Supplemental Material for Section 7.4: Exponential Functions

In this section the general exponential function, a^x is defined and some of its properties are presented. This document contains material that should have been included in the text. Recall that a^x has already been defined for many values of x. For example $a^{\frac{1}{2}}$ denotes $\sqrt{a}$. In fact a^r has been defined for any rational number $r = \frac{m}{n}$ by $a^r = (\sqrt[n]{a})^m$. Because only positive numbers have n^{th} roots, it's natural to assume that $a > 0$ which we do. To see how to use the function e^x to define a^x, note that for r a rational number, $a^r = e^{\ln a^r} = e^{r \ln a}$. The definition of a^x is accomplished by simply insisting that the same formula holds for any real number x; not just rational numbers.

Definition. *Let $a > 0$ and let x be any real number. Then $a^x = e^{x \ln a}$.*

The basic laws of exponents will now be established from this definition using the properties of the function e^x and the laws of logarithms. We first note that for $a > 0$ and any real number x, $\ln a^x = \ln e^{x \ln a} = x \ln a$ by the definition of a^x and that $\ln e^y = y$ for any number y.

Theorem 1. *Let $a > 0$, $b > 0$ and let x and y be real numbers. Then:*

1. $a^x a^y = a^{x+y}$
2. $(a^x)^y = a^{xy}$
3. $a^x b^x = (ab)^x$

Proof. 1. $a^x a^y = e^{x \ln a} e^{y \ln a} = e^{(x \ln a)+(y \ln a)} = e^{(x+y) \ln a} = a^{x+y}$. The last equality is the definition of a^{x+y}.

2. $(a^x)^y = e^{y \ln a^x} = e^{y(x \ln a)} = e^{(xy) \ln a} = a^{xy}$. The last equality is the definition of a^{xy}.

3. $a^x b^x = (e^{x \ln a})(e^{x \ln b}) = e^{(x \ln a)+(x \ln b)} = e^{x(\ln a+\ln b)} = e^{x \ln(ab)} = (ab)^x$. The last equality is the definition of $(ab)^x$. □

Using $-x$ for y in 1. we obtain that $a^x a^{-x} = a^0 = e^{0 \ln a} = e^0 = 1$. Consequently $e^{-x} = \frac{1}{e^x}$. From this fact and 1. we obtain the familiar formula $\frac{a^x}{a^y} = a^{x-y}$.