# COMBINATORIAL INTERPRETATIONS OF BINOMIAL COEFFICIENT ANALOGUES RELATED TO LUCAS SEQUENCES

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*Received: , Revised: , Accepted: , Published:*

## Abstract

Let $s$ and $t$ be variables. Define polynomials $\{n\}$ in $s, t$ by $\{0\} = 0$, $\{1\} = 1$, and $\{n\} = s\{n-1\} + t\{n-2\}$ for $n \geq 2$. If $s, t$ are integers then the corresponding sequence of integers is called a *Lucas sequence*. Define an analogue of the binomial coefficients by

$$\left\{ {n \atop k} \right\} = \frac{\{n\}!}{\{k\}!\,\{n-k\}!}$$

where $\{n\}! = \{1\}\{2\}\cdots\{n\}$. It is easy to see that $\left\{ {n \atop k} \right\}$ is a polynomial in $s$ and $t$. The purpose of this note is to give two combinatorial interpretations for this polynomial in terms of statistics on integer partitions inside a $k \times (n-k)$ rectangle. When $s = t = 1$ we obtain combinatorial interpretations of the fibonomial coefficients which are simpler than any that have previously appeared in the literature.

[1]Work partially done while a Program Officer at NSF. The views expressed are not necessarily those of the NSF.

[2]Partially supported by NSA grant H98230-08-1-0072

## 1. Introduction

Given variables $s, t$ we define the corresponding sequence of *Lucas polynomials*, $\{n\}$, by $\{0\} = 0$, $\{1\} = 1$, and for $n \geq 2$:

$$\{n\} = s\{n-1\} + t\{n-2\}. \tag{1}$$

When $s, t$ are integers, the corresponding integer sequence is called a *Lucas sequence* [10, 11, 12]. These sequences have many interesting number-theoretic and combinatorial properties.

Define the *lucanomials* for $0 \leq k \leq n$ by

$$\left\{ {n \atop k} \right\} = \frac{\{n\}!}{\{k\}!\,\{n-k\}!} \tag{2}$$

where $\{n\}! = \{1\}\{2\}\cdots\{n\}$. It is not hard to show that $\left\{ {n \atop k} \right\}$ is a polynomial in $s$ and $t$. (This follows from Proposition 2.2 below). The purpose of this note is to give two simple combinatorial interpretations of the lucanomials. They are based on statistics associated with integer partitions $\lambda$ inside a $k \times (n-k)$ rectangle. More specifically, we will show that $\left\{ {n \atop k} \right\}$ is the generating function for certain tilings of such $\lambda$ and their complements with dominos and monominos.

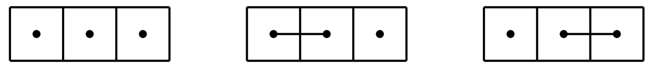
Various specializations of the parameters $s$ and $t$ are of interest. When $s = t = 1$, $\{n\}$ becomes the $n$th Fibonacci number, and $\left\{ {n \atop k} \right\}$ is known as a fibonomial coefficient. Gessel and Viennot [8] gave an interpretation of the fibonomials in terms of nonintersecting lattice paths and asked for a simpler one. Benjamin and Plott [2, 3] gave another interpretation in terms of tilings, but it is not as straight-forward as ours. When $s = \ell$ and $t = -1$ we get two new interpretations of the $\ell$-nomial coefficients of Loehr and Savage [9]. This case is of interest because of its connection with the Lecture Hall Partition Theorem introduced by Bousquet-Mélou and (Kimmo) Eriksson [5, 6, 7]. Finally, letting $s = q + 1$ and $t = -q$ one gets new interpretations for the classical $q$-binomial coefficients. For more information about these important polynomials, see the text of Andrews [1].

We should mention that it is possible to derive our results from $q$-binomial coefficient identities using algebraic manipulations, analogous to what is done for $\ell$-nomial coefficients in Section 3.3 of [9]. However, we wish to demonstrate how these results follow from simple, combinatorial arguments.

## 2. Recursions

In this section we will present the recurrence relations we will need for our combinatorial interpretations. To obtain these results, we will use a tiling model that is often useful when dealing with Lucas sequences. See the book of Benjamin and Quinn [4] for more details.

$\mathcal{L}_3$ :



$\{4\} = s^3 + st + st$

Figure 1: The tilings in $\mathcal{L}_3$ and corresponding weights

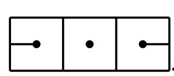
Suppose we have $n$ squares arranged in a $1 \times n$ rectangle. We number the squares $1, \ldots, n$ from left to right and also number the vertical edges of the squares $0, \ldots, n$ left to right. A *linear tiling*, $T$, is a covering of the rectangle with disjoint dominos (covering two squares) and monominos (covering one square). Let $\mathcal{L}_n$ be the set of all such $T$. The tilings in $\mathcal{L}_3$ are drawn in Figure 1 where a dot in a square represents a monomino while two dots connected by a horizontal line represent a domino.

Let the weight of a tiling be $w(T) = s^m t^d$ where $m$ and $d$ are the number of monominos and dominos in $T$, respectively. We will use the same weight for all other types of tilings considered below. Since the last tile in any tiling must be a monomino or domino, the initial conditions and recursion (1) immediately give

$$\{n+1\} = \sum_{T \in \mathcal{L}_n} w(T).$$

See Figure 1 for an illustration of the case $n = 3$.

For our second combinatorial interpretation, we will need another sequence of polynomials closely related to the $\{n\}$. Define $\langle n \rangle$ using recursion (1) but with the initial conditions $\langle 0 \rangle = 2$ and $\langle 1 \rangle = s$. If $s = t = 1$ then $\langle n \rangle$ is the $n$th Lucas number. A combinatorial interpretation for these polynomials is obtained via another type of tiling. In a *circular tiling* of the $1 \times n$ rectangle, the edges labeled 0 and and $n$ are identified so that it is possible to have a domino crossing this edge and covering the first and last squares. Such a domino, if it exists, will be called the *circular domino* of the tiling. Let $\mathcal{C}_n$ be the set of circular tilings of a $1 \times n$ rectangle. So $\mathcal{L}_n \subseteq \mathcal{C}_n$ is the subset of all circular tilings with no circular domino. For example, $\mathcal{C}_3$ consists of the tilings in $\mathcal{L}_3$ displayed previously together with



Now for $n \geq 1$ we have

$$\langle n \rangle = \sum_{T \in \mathcal{C}_n} w(T). \tag{3}$$

Indeed, to show that the sum satisfies (1) first note that we already have a weight-preserving bijection for the linear tilings involved. And if $T \in \mathcal{C}_n$ has a circular edge, then removal of the tile covering square $n-1$ will take care of the remainder. In order to make (3) also hold for $\mathcal{C}_0$, we give the empty tiling $\epsilon$ of the $1 \times 0$ box weight $w(\epsilon) = 2$. Bear in mind that $\epsilon$

considered as an element of $\mathcal{L}_0$ still has $w(\epsilon) = 1$. Context will always make it clear which weight we are using.

We start with two recursions for the Lucas polynomials. These are well known for Lucas sequences; see [4, p. 38, Identity 73] for (4) and [4, p. 46, Identity 94] or [11, p. 201, Equation 49] for (5). Also, the proofs in the integer case generalize to variable $s$ and $t$ without difficulty. But we will provide a demonstration for completeness and to emphasize the simplicity of the combinatorics involved.

**Lemma 2.1.** *For $m \geq 1$ and $n \geq 0$ we have*

$$\{m+n\} = \{n+1\}\{m\} + t\{m-1\}\{n\}. \tag{4}$$

*For $m, n \geq 0$ we have*

$$\{m+n\} = \frac{\langle n \rangle}{2}\{m\} + \frac{\langle m \rangle}{2}\{n\}. \tag{5}$$

*Proof.*For the first identity, the left-hand side is the generating function for $T \in \mathcal{L}_{m+n-1}$. The second and first terms on the right correspond to those tilings which do or do not have a domino crossing the edge labeled $n$, respectively. To illustrate, if $m = n = 2$ then the tilings in Figure 1 are counted by $\{m+n\}$. The first two tilings do not have a domino crossing the edge labeled 2 and so are counted by $\{n+1\}\{m\}$. The edge of the third tiling does cross that edge and is counted by $t\{m-1\}\{n\}$.

Multiply the second equation by 2 and consider two copies of $\mathcal{L}_{m+n-1}$. In each tiling in the first copy distinguish the edge labeled $m-1$, and do the same for the edge labeled $m$ in the second copy. The set of tilings in both copies where a domino does not cross the distinguished edge accounts for the terms corresponding to linear pairs in $\langle n \rangle \{m\} + \langle m \rangle \{n\}$. If a domino crosses the distinguished edge $m-1$ in a tiling $T$, then consider the restriction $T'$ of $T$ to the first $m$ squares as a circular tiling by shifting it so that the domino between squares $m-1$ and $m$ becomes the circular edge. Also let $T''$ be the restriction of $T$ to the last $n-1$ squares considered as a linear tiling. The pairs $(T', T'')$ account for the remaining terms in $\langle m \rangle \{n\}$. A similar bijection using the tilings with a domino crossing the distinguished edge $m$ accounts for the rest of the terms in $\langle n \rangle \{m\}$, completing the proof. □

We can use the recurrence relations in the previous lemma to produce recursions for the lucanomials.

**Proposition 2.2.** *For $m, n \geq 1$ we have*

$$\begin{aligned} \left\{ \begin{matrix} m+n \\ m \end{matrix} \right\} &= \{n+1\} \left\{ \begin{matrix} m+n-1 \\ m-1 \end{matrix} \right\} + t\{m-1\} \left\{ \begin{matrix} m+n-1 \\ n-1 \end{matrix} \right\} \\ &= \frac{\langle n \rangle}{2} \left\{ \begin{matrix} m+n-1 \\ m-1 \end{matrix} \right\} + \frac{\langle m \rangle}{2} \left\{ \begin{matrix} m+n-1 \\ n-1 \end{matrix} \right\}. \end{aligned}$$

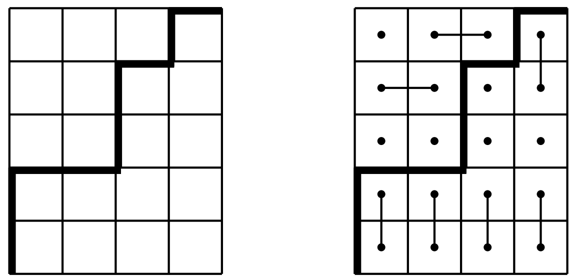


Figure 2: A partition $\lambda$ contained in $5 \times 4$ and a tiling

*Proof.*Given $m, n$ and any polynomials $p$ and $q$ such that $\{m+n\} = p\{m\} + q\{n\}$, straightforward algebraic manipulation of the definition of lucanomials yields

$$\left\{ \begin{matrix} m+n \\ m \end{matrix} \right\} = p \left\{ \begin{matrix} m+n-1 \\ m-1 \end{matrix} \right\} + q \left\{ \begin{matrix} m+n-1 \\ n-1 \end{matrix} \right\}.$$

Combining this observation with Lemma 2.1, we are done. □

## 3. The combinatorial interpretations

Our combinatorial interpretations of $\left\{ \begin{smallmatrix} m+n \\ m \end{smallmatrix} \right\}$ will involve integer partitions. A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of nonnegative integers. The $\lambda_i$, $1 \le i \le m$, are called *parts* and note that we are allowing zero as a part. The *Ferrers diagram* of $\lambda$, also denoted $\lambda$, is an array of $m$ left-justified rows of boxes with $\lambda_i$ boxes in row $i$. We say that $\lambda$ is *contained in an* $m \times n$ *rectangle*, written $\lambda \subseteq m \times n$, if it has $m$ parts and each part is at most $n$. In this case, $\lambda$ determines another partition $\lambda^*$ whose parts are the lengths of the columns of the complement of $\lambda$ in $m \times n$. The first diagram in Figure 2 shows $\lambda = (3, 2, 2, 0, 0)$ contained in a $5 \times 4$ rectangle with complement $\lambda^* = (5, 4, 2, 2)$.

A *linear tiling of* $\lambda$ is a covering of its Ferrers diagram with disjoint dominos and monominos obtained by linearly tiling each $\lambda_i$. The set of such tilings is denoted $\mathcal{L}_\lambda$. Note that if $\lambda \subseteq m \times n$ then $T \in \mathcal{L}_\lambda$ gives a tiling of each of its rows, while $T \in \mathcal{L}_{\lambda^*}$ gives a tiling of each column of its complement. We will also need $\mathcal{L}'_n$ which is the set of all tilings in $\mathcal{L}_n$ which do not begin with a monomino. This is equivalent to beginning with a domino if $n \ge 2$, and for $n < 2$ yields $\mathcal{L}'_0 = \{\epsilon\}$ and $\mathcal{L}'_1 = \emptyset$. We define $\mathcal{L}'_\lambda$ similarly. The second diagram in Figure 2 shows a tiling in $\mathcal{L}_\lambda \times \mathcal{L}'_{\lambda^*}$. In a *circular tiling of* $\lambda$ we use circular tilings on each $\lambda_i$. So if $\lambda_i = 0$ then it will get the empty tiling which has weight 2. The notation $\mathcal{C}_\lambda$ is self-explanatory. If one views the tiling $T$ in Figure 2 as an element of $\mathcal{L}_\lambda \times \mathcal{L}'_{\lambda^*}$ then it has weight $w(T) = s^6 t^7$. But as an element of $\mathcal{C}_\lambda \times \mathcal{C}_{\lambda^*}$ it has $w(T) = 4s^6 t^7$. As usual, context will clarify which weight to use. We are now in a position to state and prove our two combinatorial interpretations for the lucanomials. The first has the nice property that it is multiplicity free. The second is pleasing because it displays the natural symmetry of $\left\{ {m+n \atop m} \right\}$.

**Theorem 3.1.** *For $m, n \geq 0$ we have*

$$\left\{ {m+n \atop m} \right\} = \sum_{\lambda \subseteq m \times n} \sum_{T \in \mathcal{L}_\lambda \times \mathcal{L}'_{\lambda^*}} w(T), \tag{6}$$

*and*

$$2^{m+n} \left\{ {m+n \atop m} \right\} = \sum_{\lambda \subseteq m \times n} \sum_{T \in \mathcal{C}_\lambda \times \mathcal{C}_{\lambda^*}} w(T). \tag{7}$$

*Proof.*We will show that the right-hand side of (6) satisfies the first recursion in Proposition 2.2 as the initial conditions are easy to verify. Given $\lambda \subseteq m \times n$ there are two cases. If $\lambda_1 = n$ then the generating function for tilings of the first row of $\lambda$ is $\{n+1\}$, and $\left\{ {m+n-1 \atop m-1} \right\}$ counts the ways to fill the rest of the rectangle. If $\lambda_1 < n$ then $\lambda_1^* = m$. The generating function for $\mathcal{L}'_m$ is $t\{m-1\}$ and $\left\{ {m+n-1 \atop n-1} \right\}$ takes care of the rest.

Proving that both sides of (7) is similar using the fact that if we let $f(m,n) = 2^{m+n} \left\{ {m+n \atop m} \right\}$ then, by the second recursion in Proposition 2.2, we have $f(m,n) = \langle n \rangle f(m-1,n) + \langle m \rangle f(m, n-1)$. This completes the proof. □

We end by noting that it would be interesting to find analogues for $\left\{ {n \atop k} \right\}$ of various known identities for ordinary binomial coefficients. It might then be possible to use the previous theorem to provide combinatorial proofs. One approach would be to apply algebraic manipulations to the corresponding results for $q$-binomial coefficients $\left[ {n \atop k} \right]_q$. For example, this method was used in [9] to give an analog of the Chu-Vandermonde summation for $s = \ell$, $t = -1$. For general $s$ and $t$ it is easy to see that

$$\left\{ {n \atop k} \right\} = y^{k(n-k)} \left[ {n \atop k} \right]_{x/y}$$

where

$$x = \frac{s + \sqrt{s^2 + 4t}}{2} \quad \text{and} \quad y = \frac{s - \sqrt{s^2 + 4t}}{2}.$$

Unfortunately, this approach tends to introduce algebraic functions of $s$ and $t$ for which a combinatorial interpretation is unclear.

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