_{\delta}, \varepsilon', J_{\delta})$ satisfy

$$GW^E(\mathcal{O}_{-\delta}) \approx -GW^E(\mathcal{O}_{\delta}).$$

Proof. Choose an $L$ satisfying (4.10), take $K = J_0$ and consider the local model $\mathcal{M}^S \to S$ given by (4.14) with projection (??). The local model (4.15), which is two lines crossing at the origin, is the level set $\{z = 0\}$. We will perturb this level set in two opposite directions.

In the chart (4.11), $(C_{-\delta}, J_{-\delta})$ has coordinates $(0, -\delta, 0)$, so can be perturbed to $(C_{-\delta,s}, J_{-\delta,s})$ with coordinates $(x(s), -\delta, s)$ where $x \approx 0$ satisfies $s = x(ax - b\delta)$. The curves $(C_{-\delta,s}, J_{-\delta})$ and $(C_{\delta,s}, J_{\delta})$ can be similarly perturbed. By Lemma 2.3 the $GW$ invariants of the corresponding clusters

$$A_s = (C_{-\delta,s}, \varepsilon, J_{-\delta,s}) \quad B_s = (C_{-\delta,s}, \varepsilon, J_{-\delta,s}) \quad C_s = (C_{\delta,s}, \varepsilon, J_{\delta,s})$$

are locally constant in $s$: for sufficiently small $s$ we have

$$GW^E(A_s) = GW^E(\mathcal{O}_{-\delta}) \quad GW^E(B_s) = GW^E(\mathcal{O}_{\delta}) \quad GW^E(C_s) = GW^E(\mathcal{O}_{\delta}).$$

Assume $a > 0$ (else change $s \to -s$), and that $b > 0$ (else change $t \to -t$). The moduli space $\mathcal{M}^S$ over $S$ is locally near $C_0$ the level set $\{(x, t, s) \mid s = x(ax - bt)\}$. For each fixed $s, t$ small, this quadratic equation in $x$ has either no solution or two solutions, except at a single point $x = -bt/2a$ where the tangent is in the kernel of the projection to $\mathcal{J}$, which means that this point lies on the wall and is a non-degenerate critical point of (5.4).

Fixing a small positive $s$, the moduli space over $\gamma_s(t) = (t, s), -\delta \leq t \leq \delta$ therefore contains a path in $\mathcal{M}$ from the core of cluster $C_s$ to the core of $B_s$ that does not cross the wall. After a small perturbation
6. The structure of the GW invariants

The isotopy results of the previous section lead quickly to a formula (6.1) that shows that the GW invariants have a remarkably simple structure. This formula is compatible with a simple geometric picture: if one could find a \( J \in \mathcal{F} \) so that all \( J \)-holomorphic curves in \( X \) were elementary, then \( GW(X) \) would have exactly the form (6.1), with \( e_{\alpha,g} \) equal to the signed count of \( J \)-holomorphic curves with homology class \( A \) and genus \( g \). However, it is far from clear whether any such \( J \) exists. Thus the coefficients \( e_{\alpha,g}(X) \) can be regarded as the virtual count of elementary curves in \( X \).

**Theorem 6.1.** For any symplectic Calabi-Yau 6-manifold \( X \), there exist unique integer invariants \( e_{\alpha,g} \) such that

\[
GW(X) = \sum_{\alpha \neq 0} \sum_{g \geq 0} e_{\alpha,g}(X) \cdot GW_{g}^{elem}(q^{A}).
\]

**Proof.** The uniqueness of the coefficients in (6.1) is easily shown because the collection of series

\[
GW_{g}^{elem}(q^{k\beta}) = t^{2g-2}q^{k\beta}(1 + \text{higher order})
\]

for primitive classes \( \beta, g \geq 0 \) and \( k \geq 1 \) are all linearly independent. For existence, fix \( E \), choose any parameter \( J \in \mathcal{F}_{E} \), and use Theorem 2.4 to write the \( GW^{E} \) as a sum of finitely many cluster contributions. Formula (6.1) follows from the corresponding formula for each cluster, which is proved in Lemma 5.2 below, and then taking \( E \to \infty \).

**Lemma 6.2.** For any regular \( E \)-cluster \( O \) centered at a genus \( g \) curve \( C \) there exist unique integers \( e_{d,h} \), beginning with \( w_{1,g} = \text{sign}(C) \), such that

\[
GW^{E}(O) = \sum_{d \geq 1} \sum_{h \geq g} e_{d,h}(O) \cdot GW_{g}^{elem}(q^{dC}).
\]

**Proof.** Because all \( J \)-holomorphic maps in \( O \) represent \( k[C] \) and have genus at least \( g \), \( GW^{E}(O) \) has the form

\[
GW^{E}(O) = \sum_{k \geq 1} \sum_{h \geq g} GW_{k,h}^{E}(O) \cdot q^{kC}t^{2h-2}
\]

with \( k\omega(C) \leq E \) and \( h \leq E \). Define the relative level to be \( \Omega(k) + h - g \geq 0 \) and let \( GW_{m}^{E}(O) \) denote the sum of the terms in (6.4) with \( \Omega(k) + h - g \leq m \). The expansion (6.2) implies that \( GW_{m}^{E}(O) \) involves only weights \( w_{d,h} \) of level \( \Omega(d) + h \) at most \( m \).

We will determine the weights in (6.3) by complete induction on \( m \), starting with the trivial case \( m = -1 \). Thus we assume that for any regular cluster \( O \), whose central curve has any \( (A,g) \), there is an expansion (6.4) up to relative level \( m - 1 \), that is, for \( \Omega(k) + h \leq m - 1 \). Now by Corollary 5.1 we have

\[
GW^{E}(O) = \pm GW_{g}^{elem,E}(q^{C}) + \sum_{i \in I} \pm GW^{E}(O_{i})
\]
where the $\mathcal{O}_i$ are clusters whose central curves $C_i$ have $[C_i] = k_i[C]$, genus $g_i \geq g$ and relative level $m_i = \Omega(k_i) + g_i - g > 0$. When $GW^E(\mathcal{O})$ is truncated at relative level $m$, each $GW^E(\mathcal{O}_i)$ is truncated at level $m - m_i < m$ so by induction,

$$GW^E(\mathcal{O}) = \pm GW^\text{elem,E}_g(qC) + \sum_{i \in I} \pm GW^\text{elem,E}_{g_i}(q^{k_i}C_i)$$

hold when truncated at level $m$. This completes the induction step. □

In fact, we get the following result for any closed symplectic 6-dimensional manifold $X$ as long as we restrict to the GW invariants coming only from $c_1(X)A = 0$ classes.

**Theorem 6.3.** Assume $X$ is any closed symplectic 6-manifold. Then there there exit unique integer invariants $e_{A,g}(X)$ such that the GW invariant of $X$ satisfies

$$\sum_{A \neq 0} \sum_{c_1(A) = 0} GW_{A,g}(X) t^{2g-2} q^A = \sum_{A \neq 0} \sum_{c_1(A) = 0} e_{A,g}(X) \cdot GW^\text{elem}_g(q^A).$$

(6.5)

**Proof.** The dimension (1.2) is $2c_1(A)$, independent of the genus. It suffices to check that all the results in Section 1-2 continue to hold as long as we replace everywhere $\mathcal{M}(X)$ by the union of its zero dimensional pieces

$$\bigcup_{A \neq 0 \ c_1(A) = 0} \mathcal{M}_{A,g}(X)$$

(6.6)

A dimension count shows that for generic $J$ the limit points of (6.6) in the rough topology (after restricting below fixed energy level $E$) can only be multiple covers of points of (6.6), and not of points with $c_1(X)A \neq 0$. The rest is straightforward, and we leave the details to the reader. □

7. **Proof of the GV Conjecture**

The GV conjecture follows easily from Theorem 6.1 and the explicit form of the GW invariant of an elementary cluster. For simplicity, set

$$E_h(t,q) = \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \frac{k \beta}{2}\right)^{2h-2} q^k$$

With this notation, the GW invariant (3.3) of a genus $g$ elementary curve is

$$GW^\text{elem}_g = \sum_{d \neq 0} \sum_{h} n_{d,h}(g) E_h(t,q^d)$$

(7.1)

and the GV Conjecture takes the following form.

**Theorem 7.1.** Let $X$ be any symplectic Calabi-Yau 6-manifold. Then there are unique integers $n_{A,h}(X)$ such that

$$GW(X) = \sum_{A \neq 0} \sum_{h} n_{A,h}(X) E_h(t,q^A)$$

(7.2)

In fact, these BPS numbers $n_{A,h}(X)$ are obtained from the virtual counts $e_{A,h}(X)$ by the universal formula involving the coefficients in (7.1):

$$n_{A,h}(X) = \sum_{d \geq 1 \ g = 0} e_{A/d,g}(X) \cdot n_{d,h}(g) \in \mathbb{Z}$$

(7.3)
Proof. This follows immediately by combining (6.5) and (7.1) and rearranging the sums:

\[ GW(X) = \sum_{A \neq \emptyset} \sum_{g \geq 0} e_{A,g}(X) \sum_{d \geq 1} \sum_{h \geq g} n_{d,h}(g) \mathcal{E}_h(t, q^{dA}) = \sum_{A \neq \emptyset} \sum_{d \geq 1} \sum_{h \geq g} \left( \sum_{g} e_{A,g}(X) n_{d,h}(g) \right) \mathcal{E}_h(t, q^{dA}) \]

Note that the sum over \( g \) is finite because \( n_{d,h}(g) = 0 \) unless \( h \geq g \). The rearrangements are justified by first working below an energy level \( E(A, g) \leq E \), where all sums are finite. \( \square \)

8. Extensions of the GV structure theorem

This section extends Theorem 7.1 in two different directions: to general symplectic 6-manifolds, and to the genus 0 GW invariants of closed symplectic manifolds that are semipositive (as defined in [MS]), a class that includes symplectic Calabi-Yau manifolds of any dimension. In fact, all transversality results were proved for simple maps in index zero moduli space. The Cluster Decomposition Theorem (2.4) holds provided the underlying curve map (1.1) does not increase the dimension of such moduli spaces in the sense described below.

We restrict to the primary GW invariants of \( X \), which are defined using the evaluation map (but not the stabilization map) in (1.1). For each collection \( \{ \gamma_i \} \in H^*(X, \mathbb{Z}) \) consider the generating function

\[ GW^X(\gamma_1, \ldots, \gamma_n) = \sum_{A \neq \emptyset} \sum_{g \geq 0} (\mathcal{M}_{A,g,n}(X))^{vir} \ev^*\left( \gamma_1 \times \ldots \times \gamma_n \right) q^{A^2g-2}. \tag{8.1} \]

The pairing is defined to be zero unless the formal dimension is zero, that is, unless \( \iota = 0 \) where

\[ \iota = 2c_1(X)A + (\dim X - 6)(1 - g) + 2n - \sum_{i=1}^n \dim \gamma_i. \tag{8.2} \]

As usual, the pairing vanishes unless \( \dim \gamma_i \geq 2 \) for each \( i \).

The coefficients in (8.1) are obtained by fixing pseudo-manifolds \( B_i \) in \( X \) representing the Poincare duals of \( \gamma_i \) and restricting to the constrained moduli spaces

\[ \overline{\mathcal{M}}_{A,g,B}(X) = \ev^{-1}(B_1 \times \ldots \times B_n) \subset \mathcal{M}_{A,g,n}(X) \tag{8.3} \]

When \( \iota = 0 \), the moduli space (8.3) gives rise in the usual way to the primary GW invariant that appears as the coefficient in (8.1).

Theorem 7.1 remains true for these constrained moduli spaces. Let \( D \) be the (countable) set consisting of the indexing data \((A, g, \gamma)\) appearing in (8.1). Choose a representative \( B_\gamma \) of the Poincare dual of each \( \gamma \in H^*(X, \mathbb{Z}) \). Standard transversality results show that, for each element of \( D \), the constrained universal moduli space \( \overline{\mathcal{M}}_{A,g,B}^{simple}(X) \to \mathcal{J} \) is a manifold. The Sard-Smale theorem gives a Baire set of regular points in \( \mathcal{J} \); intersecting over the elements of \( D \) and using also the other two transversality arguments used to prove Theorem 7.1 gives a single Baire set \( \mathcal{J}^* \) of \( \mathcal{J} \) such that for each \( J \in \mathcal{J}^* \) all simple \( J \)-holomorphic maps in (8.3) are regular, and all index zero moduli spaces (8.3) satisfy:

(a) All simple \( J \)-holomorphic maps are embeddings whose image is transverse to each \( B_\gamma \) at a regular point of \( B_\gamma \).

(b) The projection \( \pi \) in (1.1) is a local diffeomorphism.

In particular, for each \( J \in \mathcal{J}^* \), there are no simple \( J \)-holomorphic maps in those spaces (8.3) with \( \iota < 0 \).

Each \( J \)-holomorphic map \( f : C \to X \) has an associated "reduced map" \( \varphi \), defined by noting that the smooth resolution \( \tilde{f} : \tilde{C} \to X \) decomposes as \( \tilde{f} = \varphi \circ \rho \) where \( \varphi : \Sigma \to X \) is a simple \( J \)-holomorphic map with smooth, but perhaps not connected domain and \( \rho : \tilde{C} \to \Sigma \) is a map of complex curves. When \( f \) is simple, \( \varphi \) is the smooth resolution of \( f \). When \( f \) is a multiple cover, \( \varphi \) is the smooth resolution of the
image curve. In all cases, \( f \) and \( \varphi \) have the same image under the underlying curve map \( c \), but \( f \) and \( \varphi \) can be elements of moduli spaces of with different formal dimension \( \varepsilon \).

This defines another filtration on both the domain and the image of the map \( c \) by the formal dimension of the moduli space containing \( f \in \overline{\cal M}(X) \) and respectively that containing its image, the reduced map \( \varphi \), for the image \( C(X) \).

The universal moduli space constrained by \( B \) is

\[
\overline{\cal M}_B(X) = \bigsqcup_{A,g} \overline{\cal M}_{A,g,B}(X)
\]

where \( \iota \) in \( \varepsilon \) is zero. Consider the corresponding restriction of the map \( c \):

\[
c : \overline{\cal M}_B(X) \to C_B(X)
\]

The examples below give structure theorems in cases where the map \( \varepsilon \) preserves the filtration by formal dimension.

Under this assumption, we can replace \( \overline{\cal M}(X) \) everywhere by \( \varepsilon \) and all proofs in Sections 1-6, except those in Section 3, hold without change. In particular, there is a dense, path-connected set \( J_{\text{isol}}(B) \) corresponding to \( J_{\text{isol}} \) in Definition 1.2 but involving only maps in \( \overline{\cal M}_B(X) \). Lemma 1.5 holds under the assumption above with \( C \) replaced by the image of \( \varepsilon \).

To finish, we must expand the definitions of “cluster” and “elementary cluster”. Define a \( B \)-constrained cluster exactly as in Definition 2.2 but using only maps in \( \overline{\cal M}_B(X) \). The contribution of a \( B \)-constrained cluster \((C,J,\varepsilon)\) to \( GW(\gamma) \) depends only on the restriction of \( J \) to the \( \varepsilon \) neighborhood of the core curve. Up to a diffeomorphism, this can be identified with an \( \varepsilon \)-disk bundle of the normal bundle \( N_C \to C \), with \( C \) mapping to the zero section and the constraints \( B_i \) mapping into fibers over points \( p_i \in C \). One can then declare certain \( B \)-constrained clusters to be “elementary”. In both of the examples below there is a simple, natural way of doing this.

8.1. **GV-formula for general symplectic 6-manifolds.** For any closed symplectic 6-manifold \( X \), each class \( A \in H_2(X,\Z) \) has a chern number \( c_1(A) = c_1(X)A \) and the dimension \( \varepsilon \) is \( \iota = 2c_1(A) + \sum(2 - \dim \gamma_i) \), independent of the genus. The invariants \( GW_{A,g} \) are zero for all classes \( A \) with \( c(A) < 0 \) (the moduli space without constraints is empty for \( J \in J^* \)). For the classes \( A \) with \( c_1(A) = 0 \) consider the GV-transform

\[
\sum_{A,A,g} GW_{A,g} q^A t^{2g-2} = \sum_{A,A,g} n_{A,g} \sum_{k=1}^{\infty} \left( \frac{kt}{2} \right)^{2g-2} q^k t^{-c(A)q^A}
\]

For classes with \( c_1(A) > 0 \) consider the following variation of the GV transform:

\[
\sum_{A,A,g} GW_{A,g} (\gamma_1,\ldots,\gamma_k) t^{2g-2} = \sum_{A,A,g} n_{A,g} (\gamma_1,\ldots,\gamma_k) \left( \frac{t}{2} \right)^{c_1(A)+2g-2} t^{-c(A)q^A}
\]

for each collection \( \{\gamma_i\} \subset H^*(X,\Z) \). The GV invariants vanish by definition for \( c_1(X)A < 0 \).

**Theorem 8.1.** Under the transforms \( \varepsilon \) and \( \varepsilon \), the primary GW series of any closed symplectic 6-dimensional manifold \( X \) becomes a sum with integral coefficients

\[
n_{A,g} \in \Z \quad \text{and} \quad n_{A,g} (\gamma_1,\ldots,\gamma_k) \in \Z
\]

for all \( \gamma_1,\ldots,\gamma_k \in H^*(X,\Z) \) and \( k \geq 0 \).

**Proof.** Fix \( \gamma = \{\gamma_i\} \), corresponding constraints \( B = \{B_i\} \), and the class \( A \) so that \( \iota = 2c_1(A) + \sum(2 - \dim \gamma_i) \) is zero. For each \( J \in J^* \), the resolution of each \( J \)-holomorphic map \( f \) factors as \( \varphi \circ \rho \) as described above. For each component \( C_i \) of \( C \), let \( d_i \) denote the degree of \( \rho|C_i \), and let \( A_i = [f(C_i)] \in H_2(X,\Z) \). Then the image of \( \varphi \) passes through all the constraints and represents \( \sum A_i/d_i \) (summed over all \( d_i \neq 0 \)). Hence \( c_1(\varphi) \leq \sum c_1(A_i) = c_1(A) \); in fact this must be an equality because the moduli space with \( c_1(A) < 0 \) are empty.
This shows that the GW series separates into a sum over the “CY classes” $c_1(A) = 0$ where the previous result apply, and a sum over the “Fano classes” $c_1(A) > 0$ that was studied by A. Zinger [Z]. Theorem 6.3 combines with the proof of Theorem 7.1 to give the integrality of $n_{A,g}$ in (8.6). The Fano case is much simpler: there is no need to consider clusters because for $f \in \mathcal{J}$, every embedded $J$-holomorphic curve $C$ with $c_1(A) > 0$ is isolated and super-rigid for the constrained moduli space, and the contribution of its multiple covers to the GW invariant is precisely

$$GW(C) = \left(2 \sin \frac{t}{2}\right)^{c_1(A)+2g-2} t^{-c_1(A)}q^C$$

(see (1.13) in [Z]). This completes the proof. □

8.2. Genus zero invariants of semipositive manifolds. There is a similar structure theorem for the rational (genus zero) GW invariant of semipositive closed symplectic manifolds. In this context, the appropriate GV transform is the sum of two transforms:

(1) For $c_1(A) = 0$, it is based on the Aspinwall-Morrison formula (see Pandharipande-Klemm):

$$\sum_{A, d \geq 1} \sum_{c_1(A) = 0} GW_{A,d}^X(\gamma_1, \ldots, \gamma_k) q^A = \sum_{A, d \geq 1} n_{A,d}^X(\gamma_1, \ldots, \gamma_k) \sum_{d \geq 1} d^{k-3} q^d$$

(8.8)

(2) For $c_1(A) > 0$, the change of variables remains that of (8.7), specialized to genus zero:

$$GW_{A,0}^X(\gamma_1, \ldots, \gamma_n) = n_{A,0}^X(\gamma_1, \ldots, \gamma_n).$$

(8.9)

As before, the invariants $GW_{A,g}$ are zero for all classes $A$ with $c(A) < 0$ (since $X$ is semipositive, there are no simple $J$-holomorphic spheres with $c(A) < 0$ for $J \in \mathcal{J}^*$).

Theorem 8.2. Under the transforms $SS_\gamma$ and $SS_\delta$, the primary genus zero GW series of any closed semipositive symplectic manifold $X$ becomes a sum with integral coefficients

$$n_{A,0}^X(\gamma_1, \ldots, \gamma_k) \in \mathbb{Z}$$

for all $\gamma_1, \ldots, \gamma_k \in H^*(X, \mathbb{Z})$ and $k \geq 0$.

Proof. Again fix $\gamma$, $B$, and $A$ so that $t = 2c_1(A) + (\dim X - 6) + \sum (2 - \dim \gamma_i)$ is zero. As before, assume $f$ is a multiple cover with $\varphi$ the reduced map, and $A_i, d_i$ are degrees of its components. If the domain of $\varphi$ has $r \geq 1$ components then its image has at least $r - 1$ self-intersection points (since the domain of $f$ was connected). For $J \in \mathcal{J}^*$, these impose $(r - 1)(\dim X - 4)$ transversely cut conditions on simple maps, so the dimension of the moduli space containing $\varphi$ is

$$\sum_{i=1}^r (2c_1(A_i) + (\dim X - 6) + (r - 1)(\dim X - 4) + \sum (2 - \dim \gamma_i)$$

Since $A = \sum d_i A_i$ and $c_1(A) \geq 0$, this is less or equal to $t - 2(r - 1) \leq t = 0$. But the moduli space is empty unless this is an equality, so we conclude that $r = 1$ and $d = 1$ whenever $c_1(A) \neq 0$. Thus the GW series again separates into a sum of $c_1(A) = 0$ classes and a sum over $c_1(A) > 0$ classes.

The Fano case [8.9] is classical: dimension counts imply that for generic $J$ the constrained moduli space consists only of simple maps, without any multiple covers thus the GW invariant is an integer.

For Calabi-Yau classes $A$, we declare a $B$-constrained cluster to be elementary if it consists of an embedded rational curve with normal bundle locally biholomorphic to $O(-1) \oplus O(-1) \oplus (\dim X - 3)O$. As proved in [PR], this curve is super-rigid (the constraints kill the kernel in the $O$ directions), and the contribution of its multiple covers to the primary GW invariant is

$$GW_{\text{elem}}(\gamma_1, \ldots, \gamma_n) = \sum_{d \geq 1} d^{n-3} q^d$$

With this, Theorem 6.1 extends to give precisely SS_\gamma.
The proof of Theorems 1.1 and 3.4 were deferred; here we give the details. In fact, much of Theorem 1.1 is a consequence of Theorem 4.1 and Theorem 6.7.7 of [MS]. Because the intersection of Baire sets is a Baire set, it remains only to show that the subset of the universal moduli space \( \mathcal{M}_{\text{simple}} \) consisting of simple maps that are either (i) not immersed or (ii) have a node (double point of their image) is a subset of codimension at least two. While we are interested in the case where \( X \) is a Calabi-Yau 6-manifold, the lemma below applies in general.

For any \( J \)-holomorphic map \( f : C \to X \), its resolution \( \tilde{f} : \tilde{C} \to X \) is \( J \)-holomorphic and has a smooth (but not necessarily connected) domain, so it suffices to work with these. For each \( k \geq 0 \), let
\[
\mathcal{M}_{k, \text{simple}} \to \mathcal{J}
\]  
the universal moduli space consisting of tuples \( (f, x_1, \ldots, x_k, J) \) where \( f \) is a simple \( J \)-holomorphic map whose domain \( \Sigma \) is smooth and compact (but not necessarily connected) with \( k \) marked points \( (x_1, \ldots, x_k) \).

For a pair of marked points, consider the evaluation map
\[
\mathcal{M}_{2, \text{simple}} \to X \times X
\]  
The inverse image \( \text{ev}^{-1}(\Delta) \) of the diagonal consists of simple \( J \)-holomorphic map \( f \) with smooth, marked domain but whose image has a double point \( f(x_1) = f(x_2) \). We can also consider the section
\[
\mathcal{M}_{1, \text{simple}} \to L_x \otimes \text{ev}_x^*TX
\]  
defined by \( \Phi(f, x) = d_x f \) (where \( L_x \) is the relative cotangent bundle to the domain at the point \( x \)). The set \( \Phi^{-1}(0) \) consists of simple, marked \( J \)-holomorphic maps \( f \) which are not immersions at the marked point \( x \).

**Lemma A.3.** For each \( k \geq 0 \), the moduli space \( \mathcal{A}_{\text{simple}} \) is a manifold and

(i) the evaluation map \( \mathcal{A}_{\text{simple}} \to \mathcal{J} \) is transverse to the diagonal.

(ii) the section \( \mathcal{A}_{\text{simple}} \to L_x \otimes \text{ev}_x^*TX \) is transverse to the zero section.

Hence if \( X \) is a symplectic Calabi-Yau 6-manifold, the subset of the moduli space \( \mathcal{M}_{\text{simple}} \) where the maps \( f \) have a double point is codimension 2 and where \( f \) fails to be an immersion is codimension 4.

**Proof.** The first statement is proved the same way as Theorem 1.1(b). The proof of Proposition 3.4.2 of [MS] implies that \( \mathcal{A}_{\text{simple}} \to \mathcal{J} \) is transverse to the diagonal and hence \( \text{ev}^{-1}(\Delta) \) is a codimension 6 submanifold of \( \mathcal{M}_{2, \text{simple}} \). For (ii), since \( f \) is simple, in a coordinate ball \( B \) around \( p \) can find a small annulus \( A \) around \( p \) on which \( f \) is an embedding (cf. Proposition 2.5.1 of [MS]). One then extends the proof of Lemma 3.4.3 of [MS] to construct a variation \( (\xi, 0, K) \in \ker L_{(f, J)}^* \) such that (i) \( K \) is supported on \( A \), (ii) \( \xi \) is supported on a small ball \( B' \subset f^{-1}(B) \) around \( x \) in \( C \) and (iii) \( \xi \) is in local coordinates around \( x \). To achieve that, one starts with a local solution \( \xi \) to the equation \( D_{(f, J)}^* \xi = 0 \) satisfying (iii), modifies it by a bump function \( \beta \) which interpolates from 1 to 0 in a small annulus \( A' \subset f^{-1}(A) \) around \( x \), and then constructs the variation in \( K \) satisfying (i) and \( Kf_{x,j} = -D_{(f, J)}(\beta \xi) \) to account for this modification.

For the dimension counts, note that in our context the fiber of \( \mathcal{A}_{\text{simple}} \) has index 2\( k \), and therefore the fiber of \( \text{ev}^{-1}(\Delta) \to \mathcal{J} \) has index \( 4 - 6 = -2 \), while that of \( \Phi^{-1}(0) \to \mathcal{J} \) has index \( 2 - 6 = -4 \).

Finally, Theorem 3.4.3 is an immediate consequence of the following lemma.

**Lemma A.4.** When \( C \) is a genus \( g \) elementary cluster, the coefficients of \( \mathcal{B}_{\text{simple}} \) satisfy

(a) (Integralty) \( n_{d, h}(g) \in \mathbb{Z} \)

(b) (Finiteness) for each \( d, g \) fixed, \( n_{d, h}(g) = 0 \) for \( h < g \) or \( h \) large.

(c) For \( g = 0 \), all \( n_{d, h}(g) \) vanish except \( n_{1, 0}(g) = 1 \).

(d) For \( g = 1 \), all \( n_{d, h}(g) \) vanish except \( n_{d, 1}(g) = 1 \) for each \( d \geq 1 \).
Proof. When $C$ is genus zero, $O(-1) \oplus O(-1) \to C$ is an elementary cluster and its contribution to the $GW$ invariant was originally calculated by Faber and Pandharipande in [FP] to be

$$GW(C) = \sum_{k} \frac{1}{k} \left( 2 \sin \frac{kt}{2} \right)^{-2} q^k$$

Comparing with (3.2) gives (c). For genus $g = 1$, (3.1) reduces to the generating function for partitions:

$$Z(C) = 1 + \sum_{d \geq 1} p(d) \ q^d = \prod_{d=1}^{\infty} \left( \frac{1}{1-q^d} \right)$$

and hence

$$GW(C) = \log(Z(C)) = -\sum_{d=1}^{\infty} \log(1-q^d) = \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{kd}}{k}.$$ 

Comparing with (3.2) gives (d).

The higher genus case follows from an algebraic fact about power series with integral coefficients that appears in the proof of Theorem 3.19 of the paper [PT] of Pandharipande and Thomas. Making the change of variable $Q = e^{it}$, (3.2) becomes

$$\log(Z(C)) = \sum_{n \geq 1} n_{d,h}(C) \sum_{k \geq 1} \frac{(-1)^{h-1}}{k} (Q^k + Q^{-k} - 2)^{h-1} q^{kd}.$$ 

On the other hand, (3.1) becomes

$$Z(C) = 1 + \sum_{d \geq 1} \sum_{\rho \neq d} \prod_{e \rho} (-1)^{g-1} (Q^{-h(\square)} + Q^{h(\square)} - 2)^{g-1} q^d = \sum_{d} \sum_{n} A_{n,d}(C) Q^n q^d \quad (A.4)$$

where, for each $d$, the inner sum is a Laurent polynomial in $Q, Q^{-1}$ with integer coefficients $A_{n,d}(C)$ as long as $g \geq 1$. The coefficients $A_{n,d}$ uniquely determine the numbers $n_{d,h}$. But the proof of Theorem 3.19 of [PT] shows that the integrality of the $A_{n,d}$ implies that all of the $n_{d,h}$ are also integers. Thus statement (a) holds.

For $g \geq 2$, the coefficient of $q^d$ in (3.1) is a Taylor series in $t^2$, $Z = 1 + t^{2g-2}q + O(t^{2g})$, so $\log Z = t^{2g-2}q + O(t^{2g})$. Comparing with (3.3) one sees that $n_{d,h}(C) = 0$ for all $h < g$, as in (b).

Finally, for genus $g \geq 2$, the inner sum in (A.4) is a Laurent polynomial in $Q$, symmetric in $Q \to Q^{-1}$, and with degree bounded by $(g-1) \sum h(\square) \leq d^2(g-1)$. This property is preserved under taking the log:

$$\log Z(C) = \log \sum_{d \geq 1} \sum_{n} a_{d,n} Q^n q^d = \sum_{d \geq 1} \sum_{h \geq 1} n_{d,h}(C) \sum_{k \geq 1} \frac{(-1)^{h-1}}{k} (Q^k + Q^{-k} - 2)^{h-1} q^{kd}$$

where $|n| \leq (g-1)d^2$. As in the proof of Lemma 3.12 of [PT], this implies the vanishing of $n_{h,d}(C)$ for large $h$. In fact, a proof by induction on $d$ using the above bound implies that $n_{d,h}(C) = 0$ for $h-1 > d^2(g-1)$.

□

REFERENCES


DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY

E-mail address: ionel@math.stanford.edu

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY

E-mail address: parker@math.msu.edu