

Practice Exam (Second Midterm), Math133

1. Which of the following functions grows faster as $x \rightarrow \infty$?

$$f(x) = x, \quad g(x) = \frac{1}{\sin^{-1} \frac{1}{x^2}}$$

2. Evaluate the following integrals

(a)

$$\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx.$$

(b)

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x-12}}.$$

(c)

$$\int e^x \sin x dx$$

3. (a) Rewrite $(\sinh x + \cosh x)^5$ in terms of exponential functions and simplify as much as you can.

(b) Evaluate

$$\int_0^{\pi/2} 2 \sinh(\sin \theta) \cos \theta d\theta$$

4. Compute the derivative of the function

$$y = \sin^{-1}(\tan^{-1}(\ln x))$$

5. Expand the following quotient by partial fractions:

$$\frac{t}{t^4 - t^2 - 2}$$

Solutions: 1.) We use l'Hôpital's rule

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\sin^{-1} \frac{1}{x^2}}{\frac{1}{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{-\frac{2}{x^3} \frac{1}{\sqrt{1-\frac{1}{x^4}}}}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} -\frac{2}{x} \frac{1}{\sqrt{1-\frac{1}{x^4}}} \\
 &= 0,
 \end{aligned}$$

hence the function $g(x)$ grows faster.

2a.) We substitute $u = \tan^{-1} x$, $du = \frac{1}{1+x^2} dx$ and get

$$\begin{aligned}
 \int \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx &= \int \sqrt{u} du \\
 &= \frac{2}{3} u^{3/2} + C \\
 &= \frac{2}{3} (\tan^{-1} x)^{3/2} + C.
 \end{aligned}$$

2b.) We first complete the square

$$x^2 - 4x - 12 = x^2 - 4x + 4 - 16 = (x - 2)^2 - 16$$

so that

$$\begin{aligned}
 \int \frac{dx}{(x-2)\sqrt{x^2-4x-12}} &= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-16}} \\
 &\stackrel{u=x-2, du=dx}{=} \int \frac{du}{u\sqrt{u^2-16}} \\
 &= \frac{1}{4} \sec^{-1} \left| \frac{u}{4} \right| + C \\
 &= \frac{1}{4} \sec^{-1} \left| \frac{x-2}{4} \right| + C
 \end{aligned}$$

2c.) We perform integration by parts twice. First, we choose $u'(x) = e^x$, $v(x) = \sin x$ so that

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

Then we choose $u'(x) = e^x$, $v(x) = \cos x$, and we get

$$\begin{aligned}
 \int e^x \sin x dx &= e^x \sin x - \left(e^x \cos x - \int e^x \cdot (-\sin x) dx \right) \\
 &= e^x \sin x - e^x \cos x - \int e^x \sin x dx.
 \end{aligned}$$

Finally,

$$2 \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

and

$$\int e^x \sin x dx = \frac{1}{2}(e^x \sin x - e^x \cos x).$$

3a.) Recalling that

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

we obtain

$$(\sinh x + \cosh x)^5 = (e^x)^5 = e^{5x}.$$

3b.) We first substitute $u = \sin \theta$, $du = \cos \theta d\theta$ so that

$$\begin{aligned} \int 2 \sinh(\sin \theta) \cos \theta d\theta &= 2 \int \sinh u du \\ &= 2 \cosh u + C \\ &= 2 \cosh(\sin \theta) + C \end{aligned}$$

and finally

$$\int_0^{\pi/2} 2 \sinh(\sin \theta) \cos \theta d\theta = 2 [\cosh(\sin \theta)]_0^{\pi/2} = 2(\cosh(1) - 1)$$

4.) Recall that

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\tan^{-1} x)' = \frac{1}{1+x^2}, \quad (\ln x)' = \frac{1}{x}.$$

Using the chain rule we get

$$y' = \frac{1}{\sqrt{1 - (\tan^{-1}(\ln x))^2}} \cdot \frac{1}{1 + (\ln x)^2} \cdot \frac{1}{x}.$$

5.) The equation $t^4 - t^2 - 2 = 0$ is quadratic in t^2 , and we find $t^2 = 2$ so that there are two solutions $t = \pm\sqrt{2}$. Dividing $t^4 - t^2 - 2$ by $(t + \sqrt{2})(t - \sqrt{2})$ we obtain $1 + t^2$ so that

$$t^4 - t^2 - 2 = (1 + t^2)(t + \sqrt{2})(t - \sqrt{2}).$$

This leads to the setup

$$\frac{t}{t^4 - t^2 - 2} = \frac{A}{t + \sqrt{2}} + \frac{B}{t - \sqrt{2}} + \frac{Ct + D}{1 + t^2}.$$

Multiplying both sides with $(1 + t^2)(t + \sqrt{2})(t - \sqrt{2})$ we get the equation

$$\begin{aligned} t &= A(t - \sqrt{2})(1 + t^2) + B(t + \sqrt{2})(1 + t^2) + \\ &\quad (Ct + D)(t - \sqrt{2})(t + \sqrt{2}). \end{aligned} \tag{1}$$

Inserting $t = \sqrt{2}$ we get

$$\sqrt{2} = B \cdot 2\sqrt{2} \cdot 3 = 6\sqrt{2}B$$

and

$$B = \frac{1}{6}.$$

Inserting $t = -\sqrt{2}$ we obtain

$$-\sqrt{2} = A(-2\sqrt{2}) \cdot 3$$

and

$$A = \frac{1}{6}.$$

We now insert $t = 0$ and the above values for A, B into equation (1) so that

$$0 = -\frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} - 2D$$

and

$$D = 0.$$

Inserting A, B, D into equation (1) we obtain

$$\begin{aligned} t &= \frac{1}{6}(1+t^2)(t-\sqrt{2}+t+\sqrt{2}) + Ct(t-\sqrt{2})(t+\sqrt{2}) \\ &= \frac{1}{3}t(1+t^2) + Ct(t^2-2) \\ &= \frac{1}{3}t + \frac{1}{3}t^3 + Ct^3 - 2Ct \\ &= \left(\frac{1}{3} + C\right)t^3 + \left(\frac{1}{3} - 2C\right)t \end{aligned}$$

which leads to

$$C = -\frac{1}{3}.$$

Summarizing,

$$\frac{\mathbf{t}}{\mathbf{t}^4 - \mathbf{t}^2 - \mathbf{2}} = \frac{1}{6} \left(\frac{1}{t + \sqrt{2}} + \frac{1}{t - \sqrt{2}} \right) - \frac{t}{3(1+t^2)} = \frac{\mathbf{t}}{\mathbf{3}(\mathbf{t}^2 - \mathbf{2})} - \frac{\mathbf{t}}{\mathbf{3}(\mathbf{1} + \mathbf{t}^2)}$$