Fourth Midterm Exam - Practice Test, Math133

Show your work in all the problems and mention convergence tests which you are using.

1. Decide whether the following infinite series converge or diverge. If you encounter a convergent geometric series then compute its limit.

   (a)
   $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$$

   (b)
   $$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^4$$

   (c)
   $$\sum_{n=2}^{\infty} \frac{7}{4^n}$$

   (d)
   $$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

2. Express the number $0.\overline{123} = 0.123123123\ldots$ as the ratio of two integers.

3. Find the Taylor polynomial of order three of the function
   $$f(x) = \sin x \text{ at } a = \frac{\pi}{2}.$$

4. Find the convergence interval of the following power series. You do not need to check the endpoints of the interval.
   $$\sum_{n=0}^{\infty} \frac{\sqrt{n}}{4^n}(x+1)^n$$

5. Draw the curves given in Polar Coordinates by
   $$r = \sin 3\theta \text{ and } r = -2\sin\theta$$
   and find their intersection points.

Solutions

1. (a) We use limit comparison test with

$$a_n = \frac{\sqrt{n}}{n+1}\ ,\ b_n = \frac{1}{\sqrt{n}}.$$

First, we note that $\sum_{n=1}^{\infty} b_n = +\infty$ since it is a divergent p-series with $p = 1/2$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

hence the given series also diverges.

(b) We have

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^4 = 1$$

hence the series diverges by the n-th term test.

(c) This is a convergent geometric series. We write it in the form

$$\begin{aligned} 7\sum_{n=2}^{\infty} \frac{1}{4^n} &= \frac{7}{4}\sum_{n=2}^{\infty} \frac{1}{4^{n-1}} \\ &= \frac{7}{4}\left(\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} - 1\right) \\ &= \frac{7}{4}\left(\frac{1}{1-\frac{1}{4}} - 1\right) \\ &= \frac{7}{12}. \end{aligned}$$

(d) We perform ratio test with

$$a_n = \frac{n!}{10^n}$$

so that

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \frac{10^n}{n!} \\ &= \lim_{n\to\infty} \frac{n+1}{10} \\ &= +\infty. \end{aligned}$$

Hence the series diverges.

2. We compute

$$\begin{aligned} 0.\overline{123} &= \frac{123}{1000} + \frac{123}{1000^2} + \cdots \\ &= \frac{123}{1000}\left(1 + \frac{1}{1000} + \frac{1}{1000^2} + \cdots\right) \\ &= \frac{123}{1000}\frac{1}{1 - \frac{1}{1000}} \\ &= \frac{123}{999}. \end{aligned}$$

3. We compute

$$f'(x) = \cos x\ ,\ f''(x) = -\sin x\ ,\ f'''(x) = -\cos x$$

so that the third Taylor polynomial at $a = \pi/2$ is given by

$$\begin{aligned} & f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{1}{2}f''(\frac{\pi}{2})(x - \frac{\pi}{2})^2 + \frac{1}{6}f'''(\frac{\pi}{2})(x - \frac{\pi}{2})^3 \\ = & \ 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 \\ = & \ 1 - \frac{1}{2}x^2 + \frac{\pi}{2}x - \frac{\pi^2}{2}. \end{aligned}$$

4. Let

$$a_n = \left|\frac{\sqrt{n}}{4^n}(x+1)^n\right| = \frac{\sqrt{n}}{4^n}|x+1|^n$$

so that ratio test yields

$$\frac{a_{n+1}}{a_n} = \frac{1}{4}\sqrt{\frac{n+1}{n}}|x+1| \stackrel{n\to\infty}{\longrightarrow} \frac{|x+1|}{4}.$$

The series converges absolutely if $|x+1| < 4$ which is the same as

$$-4 < x+1 < 4 \text{ i.e. } -5 < x < 3.$$

5. The curves only intersect in the origin. The following picture shows the curves

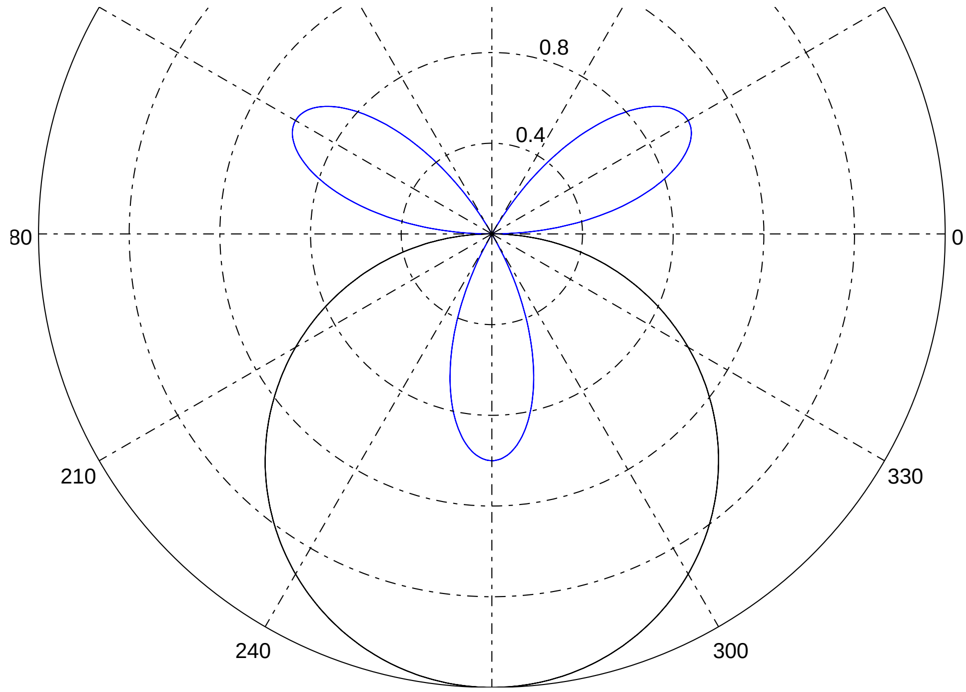

Figure 1: