LIMITS AND CONTINUITY FOR FUNCTIONS OF SEVERAL VARIABLES

We begin with a review of the concepts of limits and continuity for real-valued functions of one variable. Recall that the definition of the limit of such functions was something like what follows.

Definition 1. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. Then $\lim_{x\to a} f(x) = L$ means that $|f(x) - L|$ can be made arbitrarily small provided $0 < |x - a|$ is sufficiently small.*

The two fundamental specific limits results that follow easily from the definition are:

$$(1) \text{ If } c \in \mathbb{R}, \text{ then } \lim_{x\to a} c = c \text{ and } (2) \lim_{x\to a} x = a \text{ for any } a \in \mathbb{R}.$$

The basic facts used to compute limits are contained in the following theorem.

Theorem 1. Basic Limit Theorem: *Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then*

1. $\lim_{x\to a} f(x) + g(x) = L + M$
2. $\lim_{x\to a} f(x)g(x) = LM$ *and*
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ *provided $M \neq 0$.*

Moreover, if $\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x)$ and if $f(x) \leq h(x) \leq g(x)$, then $\lim_{x\to a} h(x) = L$. (This assertion is commonly called the Squeeze Theorem.)

From these assertions it was proved that for any rational function R (Recall that a rational function is the quotient of two polynomials.) $\lim_{x\to a} R(x) = R(a)$ for any $a \in D_R$. (Recall that the domain of a rational function is the set of numbers where the denominator is not 0.)

Continuity was defined taking a hint from the above result.

Definition 2. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and let $a \in D$. Then f is continuous at L means $\lim_{x\to a} f(x) = f(a)$.*

By the comment preceding Definition 0.2 all rational functions are continuous at each number in their domains. The same is true of all of the trigonometric functions, the logarithmic functions, the exponential functions and the inverse trigonometric functions.

From the Basic Limit Theorems and the definition of continuity we conclude immediately the Basic Continuity Theorem.

Theorem 2. Basic Continuity Theorem: *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$ and let $a \in D$. Suppose f and g are continuous at a. Then $f + g$ and fg are continuous at a as is $\dfrac{f}{g}$ provided $g(a) \neq 0$.*

Neither Theorem 0.1 nor Theorem 0.2 deal with the most important method of combining two functions; namely, the composition of two functions. The definition of that concept is recalled next.

Definition 3. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $g : E \subset \mathbb{R} \to \mathbb{R}$. Then the composition of f by g is denoted by $g \circ f$ and defined by $g \circ f(x) = g(f(x))$.*

How limits and continuity are related to composition is explained in the following two theorems.

Theorem 3. Limit Composition Theorem *Let f and g be as in Definition 0.3 with $a \in D$ and $L \in E$. Suppose $\lim_{x\to a} f(x) = L$ and suppose g is continuous at L. Then $\lim_{x\to a} g(f(x)) = g(L)$.*

The conclusion of this assertion can also be written as $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))$ which can then be remembered as, "the limit of a continuous function is the continuous function of the limit." An immediate consequence of this theorem is the following corollary.

Theorem 4. Continuity Composition Theorem: *Let f and g be as in Definition 0.3 with $a \in D$ and $f(a) \in E$. Suppose f is continuous at a and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.*

One remembers this assertion as, "The composition of two continuous functions is continuous".

This completes our review of the single variable situation. Now we take up the subjects of Limits and Continuity for real-valued functions of several variables.

Definition 4. *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$, let $P_0 \in \mathbb{R}^n$ and let $L \in \mathbb{R}$. Then $\lim_{P\to P_0} f(P) = L$ means that the distance, $|f(P) - L|$, between $f(P)$ and L can be made arbitrarily small provided the distance, $|P - P_0|$, between P and P_0 is sufficiently small for $P \in D$ and $P \neq P_0$.*

Note that the first use of vertical lines denotes absolute value while the second denotes distance between two points in $\mathbb{R}^n$.

To begin computing limits we first need some specific results similar to those for functions of one variable. The basic principle is that if a function of one variable is considered as a function of more than one variable, then the limit of the function is computed by taking the limit of the function with respect to its only variable. One specific case of this principle is stated below. Other cases are left to the reader's imagination.

Theorem 5. *Let $h : E \subset \mathbb{R} \to \mathbb{R}$ and set $f(x, y) = h(x)$. Suppose $\lim_{x\to a} h(x) = L$. Then $\lim_{(x,y)\to(a,b)} f(x, y) = L$ for any $b \in \mathbb{R}$.*

So for example $\lim_{(x,y)\to(2,9)} x^2 = \lim_{x\to 2} x^2 = 4$ and $\lim_{(x,y)\to(2,9)} \sqrt{y} = \lim_{y\to 9} \sqrt{y} = 3$.

Essentially all examples of functions of several variables we will encounter are constructed from functions of one variable by addition, multiplication, division and composition. So the following Basic Limit Theorem will permit us to compute limits.

Theorem 6. Basic Limit Theorem: *Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$. Suppose $\lim_{P\to P_0} f(P) = L$ and $\lim_{P\to P_0} g(P) = M$. Then*

1. $\lim_{P\to P_0} f(P) + g(P) = L + M$
2. $\lim_{P\to P_0} f(P)g(P) = LM$ *and*
3. $\lim_{P\to P_0} \dfrac{f(P)}{g(P)} = \dfrac{L}{M}$ *provided $M \neq 0$.*

Moreover, if $\lim_{P\to P_0} f(P) = L = \lim_{P\to P_0} g(P)$ and if $f(P) \leq h(P) \leq g(P)$, then $\lim_{P\to P_0} h(P) = L$. (The Squeeze Theorem for functions of several variables.)

Examples: Page 921; 8, 14, 18.

Doing problem 8 and others on Page 921 requires the use of the following assertion analogous to the one variable version above.

Theorem 7. Limit Composition Theorem: *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and let $h : E \subset \mathbb{R} \to \mathbb{R}$. Suppose $\lim_{P\to P_0} f(P) = L$ and suppose h is continuous at L. Then $\lim_{P\to P_0} h(f(P)) = h(L)$.*

A comment similar to the one following Theorem 0.3 applies here.

Additional examples: Page 921; 3, 10.

Having defined the limit concept for functions of several variables, the notion of continuity for such functions is defined in a fashion analogous to the one variable situation.

Definition 5. *Let $f : D \subset \mathbb{R}^n$ and let $P_0 \in D$. Then f is continuous at P_0 means $\lim_{\substack{P\to P_0 \\ P\in D}} f(P) = f(P_0)$.*

Note that contrary to the limit definition, we require that P_0 be in the domain of the function. The purpose is to guarantee that $f(P_0)$ is defined. Also according to definition of limit for functions of several variables, the only values of P that are allowed are those that are in D. We emphasize that here by adding $P \in D$ in the limit statement.

The Basic Limit Theorem and the Limit Composition Theorem yield the next two theorems.

Theorem 8. Basic Continuity Theorem: *Let f, $g : D \subset \mathbb{R}^n \to \mathbb{R}$ and let $P_0 \in D$. Suppose f and g are continuous at P_0. Then $f + g$ and fg are continuous at P_0 as is $\dfrac{f}{g}$ provided $g(P_0) \neq 0$.*

Examples: Page 922: 28, 29(b), 30, 31(a) 33, 34

Theorem 9. Continuity Composition Theorem: *Let $f : D \subset \mathbb{R}^n$, let $P_0 \in D$ and let $h : E \subset \mathbb{R} \to \mathbb{R}$. Suppose f is continuous at P_0 and that h is continuous at $f(P_0)$. Then the function $h \circ f$ defined by $(h \circ f)(P) = h(f(P))$ is continuous at P_0.*

Examples: Page 922; 27, 29(a), 31(b), 32.