

Third Midterm Exam–Practice Test, Math133

1. Solve the following differential equation

$$\frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y}$$

with the initial condition $y(0) = 1$.

2. State whether the following sequences of numbers $\{a_n\}_{n \geq 1}$ converge or diverge. If they converge, find the limit.

(a) $a_n = \sqrt[n]{2n+1}$

(b) $a_n = (-3)^n$

3. Compute the integral and simplify as much as possible

$$\int \frac{\sqrt{9-w^2}}{w^2} dw$$

(Hint: Use the identity $1 + \cot^2 \theta = 1/\sin^2 \theta$ and compute also $(\cot \theta)'$)

4. Determine whether the following improper integral converges. State the method you have used.

$$\int_1^{\infty} \frac{1}{\sqrt{e^x - x}} dx$$

5. Compute the integral

$$\int_0^1 \ln x \, dx$$

Solutions:

1. Separating the variables yields

$$\int \frac{dy}{\sqrt{y} \cos^2 \sqrt{y}} = \int dx$$

Substituting $u = \sqrt{y}$, $du = \frac{1}{2\sqrt{y}} dy$ we obtain

$$\int \frac{dy}{\sqrt{y} \cos^2 \sqrt{y}} = 2 \int \sec^2 u \, du = 2 \tan u + C = 2 \tan \sqrt{y} + C$$

and

$$2 \tan \sqrt{y} = x + C.$$

Solving for y we get

$$y = \left(\tan^{-1} \left(\frac{x}{2} + C \right) \right)^2$$

The initial condition $y(0) = 1$ leads to

$$\pm 1 = \tan^{-1}(C) \text{ and } C = \pm \tan(1)$$

so we get two solutions.

2. (a) We may consider the function $f(x) = \sqrt[3]{2x+1}$ and compute its limit for $x \rightarrow +\infty$. We first take the logarithm $g(x) = \ln f(x) = \frac{\ln(2x+1)}{3}$ and use l'Hopital's rule. We have

$$\lim_{x \rightarrow +\infty} \frac{2}{2x+1} = 0$$

and therefore $\lim_{x \rightarrow +\infty} g(x) = 0$, i.e. $\lim_{x \rightarrow +\infty} f(x) = 1$. Hence

$$\lim_{n \rightarrow +\infty} \sqrt[3]{2n+1} = 1.$$

- (b) This sequence has no limit since $|a_n| \rightarrow +\infty$ and signs of the numbers a_n alternate.

3. Substitute $w = 3 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ so that $dw = 3 \cos \theta \, d\theta$ and $\sqrt{9-w^2} = 3|\cos \theta| = 3 \cos \theta$. We obtain

$$\int \frac{\sqrt{9-w^2}}{w^2} dw = \int \frac{3 \cos \theta}{9 \sin^2 \theta} \cdot 3 \cos \theta \, d\theta = \int \cot^2 \theta \, d\theta.$$

Note that $\sin^2 \theta + \cos^2 \theta = 1$ implies

$$1 + \cot^2 \theta = \frac{1}{\sin^2 \theta}$$

and that

$$(\cot \theta)' = -\frac{1}{\sin^2 \theta}.$$

Hence

$$\int \cot^2 \theta d\theta = \int (-1 - (\cot \theta)') d\theta = -\theta - \cot \theta + C$$

and

$$\begin{aligned} \int \frac{\sqrt{9-w^2}}{w^2} dw &= -\sin^{-1}\left(\frac{w}{3}\right) - \cot(\sin^{-1}\left(\frac{w}{3}\right)) + C \\ &= -\sin^{-1}\left(\frac{w}{3}\right) - \frac{\sqrt{9-w^2}}{w} + C. \end{aligned}$$

(the simplification in the last step was done with the usual 'triangle drawing trick')

4. We remark first that the integrand is a positive function. So we have to figure out whether the improper integral equals $+\infty$ or whether it remains finite.

Solution with the Limit Comparison Test: Because e^x grows faster than x we choose

$$g(x) = \frac{1}{\sqrt{e^x}} = e^{-x/2}, \quad f(x) = \frac{1}{\sqrt{e^x - x}}.$$

We check whether $f(x)$ and $g(x)$ grow at the same rate:

$$\lim_{x \rightarrow +\infty} \frac{g(x)}{f(x)} = \lim_{x \rightarrow +\infty} \sqrt{\frac{e^x - x}{e^x}} = \lim_{x \rightarrow +\infty} \sqrt{1 - \frac{x}{e^x}} = 1$$

Hence we can use the limit comparison test with the above choice of $g(x)$.

We compute

$$\int_1^\infty g(x) dx = \int_1^\infty e^{-x/2} dx = \lim_{b \rightarrow +\infty} (-2e^{-b/2}) + 2e^{-1/2} = 0 + 2e^{-1/2} = \frac{2}{\sqrt{e}}$$

Hence, by the limit comparison test, the improper integral $\int_1^\infty f(x) dx$ converges.

Solution with the Direct Comparison Test: We note that for $x \geq 1$

$$x \leq \frac{1}{2}e^x \Rightarrow -x \geq -\frac{1}{2}e^x$$

This can be seen as follows: The left hand side $f(x) = x$ satisfies $f(1) = 1$ and it has slope 1. On the other hand, the function $g(x) = \frac{1}{2}e^x$ satisfies $g(1) = \frac{e}{2} > 1$ and also $g'(x) = \frac{e^x}{2} > 1$. So we must have $f(x) \leq g(x)$ for all $x \geq 1$. We estimate now for $x \geq 1$:

$$\sqrt{e^x - x} \geq \sqrt{e^x - \frac{1}{2}e^x} = \sqrt{\frac{1}{2}e^x} = \frac{1}{\sqrt{2}}e^{x/2}$$

and therefore

$$\frac{1}{\sqrt{e^x - x}} \leq \frac{\sqrt{2}}{e^{x/2}} = \sqrt{2}e^{-x/2}.$$

Hence

$$\begin{aligned} \int_1^\infty \frac{1}{\sqrt{e^x - x}} dx &\leq \int_1^\infty \sqrt{2}e^{-x/2} dx \\ &= \sqrt{2} \lim_{b \rightarrow +\infty} \int_1^b e^{-x/2} dx \\ &= 2\sqrt{2} \lim_{b \rightarrow +\infty} (e^{-1/2} - e^{-b/2}) \\ &= 2\sqrt{2}e^{-1/2} < +\infty. \end{aligned}$$

This means that the integral $\int_1^\infty \frac{1}{\sqrt{e^x - x}} dx$ converges.

5. We first compute the integral of the logarithm function using integration by parts:

$$\int \ln x dx = \int 1 \cdot \ln x dx = x \ln x - x + C.$$

Then

$$\begin{aligned} \int_0^1 \ln x dx &= \lim_{a \rightarrow 0} \int_a^1 \ln x dx \\ &= \lim_{a \rightarrow 0} (x \ln x - x) \Big|_a^1 \\ &= -1 - \lim_{a \rightarrow 0} (a \ln a) \\ &= -1 - \lim_{a \rightarrow 0} \frac{\ln a}{\frac{1}{a}} \\ &= -1 - \lim_{a \rightarrow 0} \frac{\frac{1}{a}}{\frac{-1}{a^2}} \\ &= -1 \end{aligned}$$