A NOTE ON V.I. ARNOLD'S CHORD CONJECTURE

CASIM ABBAS

Abstract

Let $(S^3, f\lambda)$ be the three sphere endowed with a contact form $f\lambda$, where λ is the standard tight contact form on S^3 and $f : S^3 \to \mathbf{R}\backslash\{0\}$ is a smooth function. Moreover let $\mathcal{L} \subset S^3$ be a Legendrian knot. Then V.I. Arnold raised the conjecture in 1986 that there is a characteristic chord, i.e. an orbit x of the Reeb vectorfield that intersects the knot $\mathcal{L}$ at two different times. Characteristic chords are important in the theory of contact homology, see [6], [5]. Very little is known about their existence or non-existence . Using a deep result by H. Hofer, K. Wysocki and E. Zehnder about the existence of a global surface of section in strictly convex hypersurfaces in $\mathbf{R}^4$, we observe that characteristic chords exist for certain Legendrian knots in strictly convex hypersurfaces in $\mathbf{R}^4$.

1. Introduction

This paper makes a contribution to a conjecture of V.I. Arnold in contact geometry, which he stated in his 1986 paper [4]. A *contact form* on a closed oriented three manifold M is a 1–form τ so that $\tau \wedge d\tau$ is a volume form. There is a distinguished vectorfield X_τ, called the Reeb vectorfield of τ, which is defined by $i_{X_\tau} d\tau \equiv 0$ and $i_{X_\tau}\tau \equiv 1$. The standard example on S^3 is the following: Consider the 1-form λ on $\mathbf{R}^4$ defined by

$$\lambda = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2).$$

This induces a contact form on the unit three sphere S^3. Observe that all the orbits of the Reeb vectorfield are periodic; they are the fibres of the Hopf fibration. Note that the dynamics of the Reeb vectorfield changes drastically in general if we replace λ by the contact form $f\lambda$ where f is a nowhere vanishing function on S^3 (see the example in [1], [2], [3]). A *Legendrian knot* in a contact manifold (M, τ) is a knot which is everywhere tangent to $\ker \tau$. Then V.I. Arnold stated the following

Conjecture ([4])**:** Let λ be the above contact form on the three sphere. If $f : S^3 \to (0, \infty)$ is a smooth function and $\mathcal{L}$ a Legendrian knot in S^3, then there is a *characteristic chord* for $(f\lambda, \mathcal{L})$, i.e. a trajectory x of the Reeb vectorfield and some number $T \neq 0$ so that $x(0) \in \mathcal{L}$ and $x(T) \in \mathcal{L}$.

There is almost nothing known about this problem. V.I. Arnold only mentioned the case where $f \equiv 1$: Projecting the Legendrian knot onto S^2 using the Hopf fibration, we observe that the area enclosed by this curve is a multiple of 4π, hence it must

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have a self–intersection, which in turn shows that an orbit of the Reeb vectorfield intersects the Legendrian knot at two different times.

Definition 1.1. *A three–dimensional submanifold of* $\mathbf{R}^4$ *is called a strictly convex hypersurface in* $\mathbf{R}^4$ *if it is closed and if it bounds a compact strictly convex set containing the origin.*

We will provide an existence proof for characteristic chords in the case where the contact manifold is a strictly convex hypersurface $S \subset \mathbf{R}^4$ with contact form $\lambda|_S$ and where we confine ourselves to Legendrian knots which are not linked with a certain periodic orbit of the Reeb vectorfield. Our result is based on the following crucial theorem by H. Hofer, K. Wysocki and E. Zehnder [9], [7]:

Theorem 1.2. *Let* S *be a strictly convex hypersurface in* $\mathbf{R}^4$ *and let* J *be a complex structure on the symplectic vectorbundle* $\ker\lambda \to S$ *compatible with* $d\lambda$, *i.e.* $d\lambda \circ (Id \times J)$ *should be a bundle metric on* $\ker\lambda$ *(such* J *always exist, see* [1]*). Then there exist an unknotted periodic orbit* P_0 *of the Reeb vectorfield* X_λ *with self–linking number -1 and generalized Conley–Zehnder index 3 ("binding orbit") and a diffeomorphism*

$$\Phi : S^1 \times \mathbf{C} \longrightarrow S\backslash P_0$$

with the following properties:
For every $\tau \in S^1$ *the map* $u_\tau := \Phi(\tau, .)$ *is an embedding and satisfies:*

- $u_\tau(\mathbf{C})$ *is transversal to* X_λ
- $u_\tau(Re^{2\pi it}) \to P_0(T_0t)$ *as* $R \to +\infty$ *with convergence in* C^∞ *where* T_0 *is the minimal period of* P_0
- *The maps* u_τ *satisfy the following partial differential equation:*

$$\pi_\lambda\partial_s u_\tau + J(u_\tau)\pi_\lambda\partial_t u_\tau = 0$$

$$d(u_\tau^*\lambda \circ i) = 0$$

- $d\lambda|_{u_\tau(\mathbf{C})}$ *is non-degenerate and* $\int_{\mathbf{C}} u_\tau^* d\lambda = T_0 \in (0,\infty)$.
- *Each* $u_\tau(\mathbf{C})$ *gives rise to a global surface of section, i.e. the closure of* $u_\tau(\mathbf{C})$ *is an embedded disk* D_τ *with* $\partial D_\tau = P_0$ *and for each point* $p \in S\backslash P_0$ *there are real numbers* $t^-(p) < 0$ *and* $t^+(p) > 0$ *so that*

$$\varphi_{t^-(p)}(p)\ ,\ \varphi_{t^+(p)}(p) \in \overset{\circ}{D}_\tau = u_\tau(\mathbf{C})$$

  where φ *denotes the flow of* X_λ.

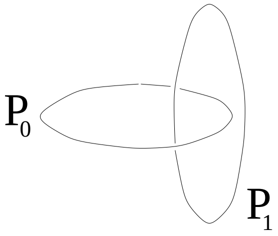
□

H. Hofer, K. Wysocki and E. Zehnder have actually shown that any periodic orbit P_0 of X_λ which is unknotted with self–linking number -1 and generalized Conley–Zehnder index 3 is a binding orbit as described in theorem 1.2 (see theorem 1.6 in [8] and section 7 of [9]). Our result is the following:

Theorem 1.3. *Let* S *be a strictly convex hypersurface in* $\mathbf{R}^4$ *equipped with the contact form* $\lambda|_S$. *Let* $\mathcal{L} \subset S$ *be a Legendrian knot with the following additional property: Assume, there is an unknotted periodic orbit* P_0 *of* X_λ *which has self–linking number -1 and generalized Conley–Zehnder index 3, so that* $\mathcal{L}$ *and* P_0 *are not linked.*
Then there exists a characteristic chord for the knot $\mathcal{L}$.

We note that such an orbit P_0 always exists in view of theorem 1.2, so that there are many knots $\mathcal{L}$ for which the assumptions are satisfied. The obvious drawback is of course that the hypothesis is in general difficult to verify.

Remark: The assumption of P_0 having generalized Conley–Zehnder index 3 can be dropped since theorem 1.2 also holds without this assumption [10]. In fact, it follows from theorem 1.2 that there always exist at least two unknotted periodic orbits P_0 and P_1 which have self–linking number -1 and are linked with $\mathrm{lk}(P_0, P_1) = 1$. In view of this remark the following is true:



Theorem 1.4. *Let S be a strictly convex hypersurface in $\mathbf{R}^4$ equipped with the contact form $\lambda|_S$. Let $\mathcal{L} \subset S$ be a Legendrian knot with the following additional property: Assume, there is an unknotted periodic orbit P_0 of X_λ which has self–linking number -1, so that $\mathcal{L}$ and P_0 are not linked.*
Then there exists a characteristic chord for the knot $\mathcal{L}$.

The relation between theorem 1.3 and V.I. Arnold's conjecture is the following: The projection along the rays issuing from the origin in $\mathbf{R}^4$ defines a diffeomorphism $\phi : S^3 \to S$ by $z \longmapsto h(z)z$ with a suitable smooth function $h : S^3 \to (0, \infty)$. Then $\phi^*\lambda|_S = h^2\lambda|_{S^3}$. Hence theorem 1.3 gives an affirmative answer to V.I. Arnold's conjecture if we restrict the class of admissible Legendrian knots and the class of contact forms $f\lambda$.

2. Proof of theorem 1.3

We assume of course that $P_0 \cap \mathcal{L} = \emptyset$ since we are done otherwise. Let us choose $\tau_0 \in S^1$ so that our given knot $\mathcal{L}$ and $u_{\tau_0}(\mathbf{C})$ intersect transversally.
We define the following map

$$\Psi : Z = D \times \mathbf{R} \longrightarrow S \backslash P_0$$

$$(x, y, z) \longmapsto \varphi_z(u_0(x, y))$$

where φ denotes the flow of X_λ, D is the open unit disk in $\mathbf{R}^2$ and $u_0 = \Phi(\tau_0, \phi(.))$ with ϕ being the diffeomorphism $\phi : D \to \mathbf{C}$, $re^{i\varphi} \longmapsto (1 - r^2)^{-1} re^{i\varphi}$. Then $\mathcal{L}$ and $\Sigma := u_0(D)$ intersect transversally.
Denote by

$$T^\pm : \Sigma \longrightarrow \mathbf{R}^\pm$$

the positive (resp. negative) return time, i.e. $T^+(p)$ (resp. $T^-(p)$) is the smallest positive (resp. largest negative) time t where we have $\varphi_t(p) \in \Sigma$. Let us denote by

$P_\Sigma : \Sigma \to \Sigma$ the Poincaré-map, i.e. $P_\Sigma(p) = \varphi_{T^+(p)}(p)$. The inverse P_Σ^{-1} is given by $p \longmapsto \varphi_{T^-(p)}(p)$. We observe that

$$\inf_{p\in\Sigma} |T^\pm(p)| > 0 \tag{1}$$

and that the derivative of Ψ is an isomorphism at every point $(x,y,z) \in Z$. Assume now

$$\Psi(x,y,z) = \Psi(x',y'.z')$$

for some points (x,y,z) , $(x',y',z') \in Z$. Then

$$\varphi_{z-z'}(u_0(x,y)) = u_0(x',y')$$

and there is some $k \in \mathbf{Z}$ so that

$$u_0(x',y') = P_\Sigma^k(u_0(x,y))$$

and

$$z' = z - \sum_{l=0}^{|k|-1} (T^\pm \circ P_\Sigma^{\pm l} \circ u_0)(x,y),$$

where $\pm =$ sign k and $P_\Sigma^0 :=$ Id_Σ. Using this and (1.2), we see that $\Psi : Z \to S\backslash P_0$ is the universal cover of $S\backslash P_0$. A straightforward calculation shows that the one form $\nu = \Psi^*\lambda$ on Z is given by

$$\nu = dz + u_0^*\lambda.$$

Moreover, it is a contact form with Reeb vectorfield

$$X_\nu(x,y,z) = \frac{\partial}{\partial z} = (0,0,1).$$

Let $\gamma : [0,1] \to S\backslash P_0$ be a parametrization of our Legendrian knot $\mathcal{L}$, i.e. $\gamma(0) = \gamma(1)$, $\gamma([0,1]) = \mathcal{L}$ and γ is an embedding. Take any lift $\tilde\gamma : [0,1] \to Z$ of γ. Note that $\tilde\gamma$ is a smoothly embedded curve since Ψ is a local diffeomorphism. Moreover $\tilde\gamma$ is Legendrian with respect to the contact structure $\ker\nu$ on Z.

The homology groups $H_1(\mathcal{L},\mathbf{Z})$ and $H_1(S\backslash P_0,\mathbf{Z})$ are isomorphic to $\mathbf{Z}$; denote generators by $[\mathcal{L}]$ and $[S\backslash P_0]$. Then the inclusion $\mathcal{L} \hookrightarrow S\backslash P_0$ induces a homomorphism $i_* : H_1(\mathcal{L},\mathbf{Z}) \to H_1(S\backslash P_0,\mathbf{Z})$, so there exists $k \in \mathbf{Z}$ so that $i_*[\mathcal{L}] = k\,[S\backslash P_0]$. Then $lk(\mathcal{L}, S\backslash P_0) = k$ is the linking number of $\mathcal{L}$ and P_0 in S. By our assumption that $\mathcal{L}$ and P_0 are not linked, we have $k = 0$.

If $\{\mathcal{L}\}$ is a generator of $\pi_1(\mathcal{L}) \approx \mathbf{Z}$ and $i_\# : \pi_1(\mathcal{L}) \to \pi_1(S\backslash P_0)$ the homomorphism induced by the inclusion i. Then

- $\mathcal{L}$ and P_0 are not linked if and only if $i_\#\{\mathcal{L}\} \in \pi_1(S\backslash P_0)$ is the trivial class,
- A lift $\tilde\gamma : [0,1] \to Z$ of γ is a closed curve if and only if $\mathcal{L}$ and P_0 are not linked.

Hence, any lift $\tilde\gamma : [0,1] \to Z = D \to \mathbf{R}$ of γ is a closed curve. Writing $\tilde\gamma(t) = (\zeta(t), z(t)) \in D \times \mathbf{R}$, we obtain

$$0 = \int_{S^1} \tilde\gamma^*\nu = \int_{S^1} \zeta^*(u_0^*\lambda).$$

We use the following lemma (proposition 5.4. of [9]):

Lemma 2.1. *Consider on D the two–form $f(x,y)dx\wedge dy$ with $f>0$ and integrable on D. Then there exists a diffeomorphism $\tau : D \to D$ satisfying*

$$\tau^*(f\,dx\wedge dy) = c\,dx\wedge dy$$

for a suitable constant $c>0$.

$\square$

Applying this to our situation, we get

$$\tau^* d(u_0^*\lambda) = c\,dx\wedge dy.$$

If $\overline{\zeta} : \overline{D} \to D$ is any smooth extension of $\zeta : S^1 \to D$ and if we write $\overline{\xi} := \tau^{-1}\circ\overline{\zeta}$, $\xi := \tau^{-1}\circ\zeta$ then

$$\begin{aligned} 0 &= \int_{S^1}\zeta^*(u_0^*\lambda) \\ &= \int_{\overline{D}}\overline{\zeta}^* d(u_0^*\lambda) \\ &= c\int_{\overline{D}}\overline{\xi}^*(dx\wedge dy) \\ &= \frac{c}{2}\int_{S^1}\xi^*(xdy-ydx). \end{aligned}$$

This means that the area of the set enclosed by the curve ξ equals zero, so ξ must have a self–intersection. this implies that ζ must also have a self–intersection, hence there is a characteristic chord for the Legendrian knot $\tilde{\gamma}$. Because $\tilde{\gamma}$ was a lift of the curve γ with respect to Ψ and because Ψ maps orbits of X_ν to orbits of X_λ, we also have obtained a characteristic chord for the knot $\mathcal{L}$. $\square$

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CASIM ABBAS, COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER STREET, NEW YORK, NY 10012