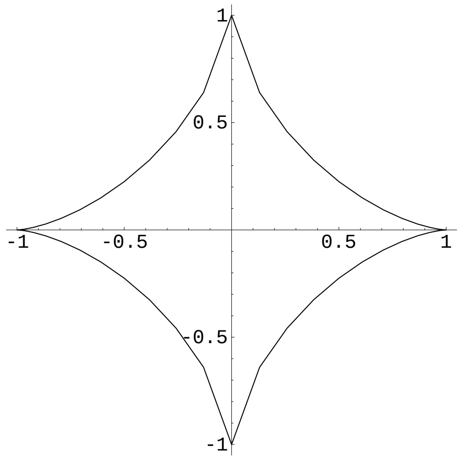
First Practice Exam, Math 133

1. Find the length of the curve given by the equation $x^{2/3} + y^{2/3} = 1$. (Hint: Consider only the part in the first quadrant, then multiply by four.) **(4 points)**



2. A particle moves in a straight line with velocity $v(t) = -t^3 + 3t^2 - 2t$. Find the total distance travelled from time $t = 0$ to time $t = 3$. **(3 points)**

3. A spring has a natural length of 10in. A 800lb force stretches the spring to 14in.

   (a) What is the spring constant ? **(1 point)**

   (b) How much work is done in stretching the spring from 10in. to 12in. ? **(1 point)**

   (c) How far beyond its natural length will a force of 1600lb stretch the spring ? **(1 point)**

4. The arch $y = \sin x$, $0 \leq x \leq \pi$ is revolved about the line $y = c$ $(0 \leq c \leq 1)$ generating a solid.

   (a) What is the volume of this solid ? **(4 points)**

   (b) Find the value c in $[0, 1]$ that minimizes the volume. **(Bonus question,2 points)**

   (Hint: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$. See pictures at the end.)

5. (a) Compute the integral

$$\int \frac{dx}{e^x + 1} \quad \textbf{(2 points)}$$

(Hint: Consider the integral $\int \frac{e^x}{e^x+1}dx$ first, and use it for the given integral)

(b) Compute the derivative of

$$f(x) = x^{\sqrt{x}} \quad (\textbf{2 points})$$

(c) Compute

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^{x/2} \quad (\textbf{2 points})$$

TOTAL: 20 POINTS

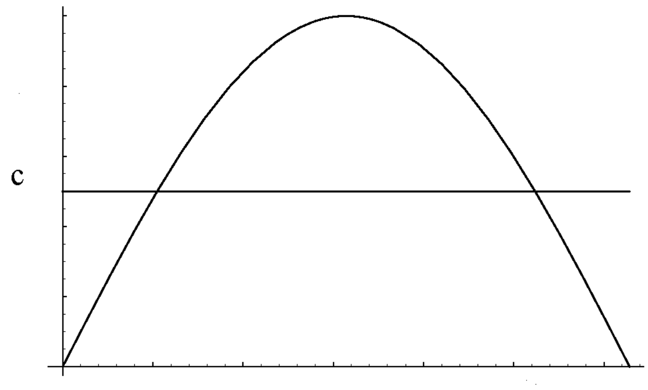


Figure 1: Problem 4–The function and the axis of rotation

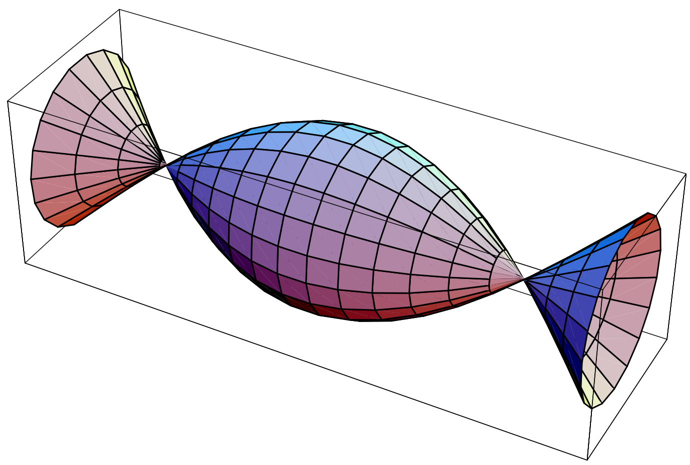


Figure 2: Problem 4–The solid of revolution

Solutions

1. We solve the equation $x^{2/3} + y^{2/3} = 1$ for y and obtain

$$y = (1 - x^{2/3})^{3/2}$$

so that

$$\begin{aligned} y' &= -\frac{2}{3}x^{-1/3} \cdot \frac{3}{2}\sqrt{1 - x^{2/3}} \\ &= -x^{-1/3}\sqrt{1 - x^{2/3}} \end{aligned}$$

and

$$\begin{aligned} \sqrt{1 + (y')^2} &= \sqrt{1 + x^{-2/3}(1 - x^{2/3})} \\ &= \sqrt{x^{-2/3}} \\ &= x^{-1/3}. \end{aligned}$$

The length of the curve in the first quadrant is then given by

$$\begin{aligned} \int_0^1 \sqrt{1 + (y')^2} dx &= \int_0^1 x^{-1/3} dx \\ &= \frac{3}{2}x^{2/3}\Big|_0^1 \\ &= \frac{3}{2} \end{aligned}$$

and the length of the whole curve then equals 6.

2. We first have to figure out where the velocity is positive and where it is negative on the interval $0 \leq t \leq 3$. We solve

$$0 = -t^3 + 3t^2 - 2t = v(t).$$

One solution is $t = 0$. In order to look for nonzero solutions we divide the above equation by t and solve

$$0 = -t^2 + 3t - 2$$

which has the solutions $t = 1$ and $t = 2$. Inserting arbitrary values from the open intervals $(0, 1)$, $(1, 2)$ and $(2, 3)$ we see that

$$v(t) \text{ is } \begin{cases} \text{negative if} & 0 < x < 1 \\ \text{positive if} & 1 < x < 2 \\ \text{negative if} & 2 < x < 3 \end{cases}.$$

The total distance travelled is then given by

$$S = -\int_0^1 v(t)dt + \int_1^2 v(t)dt - \int_2^3 v(t)dt.$$

We have

$$\int v(t)dt = -\frac{t^4}{4} + t^3 - t^2$$

so that

$$\begin{aligned}
S &= \left(\frac{t^4}{4} - t^3 + t^2\right)\Big|_0^1 + \left(-\frac{t^4}{4} + t^3 - t^2\right)\Big|_1^2 + \\
&\quad + \left(\frac{t^4}{4} - t^3 + t^2\right)\Big|_2^3 \\
&= \frac{1}{4} + 0 + \frac{1}{4} + \frac{81}{4} - 27 + 9 - 0 \\
&= \frac{83}{4} - 18 \\
&= \frac{11}{4}.
\end{aligned}$$

3. (a) The spring constant is given by

$$k = 800/4 = 200[lb/in].$$

(b)

$$W = \int_0^2 200x\,dx = 100x^2|_0^2 = 400[lb \cdot in].$$

(c) Using again $F = kx$ we obtain

$$x = F/k = 1600/200 = 8[in],$$

so that the spring is stretched 8 inches beyond its natural length.

4. (a) We move the whole picture down by c units, replacing the sine function with " $\sin x - c$". Hence the volume of the solid is given by

$$\begin{aligned}
V &= \pi \int_0^\pi (\sin x - c)^2 dx \\
&= \pi \int_0^\pi (\sin^2 x - 2c \sin x + c^2) dx \\
&= \frac{\pi}{2} \int_0^\pi (1 - \cos(2x)) dx - \\
&\quad -2c\pi \int_0^\pi \sin x dx + \pi^2 c^2 \\
&= \frac{\pi^2}{2} - \frac{\pi}{2} \int_0^\pi \cos(2x) dx + \\
&\quad +2c\pi \cos x|_0^\pi + \pi^2 c^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi^2}{2} - \frac{\pi}{4}\sin(2x)|_0^\pi + \\
&\quad +2c\pi(-1-1) + \pi^2c^2 \\
&= \frac{\pi^2}{2} - 4c\pi + \pi^2c^2.
\end{aligned}
$$

(b) The volume V depends on the parameter c,

$$V(c) = \frac{\pi^2}{2} - 4c\pi + \pi^2c^2.$$

In order to look for a minimum inside the interval $[0,1]$ we have to solve

$$0 = V'(c) = 2c\pi^2 - 4\pi.$$

This is satisfied for $c = 2/\pi$ which lies between 0 and 1. Since $V''(c) = 2\pi^2 > 0$ the volume of the solid is minimal for $c = 2/\pi$.

5. (a) Substituting $u = e^x + 1$ and $du = e^x\,dx$ we get

$$\int \frac{e^x}{e^x+1}dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C$$

since $u = e^x + 1 > 0$. Then note that

$$\frac{1}{e^x+1} = \frac{1+e^x-e^x}{e^x+1} = \frac{1+e^x}{e^x+1} - \frac{e^x}{e^x+1} = 1 - \frac{e^x}{e^x+1}$$

so that

$$\int \frac{dx}{e^x+1} = \int dx - \int \frac{e^x}{e^x+1}dx = x - \ln(e^x+1) + const.$$

(b) By definition of the exponential function we have

$$f(x) = x^{\sqrt{x}} = e^{\sqrt{x}\ln x}$$

and

$$
\begin{aligned}
f'(x) &= e^{\sqrt{x}\ln x}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right) \\
&= x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

(c) Let

$$f(x) := \left(1 + \frac{1}{x}\right)^{x/2}$$

so that

$$\ln(f(x)) = \frac{x}{2}\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{2}{x}}.$$

We compute the limit using l'Hôpital's rule:

$$\lim_{x\to+\infty} \ln(f(x)) = \lim_{x\to+\infty} \frac{\frac{-1}{x^2}\frac{1}{1+\frac{1}{x}}}{-\frac{2}{x^2}} = \lim_{x\to+\infty} \frac{1}{2+\frac{2}{x}} = \frac{1}{2}$$

so that

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^{x/2} = e^{1/2} = \sqrt{e}.$$