SUBMANIFOLDS AND HOMOLOGY OF NONSINGULAR REAL
ALGEBRAIC VARIETIES

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In this paper we study submanifolds of nonsingular real algebraic sets. We study the question of when they can be moved to algebraic sub-
sets. We then give some applications to homology theory.

In Section 2 we discuss the topology of blowing-up, and show that blowing-up turns immersed submanifolds to imbedded submanifolds (Lemma 2.3). Then in Section 3 we generalize a result of Kleiman. By ex-
tending some previous work of Schwarzenberger and Hironaka he showed that after blowing up the top Schubert cycle of the Grassmanian variety one can make the universal bundle contain a line bundle $[K_1]$. We prove some more general results of this flavour which lead to an interesting structure theorem for submanifolds (Theorem 3.4 and Theorem 3.5).

In Section 4 we first give a solution to the problem of when a codimen-
sion one smooth submanifold of a nonsingular algebraic set is isotopic to a nonsingular algebraic subset (Theorem 4.1); we identify the obstructions (transcendental homology cycles). From this we obtain an algebraic trans-
versality result (Proposition 4.3). We then turn to the question of when a map $f: M \to V$ from a smooth manifold to a nonsingular algebraic set can be made algebraic. Making $(M, f)$ algebraic means that making $M$ a non-
singular algebraic set so that the corresponding $f$ will be a polynomial map; this is equivalent to isotoping an imbedded copy of $M$ in $V \times \mathbb{R}^s$ (induced by $f$) to a nonsingular algebraic subset, for some $s$. This question reduces to the question of whether certain homology classes $\theta \in H_k(V; \mathbb{Z}/2\mathbb{Z})$ are algebraic, that is $\theta = g_* [N]$ for some algebraic $(N, g)$. We define an obstruction $\sigma(\theta)$ in terms of the previously defined codimen-
sion one obstructions such that $\sigma(\theta) = 0$ if and only if $\theta$ is algebraic (Theo-
rem 4.4). In Section 5 we give some examples of $\to V$ which cannot be made algebraic. From this we give a method of constructing inequivalent algebraic structures on smooth manifolds. We also discuss a curious ob-

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struction to reducing extra components of algebraic sets arising from Seiferts theorem [S]. This obstruction comes from homotopy groups of spheres. In Section 6 we prove a resolution theorem for $\mathbb{Z}/2\mathbb{Z}$ homology cycles (Theorem 6.2). We show that algebraic blowing up desingularizes homology cycles. We show that $\mathbb{Z}/2\mathbb{Z}$ algebraic cocycles are closed under cohomology operations (cup product, and Steenrod squares.) From these we deduce some amusing corollaries in algebraic topology.

The following are some sample results that are implied by the theorems of this paper. Let $V$ denote any nonsingular algebraic set of dimension $n$, define $AH_{n-1}(V; \mathbb{Z}/2\mathbb{Z})$ to be the subgroup of $H_{n-1}(V; \mathbb{Z}/2\mathbb{Z})$ generated by $(n-1)$-dimensional nonsingular algebraic subsets, and $H^I_{n-1}(V) = H_{n-1}(V; \mathbb{Z}/2\mathbb{Z})/AH_{n-1}(V; \mathbb{Z}/2\mathbb{Z})$ (this group can be nonzero by Theorem B(c)).

**Theorem A.** Suppose $M^{n-1} \subset V^n$ is a codimension one closed smooth submanifold. Then $M$ is isotopic to a nonsingular algebraic subset by an arbitrarily small isotopy if and only if $\alpha(M) = 0$, where $\alpha(M)$ is the image of the fundamental class $[M]$ in $H^I_{n-1}(V)$.

Let $H^A_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ be the subgroup of $H_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ generated by algebraic subsets, and let $H^A_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ be the Poincare dual of $H^A_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ assuming $V$ is compact.

**Theorem B.**

(a) $H^A_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ is closed under cohomology operations if $V$ is compact.

(b) For a given $\theta \in H_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ there exists an obstruction $\sigma(\theta)$ defined in terms of the codimension one obstruction of Theorem A, such that $\theta \in H^A_{\ast}(V; \mathbb{Z}/2\mathbb{Z})$ if and only if $\sigma(\theta) = 0$.

(c) There exist connected nonsingular algebraic sets $V$ of any dimension $n \geq 3$ such that $H_k(V; \mathbb{Z}/2\mathbb{Z}) \neq H^A_{k}(V; \mathbb{Z}/2\mathbb{Z})$, for all $k = 2, 3, \ldots, n - 1$.

(d) There are nonsingular algebraic sets $V$ and vector bundles $E$ over $V$ such that $E$ cannot be made a nonsingular algebraic set containing $V$. Furthermore $E$ can be chosen to be orientable if one wishes.

(e) For any $k$ there is a connected nonsingular algebraic set $V$ and a closed smooth codimension $k$ submanifold $M \subset V$ so that $M$ is not isotopic to a real algebraic set and in fact $(M, i)$ is not bordant to a rational function. (Here $i$ is the inclusion map.)
Call a compact nonsingular real algebraic subset (closed smooth submanifold) \( M \subset V \) a \textit{fine} algebraic subset (submanifold) if \( M \) is a component of a transversally intersecting codimension one compact nonsingular algebraic subsets (closed smooth submanifolds). Now, just as the resolution theorem of [H] turns the singular algebraic sets to nonsingular algebraic sets by blowing up along nonsingular centers, the following resolution theorem turns the nonsingular algebraic sets to fine algebraic sets by blowing along fine centers.

\textbf{Theorem C.} \textit{If} \( M \subset V \) \textit{is a compact nonsingular algebraic subset, then there exists a sequence of blowups} \( \hat{V} \xrightarrow{\pi} V \) \textit{along fine algebraic centers of dimension} \(< \dim(M)\) \textit{such that, the strict preimage} \( \bar{M} \subset \hat{V} \) \textit{of} \( M \) \textit{is a fine algebraic subset.}

Theorem C has the obvious topological version for smooth submanifolds. Since fine submanifolds are generalizations of codimension one submanifolds, Theorem C translates the problem of moving a smooth submanifold of any codimension in \( V \) to an algebraic subset, to the codimension one case of Theorem A; this explains the statement (b) of Theorem B. Fine algebraic sets can be moved around by isotopies (Proposition 4.3), hence Theorem C allows us to do transverse moves after resolution, which in turn implies that algebraic sets can be made transversal up to homology, this explains statements like (a) of Theorem B.

Seifert’s theorem [S] says that if a closed smooth submanifold \( M'' \subset \mathbb{R}'' \) has a trivial normal bundle then it is isotopic to a component of a complete intersection. The following says that there are obstructions in removing the extra components from the conclusion of Seifert’s theorem.

\textbf{Theorem D.} \textit{There exist closed smooth submanifolds} \( M \subset \mathbb{R}'' \) \textit{imbedded with trivial normal bundle such that} \( M \) \textit{cannot be isotopic to a complete intersection in} \( \mathbb{R}'' \).

Theorem A is the same as Theorem 4.1. Theorem B is a consequence of Theorems 6.6, 4.4, 6.9, 6.10 and 5.3. Theorem C is implied by Theorem 3.5. Theorem D is implied by Theorem 5.8.

1. \textbf{Definitions and the background material.}

\textit{Definition.} If \( k \) is a field a \textit{k algebraic set} is a set \( V \) of the form \( V(I) = \{ x \in k'' | p(x) = 0 \text{ for all } p \in I \} \) where \( I \) is a set of polynomial functions from \( k'' \) to \( k \).
In this paper "algebraic set" will always mean "real algebraic set" unless otherwise indicated. If $V$ is an algebraic set, $I(V)$ denotes the ideal of polynomials vanishing on $V$. If $V$ is an algebraic set one can assume that $V = p^{-1}(0)$ for a single polynomial $p$ (take for instance $p$ to be the sum of the squares of the generators of $I(V)$).

**Definition.** Let $V \subset \mathbb{R}^n$ be an algebraic set, a point $x \in V$ is called nonsingular of dimension $d$ in $V$ if there are polynomials $p_i \in I(V)$, $i = 1, \ldots, n - d$ and a neighborhood $U$ of $x$ in $\mathbb{R}^n$ with

(i) $V \cap U = U \cap \bigcap_{i=1}^{n-d} p_i^{-1}(0)$

(ii) $\nabla p_i(x), i = 1, \ldots, n - d$ are linearly independent for $x \in U$.

**Definition.** Let $V$ be an algebraic set, $\dim(V) = \max\{d \mid \text{there exists } x \in V \text{ which is nonsingular of dimension } d \}$ $\text{Nonsing}(V) = \{x \in V \mid x \text{ is nonsingular of dimension } \dim(V)\}$ $\text{Sing}(V) = V - \text{Nonsing}(V)$. We say $V$ is nonsingular if $\text{Sing}(V) = \emptyset$.

**Definition.** An algebraic set $V \subset \mathbb{R}^n$ is a complete intersection if there are polynomials $p_i, i = 1, \ldots, n - d$ with

(i) $V = \bigcap_{i=1}^{n-d} p_i^{-1}(0)$

(ii) $\nabla p_i(x), i = 1, \ldots, n - d$ are linearly independent for all $x \in V$.

In particular a complete intersection $V \subset \mathbb{R}^n$ is nonsingular, and it has a trivial normal bundle in $\mathbb{R}^n$ (a trivialization of the normal bundle is given by the vectors $\nabla p_i(x), i = 1, \ldots, n - d$).

**Definition.** Let $A \subset \mathbb{R}^n$, then the Zariski closure of $A$ is the smallest algebraic set containing $A$.

**Definition.** Given algebraic sets $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ we say a function $f: V \to W$ is an entire rational function if $f(x) = p(x)/q(x)$ where $p: \mathbb{R}^n \to \mathbb{R}^m$, $q: \mathbb{R}^n \to \mathbb{R}$ are polynomials, and $q$ doesn't vanish on $V$. We call a diffeomorphism $f: V \to W$ a birational diffeomorphism if $f$ and $f^{-1}$ are entire rational functions.

Let $Y$ be a $k$-dimensional algebraic set. $Y$ can be triangulated; then by Sullivan's local Euler characteristic condition [Su] the sum of $k$-dimensional simplexes $[Y]$ is a cycle in $H_k(Y; \mathbb{Z}/2\mathbb{Z})$ which we call the fundamental cycle. When $Y$ is connected then $H_k(Y; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $[Y]$ is the
generator. Hence every \(k\)-dimensional algebraic subset \(Y\) of an algebraic set \(V\) gives rise to a natural homology class \([Y] \in H_k(V; \mathbb{Z}/2\mathbb{Z})\) induced by the fundamental class.

**Definition.** Let \(M, L, N\) be smooth manifolds with \(L \subset N\), and let \(f: M \to N\) be a smooth map. We say \(f\) hits \(L\) cleanly if \(f^{-1}(L)\) is a smooth submanifold of \(M\) and if for every \(x \in f^{-1}(L)\) \(df_x: T_x(M)/T_x(f^{-1}(L)) \to T_y(N)/T_y(L)\) is injective where \(y = f(x)\). In other words, \(df\) injects the normal bundle of \(f^{-1}(L)\) into the normal bundle of \(L\). For instance if \(f\) is transverse to \(L\) then it hits \(L\) cleanly. We call an immersion \(f: N \to M\) a clean immersion if \(f\) hits the smooth strata of \(f(N)\) cleanly. (More precisely, there is an open cover \(\{U_\alpha\}\) of \(N\) so that \(f\) imbeds each \(U_\alpha\) and \(f\) restricted to \(U_\alpha\) hits any intersection of \(f(U_\beta)\)’s cleanly.)

**Lemma 1.1.** Let \(f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k\) be a smooth function so that \(f(\mathbb{R}^n \times 0) = 0\). Then there is a smooth map \(A: \mathbb{R}^n \times \mathbb{R}^m \to \{k \times m\text{-real matrices}\}\) so that \(f(x, y) = A_{x,y}(y)\) for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\).

**Proof.**

\[
\begin{align*}
  f(x, y) &= f(x, y) - f(x, 0) = \int_0^1 \frac{d}{dt} f(x, ty) dt \\
  &= \int_0^1 \sum_{i=1}^m \frac{\partial f}{\partial y_i} (x, ty) y_i dt = \sum_{i=1}^m y_i \int_0^1 \frac{\partial f}{\partial y_i} (x, ty) dt 
\end{align*}
\]

So by letting the \(i\)-th column of \(A_{x,y}\) be \(\int_0^1 (\partial f/\partial y_i)(x, ty) dt\) we are done. \(\square\)

**Definition.** Let \(M \subset V\) be smooth manifolds. We say \(M\) is a complete submanifold if \(M\) is the transverse intersection \(\bigcap_{i=1}^r W_i\) of closed (i.e. compact without boundary) smooth codimension one submanifolds of \(V\). We say \(M\) is a fine submanifold if it is a component of a complete submanifold. If in addition \(M, V\) and each \(W_i\) are nonsingular algebraic sets then we call \(M\) a complete algebraic subset or a fine algebraic subset instead of a complete submanifold or a fine submanifold. The normal bundle of any fine submanifold is a direct sum of line bundles. Any complete submanifold of \(\mathbb{R}^n\) is a complete intersection by [S].

**Remark 1.2.** Any closed smooth submanifold which has a trivial normal bundle is a fine submanifold. Also the natural imbeddings \(\mathbb{R} P^k \subset\)
\( \mathbb{R}P^n \) are examples of complete algebraic subsets. Not all fine submanifolds of \( \mathbb{R}^n \) are complete (see the example at the end of Section 5).

**Definition.** A nonsingular algebraic subset \( M \) of a nonsingular algebraic set \( V \) is called a stable algebraic subset if there are compact nonsingular algebraic sets \( \{ V_i \}_{i=1}^r \) with \( M = V_0 \subset V_1 \subset \cdots \subset V_r \subset V_{r+1} = V \) and \( \dim(V_{i+1}) = \dim(V_i) + 1 \) for \( i = 0, 1, \ldots, r \). Clearly every fine algebraic subset is a stable algebraic subset. If \( M \) has codimension 0 or 1 then \( M \) is automatically stable. The importance of stable algebraic sets is that they obey transversality (Proposition 4.3).

In this paper \( Cl \) denotes closure, and \( \epsilon \)-isotopy means an arbitrarily small isotopy.

2. **Blowing up.** Let \( V \) be an algebraic set and \( L \) be an algebraic subset of \( V \). Let \( B(V, L) \) denote the algebraic set obtained by blowing up \( V \) along \( L \). This is a common process in algebraic geometry. It is done as follows. Pick a polynomial map \( f:(V, L) \to (\mathbb{R}^n, 0) \) such that the coordinates of \( f \) generate \( I(L)/I(V) \) then \( B(V, L) \) is the Zariski closure of

\[
\{(x, \theta f(x)) \in V \times \mathbb{R}P^{n-1} | x \in V - L \}
\]

in \( V \times \mathbb{R}P^{n-1} \), where \( \theta: \mathbb{R}^n - \{0\} \to \mathbb{R}P^{n-1} \) is the quotient map \( \theta(x_1, \ldots, x_n) = [x_1: \cdots : x_n] \). There is the obvious projection \( \pi:B(V, L) \to V \), \( \pi \) is called the blowing up map (or projection). \( L \) is called the center of the blowup. If \( M \) is an algebraic subset of \( V \) with \( L \subset M \subset V \), then there is a natural inclusion \( B(M, L) \subset B(V, L) \), namely the Zariski closure of \( \pi^{-1}(M) - \pi^{-1}(L) \) in \( B(V, L) \).

If \( V \) is a smooth manifold and \( L \) is a codimension \( k \) smooth submanifold of \( V \), then we can define the topological blowup \( B_t(V, L) \) of \( V \) along \( L \) by

\( B_t(V, L) = (V - \text{interior } N) \cup E(N) \)

where \( N \) is a normal disc bundle of \( L \) in \( V \), and \( E(N) \) is obtained from \( N \) by replacing each fibre \( D^k \) of \( N \) by \( \mathbb{R}P^k \)-interior \( D^k \). \( E(N) \) is also an \( I \)-bundle over the projectivized normal bundle \( P(L, V) \) of \( v(L, V) \). Recall \( P(L, V) \) is the \( \mathbb{R}P^{k-1} \)-bundle over \( L \) obtained from \( v(L, V) \) by projectivizing each fibre \( \mathbb{R}^k \) to \( \mathbb{R}P^{k-1} \). There is a map \( \pi_t:B_t(V, L) \to V \) which is the identity on \( V \)-interior \( N \), and crushes \( P(L, V) \) fibrewise onto \( L \). In particular \( \pi_t^{-1}(L) \)
\( = P(L, V) \). If furthermore \( V, L \) are nonsingular algebraic sets, it is a standard fact that there is a diffeomorphism \( B_t(V, L) \approx B(V, L) \) which makes the following commute

\[
\begin{array}{ccc}
B(V, L) & \xrightarrow{\pi} & V \\
\parallel & & \downarrow \pi_t \\
B_t(V, L) & \xrightarrow{\pi_t} & \\
\end{array}
\]

Hence in this case we can omit the subscript \( t \). From now on \( B(V, L) \xrightarrow{\pi} V \) will denote \( B_t(V, L) \xrightarrow{\pi_t} V \) if \( L \subseteq V \) are smooth manifolds; and \( B(V, L) \xrightarrow{\pi} V \) will be called the topological blowup of \( V \) along \( L \). If \( L \subseteq V \) are nonsingular algebraic sets the usual algebraic blowup \( B(V, L) \xrightarrow{\pi} V \) will be referred as the algebraic blowup of \( V \) along \( L \). There is a natural equivalence \( B(V, L) \times N = B(V \times N, L \times N) \) for smooth manifolds (or algebraic sets) \( V, L, N \).

If \( L \subseteq V \) are smooth manifolds and \( M \) is a closed subset with \( L \subseteq M \subseteq V \), we define \( B(M, L) \) to be the closure of \( \pi^{-1}(M) - \pi^{-1}(L) \) in \( B(V, L) \) where \( \pi: B(V, L) \rightarrow V \) is the topological blowup. In case \( M \) is a smooth manifold then this definition coincides with the usual topological blowup of \( M \) along \( L \). In particular any inclusion of smooth manifolds \( L \subseteq M \subseteq V \) gives rise to a natural inclusion \( B(M, L) \subseteq B(V, L) \). This copy of \( B(M, L) \) in \( B(V, L) \) is called the strict preimage of \( M \) under \( \pi \). In particular \( \pi^{-1}(L) \cap B(M, L) = P(L, M) \). The following lemma makes these facts precise.

\textbf{Lemma 2.1.} Let \( M, L \) and \( N \) be smooth manifolds, with \( L \subseteq N \) and let \( f:M \rightarrow N \) be a smooth map which hits \( L \) cleanly. Then \( f \) induces a unique smooth map \( f': B(M, f^{-1}(L)) \rightarrow B(N, L) \) so that the following diagram commutes

\[
\begin{array}{ccc}
B(M, f^{-1}(L)) & \xrightarrow{f'} & B(N, L) \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{f} & N \\
\end{array}
\]
Proof. It suffices to prove existence and uniqueness locally. So let \( M = \mathbb{R}^m \times \mathbb{R}^k, f^{-1}(L) = \mathbb{R}^m \times 0, N = \mathbb{R}^n \times \mathbb{R}^p, L = \mathbb{R}^n \times 0, f(x, y) = (f_1(x, y), f_2(x, y)) \in \mathbb{R}^n \times \mathbb{R}^p \). Since \( f_2(\mathbb{R}^m \times 0) = 0 \). Lemma 1.1 implies that there is a \((p \times k)\)-matrix valued smooth function from \( \mathbb{R}^m \times \mathbb{R}^k \), \((x, y) \rightarrow A_{x,y} \) so that \( f_2(x, y) = A_{x,y}(y) \) for all \((x, y) \in \mathbb{R}^m \times \mathbb{R}^k \). The requirement that \( f \) hits \( L \) cleanly implies that \( \ker(A_{x,0}) = 0 \) for all \( x \in \mathbb{R}^m \).

\[
B(M, f^{-1}(L)) = \mathbb{R}^m \times (\mathbb{R}^p - pt)
= \{(x, [t, y])|x \in \mathbb{R}^m, t \in \mathbb{R}, y \in \mathbb{R}^k - 0\},
\]

\[
B(N, L) = \mathbb{R}^n \times (\mathbb{R}^p - pt)
= \{(u, [s, v])|u \in \mathbb{R}^n, s \in \mathbb{R}, v \in \mathbb{R}^p - 0\}
\]

and the projections to \( M \) and \( N \) are given by \((x, [t, y]) \rightarrow (x, t \theta(y))\) and \((u, [s, v]) \rightarrow (u, s \theta(v))\) respectively, where \( \theta(y) = y/|y|^2 \). In order to make our diagram commute, if \( f'(x, [t, y]) = (u, [s, v]) \) we must have

\[
f(x, t \theta(y)) = (f_1(x, t \theta(y)), A_{x,t \theta(y)}(t \theta(y)))
= (u, s \theta(v))
\]

A little calculation shows that

\[
u = f_1(x, t \theta(y))
[s, v] = \left[ |A_{x,t \theta(y)}(y)|^2 \left( \frac{t}{|y|^2} \right), A_{x,t \theta(y)}(y) \right]
\]

so we have uniqueness. We also have existence as long as we can show that \( A_{x,t \theta(y)}(y) \neq 0 \) when \( y \neq 0 \). But if \( A_{x,t \theta(y)}(y) = 0 \) then \( f_2(x, t \theta(y)) = A_{x,t \theta(y)}(t \theta(y)) = 0 \) hence \( t \theta(y) = 0 \) so \( t = 0 \). But \( A_{x,0}(y) \neq 0 \) when \( y \neq 0 \) as we wish. Hence \( f' \) exists and is unique.

Lemma 2.2. Let \( L, M, N, V \) be smooth manifolds, with \( M, N \subset V \) intersecting cleanly and \( L \subset M \cap N \). Then \( B(M, L) \) and \( B(N, L) \) intersect cleanly and \( B(M, L) \cap B(N, L) = B(M \cap N, L) \). If \( M \) and \( N \) are transverse, then \( B(M, L) \) and \( B(N, L) \) are transverse also.
**Proof.** Let \( \pi : B(V, L) \to V \) be the blowing up projection. Then

\[
B(M \cap N, L) = \text{closure } \pi^{-1}(M \cap N - L)
\]

\[
\subset \text{closure } \pi^{-1}(M - L) \cap \text{closure } \pi^{-1}(N - L)
\]

\[
= B(M, L) \cap B(N, L).
\]

But if \( x \in B(M, L) \cap B(N, L) - B(M \cap N, L) \) then \( \pi(x) \in M \cap N \) so \( x \) is a line tangent to \( M \cap N \) in \( P(V, L) \) (since \( TM \cap TN = T(M \cap N) \) by cleanliness) but this is a contradiction since if \( x \) is tangent to \( M \cap N \) it is in \( B(M \cap N, L) \). Hence \( B(M, L) \cap B(N, L) = B(M \cap N, L) \).

Cleanliness or transversality of \( B(M, L) \) can be seen by local arguments similar to those of Lemma 4.2 of [AK3].

**Definition.** Let \( V \) be a smooth manifold and \( M \subset V \) a smooth stratified subset. We call \( \tilde{V} \xrightarrow{\pi} V \) a multiblowup of \( V \) along \( M \) if \( \pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_k \) for some \( k \) where

\[
\tilde{V} = V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} V_1 \xrightarrow{\pi_0} V_0 = V
\]

such that:

(1) \( V_{i+1} = B(V_i, L_i), M_{i+1} = B(M_i, L_i), M_0 = M \) and \( L_i \) is a closed smooth submanifold of a smooth stratum of \( M_i \) with \( \dim L_i < \dim M \), \( \pi_i \) are the blowing up projections.

(2) \( M_k \) is a smooth imbedded submanifold of \( V_k \).

We usually denote \( M_k \) by \( \tilde{M} \) and call it the strict preimage of \( M \). \( L_i, i = 0, \ldots, k - 1 \) are called the centers of the multiblowup \( \tilde{V} \xrightarrow{\pi} V \). If in addition \( V \) is a nonsingular algebraic set and each \( V_{i+1} \to V_i \) is obtained by blowing up the nonsingular algebraic set \( V_i \) along a nonsingular algebraic subset \( L_i \) where \( i = 0, \ldots, k - 1 \); then we say \( \tilde{V} \xrightarrow{\pi} V \) is an algebraic multiblowup of \( V \). A multiblowup \( \tilde{V} \xrightarrow{\pi} V \) of \( V \) along \( M \) is called a complete (fine) multiblowup along \( M \) if all the submanifolds \( \tilde{M} \subset V_k, L_i \subset V_i \) for \( i = 0, \ldots, k - 1 \) are complete (fine) submanifolds. A complete (fine) algebraic multiblowup \( \tilde{V} \xrightarrow{\pi} V \) along \( M \) is an algebraic multiblowup with each \( \tilde{M} \subset \tilde{V}, L_i \subset V_i \) complete (fine) algebraic subsets. The following is a natural application of multiblowups.

**Lemma 2.3.** Let \( V \) be a smooth manifold and \( M \xrightarrow{\xi} V \) be a clean immersion of a closed smooth manifold \( M \). Then there exists a multiblow-
up \( \tilde{V} \xrightarrow{\pi} V \) of \( V \) along \( f(M) \) so that the strict preimage \( \tilde{M} \) of \( f(M) \) is an imbedded smooth submanifold of \( \tilde{V} \).

**Proof.** For \( y \in f(M) \) define

\[
n(y) = \max \{ n \mid \text{there exist } n \text{ distinct points } x_1, \ldots, x_n \in M \text{ with } f(x_i) = y \text{ for } i = 1, \ldots, n \}
\]

\[= \text{number of points in } f^{-1}(y)\]

Since \( f \) is a clean immersion of a closed smooth manifold, \( f(M) \) is a stratified set \( \bigcup M_i \). Each stratum \( M_i \) is given by \( M_i = \{ y \in f(M) \mid n(y) = i \} \). Let \( d_f = \max \{ i \mid M_i \neq \emptyset \} \) then \( M_{d_f} \) is a closed smooth manifold (of perhaps nonconstant dimension).

Denote \( M' = M_{d_f}, M'' = f^{-1}(M') \). Then \( f: M'' \to M' \) is a \( d_f \)-fold covering projection. Note that \( f \) hits \( M' \) cleanly so by Lemma 2.1 we have a unique map \( f' \) making the following diagram commutative

\[
\begin{array}{ccc}
B(M, M'') & \xrightarrow{f'} & B(V, M') \\
\downarrow\pi & & \downarrow\pi \\
M & \xrightarrow{f} & V
\end{array}
\]

It is immediate that \( f' \) is an immersion. Hence if we show that \( f' \) is clean and \( f'\^{-1}(y) \) has less than \( d_f \) points for all \( y \) we will be done by induction. The above conditions are all local in \( V \) and follow immediately from Lemma 2.2.
2.4. For any imbedding \( f: M \hookrightarrow V \) a multiblowup \( \tilde{V} \xrightarrow{\pi} V \) along \( f(M) \) factors in a natural way

\[
\begin{array}{c}
\tilde{M} \xrightarrow{i} \tilde{V} \\
\downarrow \quad \downarrow \pi \\
M \xrightarrow{f} V
\end{array}
\]

where \( \tilde{M} \to M \) is some (induced) multiblowup. This follows from repeated applications of Lemma 2.1 as in the proof of Lemma 2.3.

Another useful property of algebraic multiblowups is that they project algebraic sets to algebraic sets. More specifically if \( V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} V_0 \) is an algebraic multiblowup of a nonsingular algebraic set \( V \) and \( Z \) is an algebraic subset of \( V_k \) then \( p_k(Z) \cup \bigcup_{i=0}^k p_i(L_i) \) is an algebraic subset of \( V \) where \( p_i = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_i \) and \( p_0 = id \). This follows from the repeated applications of the following lemma.

**Lemma 2.5.** Let \( V \) be a nonsingular algebraic set and \( L \) be a nonsingular algebraic subset of \( V \). Then for any algebraic subset \( Z \) of \( B(V, L) \), \( \pi(Z) \cup L \) is an algebraic set.

**Proof.** Let \( f_1, \ldots, f_r \) be a set of generators of \( I(L) \) then \( B(V, L) = \text{Zariski closure of } Q \) where \( Q = \{ (x, [f_1(x) : \cdots : f_r(x)]) \in V \times \mathbb{R}P^{r-1} | x \in L \} \).
Let \( \varphi(x, u_1, \ldots, u_r) = 0 \) be a polynomial equation describing \( Z \) as an algebraic set in \( B(V, L) \subset V \times \mathbb{R}P^{r-1} \), \( \varphi \) is homogeneous in the variables \( u_1, \ldots, u_r \) (so \( Z = \varphi^{-1}(0) \cap B(V, L) \)). Write \( Z = Z_1 \cup Z_2 \) with \( Z_1 = Z \cap Q \) and \( Z_2 = Z - Z_1 = Z \cap (B(V, L) - Q) \). Then \( \pi(Z_2) \subseteq \pi(B(V, L) - Q) = L \), hence \( \pi(Z) \cup L = \pi(Z_1) \cup \pi(Z_2) \cup L = \pi(Z_1) \cup L = \pi\{x, [f_1(x) : \cdots : f_r(x)] \in V \times \mathbb{R}P^{r-1} | \varphi(x, f_1(x), \ldots, f_r(x)) = 0, x \in V - L\} \cup L = \{x \in V - L | \varphi(x, f_1(x), \ldots, f_r(x)) = 0\} \cup L \). Since \( u_1 = u_2 = \cdots = u_r = 0 \) implies \( \varphi(x, u_1, \ldots, u_r) = 0 \), \( \pi(Z) \cup L \) is the set of solutions of the polynomial equation \( \varphi(x, f_1(x), \ldots, f_r(x)) = 0 \). \( \square \)

For a further discussion of the topology of blowups the reader can consult [AK3].

### 3. Splitting trivial bundles by blowing up.

**Lemma 3.1.** Let \( V \) be a smooth manifold and \( M \subset V \) a smooth closed submanifold. If \( M' \subset V \) is isotopic to \( M \) by a small isotopy and \( M \) is transverse to \( M' \) then the normal bundle of \( B(M, M \cap M') \) in \( B(V, M \cap M') \) splits a trivial line bundle.

**Proof.** Let \( L = M' \cap M \). Put a Riemannian metric on \( V \). Define \( \alpha : M \times [0, 1] \to V \) by letting \( \alpha : x \times [0, 1] \to V \) be the shortest geodesic from \( x \) to the closest point to \( x \) in \( M' \). Note \( \alpha^{-1}(L) = L \times [0, 1] \) and \( \alpha \) hits \( L \) cleanly.

Hence by Lemma 2.1, \( \alpha \) induces a unique map \( \alpha' : B(M, L) \times [0, 1] \to \)}
$B(V, L)$ (observe that the left hand side is $B(M \times [0, 1], L \times [0, 1])$) so that the following commutes:

$$
\begin{array}{ccc}
B(M, L) \times [0, 1] & \xrightarrow{\alpha'} & B(V, L) \\
\downarrow_{\pi \times id} & & \downarrow_{\pi} \\
M \times [0, 1] & \xrightarrow{\alpha} & V
\end{array}
$$

Note $\alpha': B(M, L) \times 0 \to B(V, L)$ is an embedding onto $B(M, L)$. If we show that $\alpha': B(M, L) \times [0, 1] \to B(V, L)$ is an embedding we will be done.

A moment's reflection shows it suffices to prove that $\alpha'$ is an imbedding in the case $V = \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k, M = \mathbb{R}^n \times \mathbb{R}^k \times 0, L = \mathbb{R}^n \times 0 \times 0$ and geodesics perpendicular to $M$ are straight lines. Then $\alpha(x, y, s) = (x, y, s \varphi(x, y))$ for some function $\varphi: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$. The proof of Lemma 2.1 shows that

$$\alpha'(x, [t:y], s) = (x, [t\beta(x, t, y, s):y:sB_{x,t\beta(y)}(y)])$$

for some smooth $\beta: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R} - 0$ and some $(k \times k)$-matrix valued function $B_{x,y}$ so that $\varphi(x, y) = B_{x,y}(y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$. Now the facts $M'$ is transverse to $M$ and $M'$ is the graph of $\varphi$ imply that the Jacobian of $\varphi: x \times \mathbb{R}^k \to \mathbb{R}^k$ is nonsingular for each $x \in \mathbb{R}^n$. But $B_{x,0}$ equals this Jacobian so $B_{x,0}$ is nonsingular for each $x \in \mathbb{R}^n$. This easily implies that $\alpha'$ is an imbedding since the Jacobian of the map

$$\mathbb{R}^n \times (\mathbb{R} \times \mathbb{R}^k - 0) \times \mathbb{R} \to \mathbb{R}^n \times (\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k - 0)$$

induced by $\alpha'$ is the following nonsingular matrix when $t = 0$

$$
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & \beta & 0 & 0 \\
0 & 0 & I & 0 \\
* & * & * & B_{x,0}(y)
\end{pmatrix}
$$

Lemma 3.2. Let $M, M'$ and $V$ be as in Lemma 3.1 and $Q \subset L = M \cap M'$ be any closed smooth submanifold. Let $L_1 = B(L, Q), M_1 = \ldots$
$B(M, Q), M'_1 = B(M', Q)$ and $V_1 = B(V, Q)$. Then the normal bundle of $B(M_1, L_1)$ in $B(V_1, L_1)$ still splits a trivial line bundle.

**Proof.** We need to show that $M_1, M'_1$ are closely isotopic and $M_1$ is transverse to $M'_1$. That $M_1$ is transverse to $M'_1$ is easy to see. To see that $M_1$ and $M'_1$ are closely isotopic, let $\alpha : M \times I \to V$ be a small isotopy of $M$ to $M'$ rel $L = M \cap M'$ where $I = [0, 1]$. Now $\alpha$ hits $Q$ cleanly so by Lemma 2.1 we have a unique map $\alpha'$ making the following diagram commutative

$$
\begin{array}{ccc}
B(M, Q) \times I & \overset{\alpha'}{\longrightarrow} & B(V, Q) \\
\downarrow{\pi \times id} & & \downarrow{\pi} \\
M \times I & \overset{\alpha}{\longrightarrow} & V
\end{array}
$$

Note that $\alpha'$ is an isotopy from $B(M, Q)$ to $B(M', Q)$ rel $B(L, Q)$ so we are done. \(\square\)

A successive application of the proof of Lemma 3.2 gives us the following.

**Lemma 3.3.** Let $M, M'$ and $V$ be as in Lemma 3.1 and $L = M \cap M'$. Let $\tilde{V} \overset{\pi}{\rightarrow} V$ be a multiblowup of $V$ along $L$ and let $\tilde{M}$, and $\tilde{L}$ be the strict preimages of $M$, and $L$ respectively. Then the normal bundle of $B(\tilde{M}, \tilde{L})$ in $B(\tilde{V}, \tilde{L})$ still splits a trivial line bundle.

One can also give a universal proof to Lemma 3.1. We show that by blowing up the Grassmannian variety along a certain Schubert cycle we can split a trivial line bundle from the universal bundle. A nice aspect of this proof is that one can specifically write down universal equations for the split line bundle. The origin of this idea goes back to $[K_1]$.

Let $\eta_n = \{n \times n = \text{symmetric matrices}\} \approx \mathbb{R}^{(1/2)n(n+1)}$. Recall

$$
G_{n,k} = \{P \in \eta_n | P^2 = P, \text{trace } P = k\}
$$

$$
E_{n,k} = \{(P, x) \in G_{n,k} \times \mathbb{R}^n | Px = x\}
$$

$G_{n,k}$ is identified with the space of $k$-planes in $\mathbb{R}^n$ (Grassmannian). That is every $P \in G_{n,k}$ uniquely determines the plane image $(P)$, and every $k$-plane in $\mathbb{R}^n$ determines such a matrix $P$ namely the projection matrix onto that plane. Note $E_{n,k} \overset{p}{\rightarrow} G_{n,k}$ is the universal $\mathbb{R}^k$-bundle defined by $p(P, x) = P$. There is the natural imbedding $G_{n,k} \subset E_{n,k}$ in which the normal bundle corresponds to the universal bundle.
Now let $M = G_{n,k}$, $V = E_{n,k}$. Let $c \in \mathbb{R}^n - \{0\}$ be some fixed vector. Define $M' \subset E_{n,k}$ by

$$M' = \{(P, x) \in E_{n,k} \mid P(c) = x\}$$

Then $M'$ is just a copy of $M$ in $V$, obtained by moving $M$ by an isotopy. Then

$$L = M \cap M' = \{P \in G_{n,k} \mid P(c) = 0\}$$

= all $k$-planes perpendicular to $c \approx G_{n-1,k}$

$L$ is the *First Schubert Cycle*, it represents the dual of the top Steifel-Whitney class. This $L$ is also a nonsingular algebraic set. Now $I(L)/I(V)$ is generated by polynomials

$$\begin{align*}
(P, x) & \mapsto x_i \\
(P, x) & \mapsto P_i \cdot c
\end{align*}$$

where $P_i$ is the $i$-th row of $P$. Hence

$$B(V, L) = \text{Zariski closure of } \{(P, x, [x : P(c)]) \mid (P, x) \in V - L\}$$

= \{(P, x, [y : z]) \mid (P, x) \in V, P(y) = y, P(z) = z\}

$$|P(c)|^2 z = P(c)\langle c, z \rangle, |P(c)|^2 y = \langle c, z \rangle \cdot x,$$

$$|x|^2 y = x\langle x, y \rangle, |x|^2 z = P(c)\langle x, y \rangle$$

To see the last equality consider the projection

$$B(V, L) \xrightarrow{\pi} V$$

$$(P, x, [y : z]) \sim (P, x)$$

$\pi$ is an isomorphism above $V - L$, because if $(P, x) \in V - L$ then only $y = \alpha x$, $z = \alpha P(c)$ for some $\alpha$ solves the above equations. If $(P, x) \in L$ then
\[
\pi^{-1}(P, x) = \{(P, 0, [y:z])| P(y) = y, P(z) = z \}
\]

\[
\approx \mathbb{P}^{2k-1}.
\]

Likewise \(B(M, L) \hookrightarrow B(V, L)\) is given by \(B(M, L) = \{(P, 0, [0:z]) \in B(V, L)\}\). We have the specific imbedding \(\alpha : B(M, L) \times \mathbb{R} \to B(V, L)\) given by \(\alpha(P, 0, [0:z], t) = (P, P(tc), [tz:z])\) which gives the splitting of a trivial line bundle off the normal bundle of \(B(M, L)\) in \(B(V, L)\).

By putting together these results we get the following full generalized form of Lemma 3.1.

**Theorem 3.4.** Let \(M, V\) be smooth manifolds and \(M\) be closed, and let \(f : M \to V\) be a clean immersion. Then there is a fine multiblowup of \(V\) along \(f(M)\). In particular the strict preimage of \(f(M)\) is an imbedded fine submanifold.

**Proof.** Recall the stratification \(\bigcup_{i=1}^{d_f} M_i\) of \(f(M)\) discussed in Lemma 2.3, \(M_i = \{ y \in f(M) | n(y) = i \}\) where \(n(y)\) is the number of points in \(f^{-1}(y)\) and \(d_f = \max \{ i | M_i \neq \emptyset \}\). \(M_{d_f}\) is a smooth manifold. We proceed by induction, consider the following statement:

\(\mathcal{C}(m, d)\). For any manifolds \(M^m, V^n\) as above and a clean immersion \(M^m \to V^n\), there is a fine multiblowup of \(V\) along \(f(M)\) provided that \(\dim(M) = m, d = d_f\).

**Claim.** \(\mathcal{C}(m, 1)\) holds

**Proof of claim.** \(\mathcal{C}(0, 1)\) holds trivially. Assume by induction \(\mathcal{C}(m', 1)\) holds for \(m' < m\). Let \(M^m \subset V^n\) be an imbedding. Define a complexity \(C(M)\) of this imbedding to be the smallest integer \(r\) such that there is a complete submanifold \(W^r \subset V\) of dimension \(r\) with \(M^m \subset W^r \subset V\). It suffices to prove that if \(m < r\) then there is a multiblowup \(\tilde{V} \to V\) along \(M\) with fine centers such that \(C(\tilde{M}) \leq r - 1\) where \(\tilde{M}\) is the strict preimage of \(M\). If \(m < r\) by Lemma 3.1 there is a smooth manifold \(L^{2m-r} \subset M^m\) such that the normal bundle of \(B(M, L)\) in \(B(W, L)\) splits a trivial line bundle. Since \(\dim(L) < m\), by induction \(V\) admits a fine multiblowup \(V_1 \to V\) along \(L\); let \(L_1, M_1, W_1\) be the strict preimages of \(L, M, W\). By Lemma 3.3 the normal bundle of \(B(M_1, L_1)\) in \(B(W_1, L_1)\) continues to split a trivial line bundle. Let \(E\) be the total space of this line bundle; let \(h : E \to B(M_1, L_1)\).
$L_1) \times \mathbb{R} \to \mathbb{R}$ be a trivialization map. $h$ is transversal to 0 and $h^{-1}(0) = B(M_1, L_1)$. By extending $h$ over $B(V_1, L_1)$ we get a codimension one closed smooth submanifold $Z^{r-1} = h^{-1}(0) \subset B(V_1, L_1)$. Let $W_1^{r-1}$ be the transverse intersection $Z^{r-1} \cap B(W_1, L_1)$. Since $W$ is a complete submanifold so are $W_1$ and $B(W_1, L_1)$ (Lemma 2.2). Hence $W_1^{r-1}$ is a complete submanifold containing $B(M_1, L_1)$. Therefore $C(B(M_1, L_1)) \leq r - 1$ we are done.

The statement $C(m, d), d > 1$ will not be used in the paper we prove it for completeness.

Claim. $C(m, d)$ holds

By induction assume $C(m, d')$ for all $d' < d$. Let $f: M \to V$ be a clean immersion with $d_f = d$. Let $M' = M_{d'}$, $M'' = f^{-1}(M_{d'})$. By induction, i.e. $C(m, 1)$ the imbedding $M' \subset V$ admits a fine multiblowup $V_1 \xrightarrow{\pi_1} V$. Hence $f$ induces a clean immersion $f_1: M_1 \to V_1$ (as in the proof of Lemma 2.3) where $M_1$ is a multiblowup of $M$ along $M''$ such that the following commutes

\[
\begin{array}{ccc}
M_1 & \xleftarrow{f_1} & V_1 \\
\downarrow{\pi_1} & & \downarrow{\pi_1} \\
M & \xleftarrow{f} & V
\end{array}
\]

By inspection $d_{f_1} \leq d$. Let $M'_1 \subset V_1$ be the strict preimage of $M'$. Let $f_1^{-1}(M'_1) = M''_1$. Then by Lemma 2.1 there exists a unique map $f'_1$ which makes the following commute

\[
\begin{array}{ccc}
B(M_1, M''_1) & \xrightarrow{f'_1} & B(V_1, M'_1) \\
\downarrow{\pi} & & \downarrow{\pi} \\
M_1 & \xrightarrow{f_1} & V_1
\end{array}
\]

$f'_1$ is a clean immersion and has the property $d_{f'_1} < d$ (recall the proof of Lemma 2.3). Then by induction $B(V_1, M'_1)$ admits a fine multiblowup along $\text{image}(f'_1) V_2 \to B(V_1, M'_1)$. Then the fine multiblowup

\[
V_2 \to B(V_1, M'_1) \to V_1 \to V
\]

is a fine multiblowup of $V$ along $f(M)$.
The following is the algebraic version of Theorem 3.4.

**Theorem 3.5.** Suppose $V$ is a nonsingular algebraic set and $f: M \hookrightarrow V$ is an imbedding onto a compact nonsingular algebraic subset of $V$. Then there is a fine algebraic multiblowup of $V$ along $f(M)$. In particular the strict preimage of $f(M)$ is a fine algebraic subset.

*Proof.* We prove this by modifying the proof of $C(m, 1)$ of Theorem 3.4. First recall the notations of Theorem 3.4. It suffices to prove the following statement by induction.

$A(m, r)$. Any imbedding $M \hookrightarrow V$ as above admits an algebraic multiblowup so that $\tilde{M}$ is a fine algebraic subset of $\tilde{V}$ provided $\dim(\tilde{M}) = m$, $\dim(V) - \dim(M) = r$.

$A(0, r)$ and $A(m, 1)$ hold trivially. Assume $A(m', r')$ holds if $m' + r' < r + m$ or $r' + m' = r + m$ and $m' < m$. We show that $A(m, r)$ holds: By (Lemma 2.7 of [AK$_2$], or [AK$_1$]) there is an entire rational function $g: M \to G_{v, r}$ which is the classifying map of the normal bundle of $M$ in $V$. Then by Lemma 3.6 below we can choose the first Schubert cycle $\mathcal{L} \subset G_{v, r}$ such that $g$ is transversal to $\mathcal{L}$. We can take $L$ to be the nonsingular algebraic set $g^{-1}(\mathcal{L})$ and proceed exactly as in the proof of $C(m, 1)$. Observe that now we can take all the blowups to be algebraic blowups. Also by approximating $h$ by a polynomial fixing the nonsingular algebraic set $B(M_1, L_1)$ (see Lemma 2.1 of [AK$_1$]) we can assume $Z$ is a nonsingular algebraic set. Then the proof follows by induction as in the proof of $C(m, 1)$. \[\Box\]

Recall for $c \in \mathbb{R}^v - 0$, $L_c = \{P \in G_{v, r}|P(c) = 0\}$ is the first Schubert cycle and it represents the dual of the $r$-th Steifel-Whitney class. The following lemma says that for a given map to $G_{v, r}$ one can choose $c$ so that $L_c$ is transversal to this map. We are grateful to N. Goldstein for pointing out this lemma to us.

**Lemma 3.6.** (Kleiman [K$_2$]). Let $f: M \to G_{v, r}$ be an entire rational map from a nonsingular algebraic set. Then there exists $c_0 \in \mathbb{R}^v$ so that $f$ is transversal to $L_{c_0}$.

*Proof.* $O(v)$ acts transitively on $G_{v, r}$ since $G_{v, r} = O(v)/O(r) \times O(v - r)$. Pick $c \in \mathbb{R}^v - 0$ let $L_c$ be the corresponding first Schubert cycle in $G_{v, r}$ then let $\varphi: M \times O(v) \to G_{v, r}$ be the map $\varphi(x, A) = Af(x)$, $\varphi$ is onto (since $O(v)$ acts transitively) and $\varphi$ is a submersion (check). Then consider the smooth manifold $Z = \varphi^{-1}(L_c) = \{(x, A) \in M \times O(v)|Af(x) \in L_c\}$. 

Let $\pi : Z \to O(v)$ be the map induced by projection. By Sard’s theorem we can find a regular value $A_0 \in O(v)$ of $\pi$, then $\pi^{-1}(A_0)$ is a smooth manifold. In fact this means that $f$ is transversal to $A_0^{-1}L_c = L_{c_0}$, $c_0 = A_0^{-1}(c)$ and $\pi^{-1}(A_0) \approx f^{-1}(L_{c_0})$ is a nonsingular algebraic subset of $M$. □

4. Moving submanifolds to algebraic subsets. Let $V$ be a nonsingular real algebraic set of dimension $n$. Let $AH_k(V; \mathbb{Z}/2\mathbb{Z})$ be the subgroup of $H_k(V; \mathbb{Z}/2\mathbb{Z})$ generated by $k$-dimensional stable algebraic subsets of $V$. Define

$$H'_k(V) = H_k(V; \mathbb{Z}/2\mathbb{Z})/AH_k(V; \mathbb{Z}/2\mathbb{Z})$$

In particular, we call $H'_{n-1}(V)$ the group of codimension one transcendental cycles. Generally $H'_{n-1}(V)$ can be nontrivial (see Remark 5.5). For any codimension one closed smooth submanifold $M \subset V$ we associate $\alpha(M) \in H'_{n-1}(V)$ where $\alpha(M)$ is the image of the fundamental class $[M]$ under the quotient map $H_{n-1}(V; \mathbb{Z}/2\mathbb{Z}) \to H'_{n-1}(V)$. The following theorem says that $\alpha(M)$ is the only obstruction to moving $M$ to a nonsingular algebraic subset of $V$.

**Theorem 4.1.** Let $V^n$ be a nonsingular real algebraic set. Then a codimension one closed smooth submanifold $M \subset V$ is isotopic to a nonsingular real algebraic subset of $V$ by an arbitrarily small isotopy if and only if $\alpha(M) = 0$.

**Proof.** Proof in one direction is trivial. Assume $\alpha(M) = 0$ then $[M] \in AH'_{n-1}(V; \mathbb{Z}/2\mathbb{Z})$, so we can pick nonsingular codimension one algebraic subsets $V_1, \ldots, V_k$ so that $[M] = \Sigma_{i=1}^k [V_i]$ as homology classes. By induction on $k$ we may assume that $V_i$ are in general position (first isotop the $V_i$’s to $M_i$’s in general position, by the result with $k = 1$ we may $\epsilon$-isotop each $M_i$ to a nonsingular algebraic set $V_i'$ then the $V_i'$’s are in general position. Note for $k = 1$ we need not do this step). After an $\epsilon$-isotopy of $M$ we may assume $M$ is in general position with $V_i$’s also. By Lemma 1.4 of [AK5], $M \cup \bigcup_{i=1}^k V_i$ separates $V$ into two regions $N$ and $P$ with $N$ compact (so $V = N \cup P$ and $N \cap P = M \cup \bigcup_{i=1}^k V_i$ also $N = Cl \text{Int } N$ and $P = Cl \text{Int } P$).

We claim that there is a smooth function $f : V \to \mathbb{R}$ such that $N = f^{-1}(-\infty, 0]$ and $P = f^{-1}[0, \infty)$ and for each $x \in M \cup \bigcup_{i=1}^k V_i$ there is a coordinate chart $\varphi : (\mathbb{R}^n, 0) \to (V, x)$ and a smooth function $\theta : \mathbb{R}^n \to \mathbb{R} - \{0\}$ with $f \circ \varphi(x_1, \ldots, x_n) = (\Pi_{i=1}^b x_i)\theta(x_1, \ldots, x_n)$ for some $b$. (Necessarily $b$ is the number of $M$ or $V_i$’s in which $x$ is contained since $f^{-1}(0) = M \cup \bigcup V_i$). We may also assume that outside of some compactum $f = q$ for some proper polynomial $q : (V, \bigcup_{i=1}^k V_i) \to (\mathbb{R}, 0)$. 


Given these claims, we may approximate \( f \) by a rational function \( p : (V, \cup V_i) \to (\mathbb{R}, 0) \) (see [AK_1] Lemma 2.1 for example). We claim that \( Cl(p^{-1}(0) - \cup V_i) \) is a nonsingular real algebraic set which is \( \epsilon \)-isotopic to \( M \).

First we prove that \( Cl(p^{-1}(0) - \cup V_i) \) is a nonsingular real algebraic set. Pick any \( x \in M \cap \cup V_i \). After renumbering, assume that \( x \in \cap_{i=1}^b V_i - \cup_{i=b+1}^k V_i \). Pick polynomials \( r_i : (V, V_i) \to (\mathbb{R}, 0) \), \( i = 1, \ldots, b \) so that \( (dr_i)_x \neq 0 \). By Lemma 2.10 of [M] there is a rational function \( s_1 : V - U_1 \to \mathbb{R} \) such that \( p \vert_{V-U_1} = r_1 \cdot s_1 \) where \( U_1 \) is some algebraic set with \( x \notin U_1 \). Notice that \( \cup V_i \subset p^{-1}(0) \) and \( r_1^{-1}(0) \) is just \( V_1 \) near \( x \) and \( V_i's \) are irreducible, so \( s_1 \) must vanish on \( \cup_{i=2}^b V_i \). Hence \( s_1 = r_2 \cdot s_2 \) for some rational function \( s_2 \) on \( V - U_2 \) where \( U_2 \) is some algebraic set with \( x \notin U_2 \). By repeating this process we find a rational function \( s : V - U \to \mathbb{R} \) where \( U \) is a some algebraic set with \( x \notin U \) and \( p \vert_{V-U} = s \cdot r_1 \cdot r_2 \cdots r_b \). Let \( \varphi(\mathbb{R}^n, 0) \to (V, x) \) be local coordinates and let \( \theta: \mathbb{R}^n \to \mathbb{R} - \{0\} \) with \( f \circ \varphi(x) = (\Pi_{i=1}^{b+1} x_i) \theta(x) \), \( \varphi^{-1}(V_i) = \{ x \vert x_i = 0 \} \), \( i = 1, \ldots, b \) and \( \varphi^{-1}(M) = \{ x \vert x_{b+1} = 0 \} \), \( x = (x_1, \ldots, x_n) \). Then we must have \( r_i \circ \varphi(x) = x_i \cdot \psi_i(x) \) for some smooth function \( \psi_i: \mathbb{R}^n \to \mathbb{R} - \{0\} \), hence \( f \circ \varphi(x) = (\Pi_{i=1}^{b+1} r_i \circ \varphi(x)) \cdot x_{b+1} \cdot \theta'(x) \) for some \( \theta' : \mathbb{R}^n \to \mathbb{R} - \{0\} \). Since \( p \) approximates \( f \), \( p \circ \varphi = (s \circ \varphi) \cdot \Pi r_i \circ \varphi \) approximates \( (x_{b+1} \cdot \theta') \Pi r_i \circ \varphi = f \circ \varphi \) so \( s \circ \varphi \) approximates \( x_{b+1} \cdot \theta' \). In particular \( ds_x \neq 0 \) and \( s^{-1}(0) \) is a manifold close to \( M \) near \( x \) (in the sense that the closest point map from \( s^{-1}(0) \) to \( M \) is a diffeomorphism near \( x \)). Note that near \( x \), \( s^{-1}(0) \) is \( Cl(p^{-1}(0) - \cup V_i) \). Thus we have shown above that if \( X \) is the Zariski closure of \( p^{-1}(0) - \cup V_i \) then \( Cl(p^{-1}(0) - \cup V_i) \subset \text{Nonsing} \ X \). Also we may isotop \( M \) to \( Cl(p^{-1}(0) - \cup V_i) \) along geodesics connecting points in \( Cl(p^{-1}(0) - \cup V_i) \) to the closest point in \( M \).

We claim \( Cl(p^{-1}(0) - \cup V_i) = X \). If this is not true then let \( x \in X' - Cl(p^{-1}(0) - \cup V_i) \) where \( X' \) is an irreducible component of \( X \). Then we must have \( x \in \cup V_i \) so after renumbering assume \( x \in \cap_{i=1}^b V_i - \cup_{i=b+1}^k V_i \). Pick any \( y \in X' - \cup V_i \). Then we may pick polynomials \( r_i : (V, V_i) \to (\mathbb{R}, 0) \) \( i = 1, \ldots, b \) so that \( (dr_i)_x \neq 0 \) and \( r_i(y) \neq 0 \). Then as above we have a rational function \( s : V - U \to \mathbb{R} \) for some algebraic set \( U \) with \( x \notin U \) and \( p \vert_{V-U} = s \cdot r_1 \cdots r_b \) and \( s(x) \neq 0 \). Then \( p^{-1}(0) \cup U = \cup_{i=1}^b r_i^{-1}(0) \cup s^{-1}(0) \cup U \). Now a neighborhood of \( y \) in \( \text{Nonsing} \ X \) is in \( p^{-1}(0) - \cup_{i=1}^b r_i^{-1}(0) \) hence \( X' \subset s^{-1}(0) \cup U \). But \( x \notin s^{-1}(0) \cup U \) contradiction. So \( X = Cl(p^{-1}(0) - \cup V_i) \).

It remains to demonstrate the existence of the function \( f \). First, it suffices to construct \( f \) locally and piece together with a partition of unity. To
see this take charts \( \varphi_i: (\mathbb{R}^n, 0) \to (V, x) \) and smooth functions \( \theta_i: \mathbb{R}^n \to \mathbb{R} - \{0\} \) and \( f_i: \text{Im}(\varphi_i) \to \mathbb{R} \) and \( \psi_i: \text{Im}(\varphi_i) \to [0, 1] \) so that \( f_i \circ \varphi_i(x_1, \ldots, x_n) = (\prod_{i=1}^b x_i) \theta_i(x_1, \ldots, x_n) \), \( f_i^{-1}(-\infty, 0] = N \cap \text{Im}(\varphi_i) \) and \( f_i^{-1}[0, \infty) = P \cap \text{Im}(\varphi_i) \) \( i = 0, 1 \). Also assume that \( \psi_0 + \psi_1 > 0 \) on \( U = \text{Im}(\varphi_0) \cap \text{Im}(\varphi_1) \). After renumbering we may suppose that \( \varphi_0^{-1}\circ \varphi_1(x_1, \ldots, x_n) \) is \( x_i \cdot \eta_i(x_1, \ldots, x_n) \). But then

\[
\psi_0 f_0 + \psi_1 f_1 \varphi_1(x_1, \ldots, x_n) = (\prod_{i=1}^b x_i) \theta(x_1, \ldots, x_n)
\]

Hence \((\psi_0 f_0 + \psi_1 f_1)\varphi_1(x_1, \ldots, x_n) = (\prod_{i=1}^b x_i) \theta(x_1, \ldots, x_n)\) for some \( \theta: \mathbb{R}^n \to \mathbb{R} - \{0\} \). So we can glue local \( f \)'s together. It remains to construct \( f \) locally. For this just take a local coordinate chart \( \varphi: (\mathbb{R}^n, 0) \to (V, x) \) so that \( \varphi^{-1}(M \cup \bigcup_{i=1}^b V_i) = \bigcup_{i=1}^b \{ x_i = 0 \} \) and let \( f \circ \varphi(x_1, \ldots, x_n) = \pm \prod_{i=1}^b x_i \) where \( \pm \) sign is determined so that \( f^{-1}(-\infty, 0] = N \cap \text{Im}(\varphi) \). \( \Box \)

**Remark 4.2.** If in the statement of Theorem 4.1 in addition we have a nonsingular algebraic set \( L \) with \( L \subset M \). Then \( M \) can be isotoped to a nonsingular algebraic subset of \( V \) fixing \( L \) (by a small isotopy) if and only if \( \alpha(M) = 0 \). The proof of this is similar to that of (Lemma 2.2 [AK1]), i.e. we just pick the rational function \( p \) in the above proof so that \( p(L) = 0 \) and then proceed exactly the same.

A useful corollary to Theorem 4.1 is that any codimension one nonsingular algebraic subset can be moved around by isotopies (first move it by any isotopy to a smooth manifold, then \( \epsilon \)-isotop this manifold to a nonsingular algebraic subset by using the theorem). The following is a useful generalization of this fact.

**Proposition 4.3.** (Algebraictransversality). Let \( V \) be a nonsingular algebraic set and \( M \subset V \) be a stable algebraic subset. Let \( N \) be a smooth subcomplex of \( V \) (i.e. a subcomplex of some \( C^\infty \) triangulation of
Then there exists an arbitrarily small smooth isotopy $f_i: M \to V$ with $f_0(M) = M$ and $f_1(M)$ a stable algebraic subset of $V$ transverse to $N$.

Likewise, if $g: W \to V$ is a smooth map from a smooth manifold then there exists an arbitrarily small smooth isotopy $f_i: M \to V$ so that $f_0 = \text{identity}$ and $f_1(M)$ is a stable algebraic subset of $V$ transverse to $g$.

**Proof.** Recall $M$ stable means that there are compact nonsingular algebraic sets $\{V_i\}_{i=1}^r$ with $M = V_0 \subset V_1 \subset \cdots \subset V_r \subset V_{r+1} = V$ and $\dim(V_{i+1}) = \dim(V_i) + 1$.

Let $\mathcal{F}_r$ be the set of all nonsingular algebraic sets $\{V_i\}_{i=0}^r$ with $V_0 \subset V_1 \subset \cdots \subset V_{r+1}, \dim(V_{i+1}) = \dim(V_i) + 1$ and each $V_i$ is compact for $0 \leq i \leq r$. Consider the following statement where $\emptyset$ means transverse to $C_r$: For any $\{V_i\}_{i=0}^r \in \mathcal{F}_r$ and a smooth subcomplex $N \subset V_{r+1}$, there exists an arbitrarily small ambient isotopy of $V_{r+1}$ taking $\{V_i\}$ to nonsingular algebraic sets $\{V_i\}$ such that $V'_i \emptyset N$ for all $i$ (in particular $V'_{r+1} = V_{r+1}$).

$C_0$ holds by smooth transversality and Theorem 4.1. It suffices to show inductively that $C_r$ for all $r' < r = C_r$. Now assume $C_{r-1}$ and pick $\gamma \in \mathcal{F}_r$ where $\gamma = \{V_i\}_{i=0}^r$ and $N \subset V_{r+1}$ is a smooth subcomplex.

First suppose that $V_i \emptyset N$. Then we may apply $C_{r-1}$ to $\{V_i\}_{i=0}^r$ and perform a small isotopy to $\{V'_i\}_{i=0}^r$ so that $V_r = V'_r$ and $V_i' \emptyset (N \cap V_r)$. But $V_r \emptyset N$ and $V_i' \emptyset (N \cap V_r)$ implies that $V_i' \emptyset N$.

So it suffices to show that we may isotop so that $V_i'$ is transverse to $N$. Suppose now that $(V_r - U) \emptyset N$ where $U$ is a small neighborhood of an algebraic set $Z \subset V_r$. Here small means that the distance of each point of $U$ to $Z$ is small compared to how much we are willing to allow $f_i$ to move. We will show that we may perform a small isotopy of $\{V_i\}_{i=0}^r$ to $\{V'_i\}_{i=0}^r$ so that $(V'_r - U') \emptyset N$ where $U'$ is a small neighborhood of an algebraic set $Z' \subset V'_r$ and either $U'$ is empty or $\dim Z' < \dim Z$. Since we may start out with $Z = U = V_r$, this proves the theorem since eventually $U'$ is empty i.e. $V'_r \emptyset N$.

By $C_{r-1}$ applied to $V_0 \subset V_1 \subset \cdots \subset V_r$ we may find a small isotopy of the $V_i$'s to algebraic sets $V_i'$ so that $V_i' \emptyset (N \cap V_r)$ and $V_i' \emptyset Z$ in $V_r$. Notice now that $V'_r$ is transverse to $N$ in $V_{r+1}$ except on a small neighborhood of $V'_r \cap Z$. Now by Remark 4.2 we may perform a small isotopy of $V_r$, fixing $V_{r-1}$, to an algebraic set $V'_r$ so that $V'_r$ is transverse to $N$ except on a small neighborhood of $V'_{r-1} \cap Z$.

Let $Z' = V'_{r-1} \cap Z$. Then since $V'_{r-1} \emptyset Z$, $\dim Z' < \dim Z$. In addition, if $\dim Z = 0$, i.e. $Z$ is a finite set of points, $V'_{r-1}$ will miss a small neighborhood of $Z$ entirely, so $V'_r \emptyset N$. 

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The proof of transversality to the map $g$ is similar to the above.

**Definition.** For a given map $f: M \to V$ from a closed smooth manifold to a nonsingular algebraic set, we say $(M, f)$ is *algebraic* if there is a nonsingular algebraic set $M'$, a polynomial map $f'$, and a diffeomorphism $\varphi$ making the following commute

\[
\begin{array}{ccc}
M & \xrightarrow{f} & V \\
\downarrow & \searrow & \downarrow \\
M' & \xrightarrow{\varphi} & V
\end{array}
\]

Clearly $(M, f)$ is algebraic if and only if there is a nonsingular algebraic subset $Z \subset V \times \mathbb{R}^s$ for some $s$ and a diffeomorphism $\psi$ making the following commute

\[
\begin{array}{ccc}
M & \xrightarrow{f} & V \\
\downarrow & \searrow & \downarrow \\
Z & \xrightarrow{\psi} & V \times \mathbb{R}^s
\end{array}
\]

where $\pi$ is the projection. This is because the graph of $f'$ is a nonsingular algebraic subset of $V \times \mathbb{R}^s$ and diffeomorphic to $M$ (for some $s$).

Recall by Steenrod representibility theorem [T] the map $\eta_*(V) \xrightarrow{ev} H_k(V; \mathbb{Z}/2\mathbb{Z})$ is onto where $\eta_*(V)$ is the unoriented smooth bordism group of $V$ and $ev(M \xrightarrow{f} V) = f_\# [M]$. We say $\theta \in H_k(V; \mathbb{Z}/2\mathbb{Z})$ is *algebraic* if $ev[M, f] = \theta$ for some algebraic $M \xrightarrow{f} V$. Let $f: M \to V$ be any map from a closed smooth manifold to a nonsingular algebraic set. It is well known that $f \times 0$ can be approximated by an imbedding $F: M \hookrightarrow V \times \mathbb{R}^s$ with $F(M)$ a nonsingular algebraic subset (for some $s$) if and only if the bordism class $[M, f]$ in $\eta_*(V)$ contains an algebraic representative (see Propositions 0.2 and 2.3 of [AK3]). Recall $\eta_*(V) \cong H_*(V; \mathbb{Z}/2\mathbb{Z}) \otimes \eta_*(\text{point})$. Let $[M, f] = \Sigma \theta_i(f) \otimes U_j(f)$ then $[M, f]$ contains an algebraic representative if each $\theta_i(f) \in H_i(V; \mathbb{Z}/2\mathbb{Z})$ is algebraic (see Lemma 2.5 of [AK1] or Lemma 2.8 of [AK2]). Let $H^A_k(V, \mathbb{Z}/2\mathbb{Z})$ be the subgroup of $H_k(V, \mathbb{Z}/2\mathbb{Z})$ generated by algebraic elements, i.e. it is generated by $g_\# [N]$ with $N \xrightarrow{g} V$ algebraic. Hence $(M, f)$ is algebraic after a small isotopy of $f$ if each $\theta_\#(f) \in H^A_*(V, \mathbb{Z}/2\mathbb{Z})$.

For a given $\theta \in H_k(V, \mathbb{Z}/2\mathbb{Z})$ by using Theorem 3.4 one can write down obstructions $\sigma(\theta)$ in terms of codimension one obstructions (Theorem 4.1) such that $\sigma(\theta) = 0$ if and only if $\theta \in H^A_k(V, \mathbb{Z}/2\mathbb{Z})$. For a given fine sub-
manifold $M' \subset V'$ of a nonsingular algebraic set define $\sigma(M) = 0$ if $M$ is isotopic to a fine algebraic subset otherwise let $\sigma(M) = 1$. Recall $M''$ is a component of $W''$ where $W''$ is the transverse intersection $\bigcap_{i=1} W_i$ of codimension one closed smooth submanifolds. Clearly if $\alpha(W_i) = 0$ in $H_{v-1}^i(V)$ for all $i$ then $W$ is a complete algebraic subset (Theorem 4.1) and if also $\alpha(M) = 0$ in $H^m_m(W'')$ then $M$ is a fine algebraic subset (it is a component of $W$). In general for a given imbedding $f:M \hookrightarrow V$ we let $\mathcal{F}(M, f)$ be the set of all fine multiblowups $\tilde{V} \xrightarrow{\pi} V$ along $f(M)$ (see Section 2). $\mathcal{F}(M, f) \neq \emptyset$ by Theorem 3.4. Let $(\tilde{V}, \pi) \in \mathcal{F}(M, f)$, i.e.

$$\tilde{V} = V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} V_0 = V$$

$V_i = B(V_{i-1}, L_{i-1})$ where $L_i \subset V_i$ and $\tilde{M} \subset V_k$ are all fine submanifolds for all $i = 0, 1, \ldots, k - 1$. Make the convention that $L_k = \tilde{M}$. Then by the previous discussion if $\sigma(L_0) = 0$ then $L_0$ can be isotoped to a fine algebraic subset of $V$ by a small isotopy. Hence $V_1 \rightarrow V$ can be taken to be an algebraic blowup of $V$ along $L_0$, so this makes $V_1$ a nonsingular algebraic set and therefore $\sigma(L_1)$ is defined. By iterating this process we get: if $\sigma(L_0) = \sigma(L_1) = \cdots = \sigma(L_{i-1}) = 0$ then $V_i \rightarrow V_{i-1} \rightarrow \cdots \rightarrow V_0$ is an algebraic multiblowup along fine algebraic centers, and $\sigma(L_i)$ is defined. Let

$$\sigma(\tilde{V}, \pi) = \inf\{k - n \mid \sigma(L_i) = 0 \text{ for } i \leq n\}$$

Then clearly $\sigma(\tilde{V}, \pi) = 0$ if and only if $\tilde{V} \xrightarrow{\pi} V$ is a fine algebraic multiblowup along $f(M)$. Now let

$$\sigma(f) = \inf\{\sigma(\tilde{V}, \pi) \mid (\tilde{V}, \pi) \in \mathcal{F}(M, f)\}$$

For a given $f:M \rightarrow V$ which is not necessarily an imbedding define

$$\sigma(f) = \inf\left\{\sigma(f') \mid f':M \hookrightarrow V \times \mathbb{R}^s \text{ is an imbedding which is homotopic to } f \times 0 \text{ for some } s\right\}$$

For $\theta \in H_k(V; \mathbb{Z}/2\mathbb{Z})$ let

$$\sigma(\theta) = \inf\{\sigma(f) \mid [M, f] \in \eta_*^*(V), f_*[M] = \theta\}$$

**Theorem 4.4.** Let $\theta \in H_k(V; \mathbb{Z}/2\mathbb{Z})$, then $\sigma(\theta) = 0$ if and only if $\theta$ is algebraic.
Proof. If \( \sigma(\theta) = 0 \) then there is \([M, f]\) with \( f_*[M] = \theta \) and a fine algebraic multiblowup \( \overline{V} \times \mathbb{R}^s \to V \times \mathbb{R}^s \) along \( f'(M) \); then \( (\tilde{M}, p \circ \pi) \) is algebraic and \( ev(\tilde{M}, p \circ \pi) = \theta \) where \( \tilde{M} \) is the strict preimage of \( f'(M) \) and \( p \) is the projection \( V \times \mathbb{R}^s \to V \) (since \( \pi: \tilde{M} \to M \) is degree 1). Conversely if \( \theta \) is algebraic then \( \theta \) has an algebraic representative \((M, f)\) with \( ev(M, f) = \theta \), and by the above remarks \( f \times 0 \) is homotopic to an embedding \( f': M \hookrightarrow V \times \mathbb{R}^s \) onto a nonsingular algebraic subset for some \( s \). By Theorem 3.5 there is a fine algebraic multiblowup \( \overline{V} \times \mathbb{R}^s \to V \times \mathbb{R}^s \) along \( f'(M) \). Hence \( \sigma(\overline{V} \times \mathbb{R}^s, \pi) = 0 \), i.e. \( \sigma(\theta) = 0 \). \( \square \)

Remark 4.5. Even though \( \sigma(\theta) \) is not practical to compute it is interesting to note that it is a codimension 1 obstruction measuring whether duals of first cohomology of certain associated algebraic sets are transcendental.

Remark 4.6. Given an imbedding \( f': M \hookrightarrow V \times \mathbb{R}^s \) as above, i.e. \( f' \) is homotopic to \( f \times 0 \) and \( ev(M, f) = \theta \). It is interesting to note that if a fine multiblowup along \( f'(M) \) is a complete multiblowup (a topological condition) and \( H^n_{n-1}(V) = 0, n = \dim(V) \) then \( \theta \) is algebraic (exercise).

5. Obstructions. Here we give examples of maps \( f: M \to V \) from closed smooth manifolds to nonsingular algebraic sets such that \( (M, f) \) cannot be made algebraic by an isotopy of \( f \), i.e. we realize the obstructions discussed in Section 4. As an application we show that this gives rise to many smooth manifolds carrying inequivalent algebraic structures (Remark 5.6). We conclude this section by a discussion of an amusing obstruction (arising from homotopy groups of spheres) to reducing number of components of some algebraic sets. We first start with a generalized version of a lemma of Tognoli and Benedetti.

Lemma 5.1. Let \( V \) be an irreducible real algebraic set, let \( f: V \to \mathbb{R}^n \) be an entire rational function, with the property that \( \chi(f^{-1}(x)) \) is odd for a dense set of points \( x \in f(V) \). Then \( \dim(f(V) - f(V)) < \dim f(V) \) where \( f(V) \) is the Zariski closure of \( f(V) \), and \( \chi \) denotes Euler characteristic.

Proof. This follows immediately from Lemma 5.2 below by setting \( W = f(V) \), since \( \delta(f) = 1 \) and we must have \( f(V) - f(V) \subset X \). \( \square \)

Lemma 5.2. Let \( V \) and \( W \) be real algebraic sets and let \( f: V \to W \) be an entire rational function. Assume \( W \) is irreducible. Then there is a \( \delta(f) = 0 \) or \( 1 \) and a real algebraic set \( X \subset W \) so that \( \dim X < \dim W \) and so that for each \( z \in W - X \), the Euler characteristic of \( f^{-1}(z) \) is congruent to \( \delta(f) \) mod 2. We call \( \delta(f) \) the degree of \( f \).
Proof. By taking the graph of \( f \) we may as well assume that \( V \subset W \times R^k \) and \( f \) is induced by projection. Suppose \( W \subset R^n \). Let \( W' \) and \( V' \) be the Zariski closures of \( W \) and \( V \) in \( CP^n \) and \( CP^n \times CP^k \), respectively. In particular, \( W' \) and \( V' \) are complex algebraic sets. Take a Whitney stratification of \( V' \) so that each \( i \)-skeleton \( V'_i \) is a complex algebraic subset of \( V' \) and so \( V' \cap CP^n \times C^k \) is a union of strata. Let \( \dim W = \dim W' = m \). For each \( i \) let \( Y_i = \{ y \in V'_i \mid V'_{i-1} \mid \pi(y) \in \text{Nonsing} \ W' \text{ and } (d\pi)_y \text{ has rank } \leq m \} \) where \( \pi: V' \to W' \) is induced by projection. Now \( \bigcup \pi(Y_i) \) is a constructible set, but it has measure 0 by Sards Theorem. Hence, if we let \( X' = \text{Sing} W' \cup \text{the Zariski closure of} \ \bigcup \pi(Y_i) \), then \( \dim X' \leq \dim W' \).

By Thoms first isotopy lemma, \( \pi: (W' - X') \times C^k \cap V' \to W' - X' \) is a fibration. Since \( W' \) is irreducible we know that \( W' - X' \) is connected, so all the \( z \times C^k \cap V' \) are homeomorphic for \( z \in W' - X' \). Let \( \delta(f) = \) the Euler characteristic of \( z \times C^k \cap V' \) for \( z \in W' - X' \) and let \( X = X' \cap R^n \). Then \( X \) is a real algebraic subset of \( W \) and \( \dim X \leq \dim X' < \dim W' = \dim W \). Let \( \sigma: CP^n \times CP^k \to CP^n \times CP^k \) be complex conjugation on each coordinate. Note that \( \sigma(V') = V' \) and, furthermore, \( V \) is the fixed point set of the involution \( \sigma: V' \cap C^n \times C^k \to V' \cap C^n \times C^k \). Hence, if \( z \in W, f^{-1}(z) \) is the fixed point set of the involution \( \sigma: (z \times C^k \cap V') \to z \times C^k \cap V' \). Thus, \( \chi(f^{-1}(z)) = \chi(z \times C^k \cap V') = \delta(f) \mod 2 \) if \( z \in W - X \).

Note that if \( f: V \to W \) and \( g: W \to Z \) then \( \delta(gf) = \delta(g) \cdot \delta(f) \). Also, if \( f: V \to W \) and \( V = Y \cup Z \) then \( \delta(f) = \delta(f|_Y) + \delta(f|_Z) + \delta(f|_{Y \cap Z}) \).

Theorem 5.3. For any \( k \) there exists a connected nonsingular algebraic set \( V \) and a closed smooth codimension \( k \) submanifold \( M \hookrightarrow V \) such that \( (M, f) \) cannot be made algebraic by a small isotopy of \( f \) where \( f \) is the inclusion.

Proof. Pick \( \nu \) so that \( \nu - k \) is even and greater than 1 then let \( W = R^\nu \) and \( X = V(x_1^2 + (x_1^2 - 1)(x_1^2 - 4), x_3, x_4, \ldots, x_n) \). \( X \) is an irreducible algebraic subset of \( W \) with two components \( X_0, X_1 \), with each \( X_i \) diffeomorphic to a circle, \( i = 0, 1 \). Take \( N^m \), any smooth closed submanifold of \( R^\nu \) with \( N \cap X = X_0 \), and \( m = \nu - k \). Then let \( M = B(N, X_0) \), \( V = B(W, X) \), and \( f \) be the inclusion \( M \hookrightarrow V \) where \( \pi: B(W, X) \to W \) is the algebraic blowing up of \( W \) along \( X \). Then \( (M, f) \) cannot be algebraic (even the bordism class cannot contain an algebraic representative) because if this were the case there would be a nonsingular algebraic subset \( Y \) of \( V \times R^s \) which is isotopic to \( M \times \{ 0 \} \) by a small isotopy. Let \( p: V \times R^s \to V \) be the
projection, then we claim that by applying Lemma 5.1 to \( \pi \circ p : Y \cap (\pi \circ p)^{-1}(X) \rightarrow \mathbb{R}^v \) we get a contradiction. This is because \( \pi \circ p \) is an entire rational function \( \pi \circ p(Y \cap (\pi \circ p)^{-1}(X)) = \pi \circ p((M \times 0) \cap (\pi \circ p)^{-1}(X)) = X \) by transversality, and \( X_0 = X \). Also for all \( x \in X_0 \)

\[
Y \cap (\pi \circ p)^{-1}(x) \approx M \times 0 \cap (\pi \circ p)^{-1}(x) \text{ by transversality}
\]

\[
\approx B(N, X_0) \cap \pi^{-1}(x)
\]

\[
\approx \mathbb{R}P^{m-2}
\]

hence has odd Euler characteristic (recall \( m \) is even).

**Remark 5.4.** By projectivizing \( W \) to \( \mathbb{R}P^v \) we can make \( V \) compact in the above examples.

**Remark 5.5.** By taking \( k = 1 \) in Theorem 5.3 we get a nonsingular algebraic set \( V \) with \( H^l_{v-1}(V) \neq 0 \), because otherwise \( M^{l-1} \hookrightarrow V^l \) would be isotopic to a nonsingular algebraic subset of \( V \) by a small isotopy by Theorem 4.1 (cf. Theorem 6.9).

**Remark 5.6.** Theorem 5.3 gives us a way of constructing distinct algebraic structures on smooth manifolds as follows: Let \( V \) be a smooth manifold, and

\[
S_{\text{Alg}}(V) = \left\{ (V', g) \left| \begin{array}{c}
V' \text{ is a nonsingular algebraic set} \\
g : V' \rightarrow V \text{ is a diffeomorphism}
\end{array} \right. \right\} / \sim
\]

\( \sim \) is the equivalence relation: \( (V', g) \sim (V'', h) \) if there is a birational diffeomorphism \( \gamma \) making the following commute.
$S_{\text{Alg}}(V)$ is the set of different algebraic structures on $V$ (a moduli space of $V$).

Choose $V'$ to be the algebraic set of Theorem 5.3 with the imbedding $M \hookrightarrow V'$ such that $(M, f)$ is not algebraic. Call the underlying smooth manifold of $V'$ by $V$ and let $g : V' \to V$ be the identity map (forgetful map). Then by [AK₂] there is a diffeomorphism $h : V'' \to V$ from some nonsingular algebraic set $V''$ such that $h^{-1}(g \circ f(M))$ is a nonsingular algebraic subset of $V''$. Then $(V'', h)$ is not equivalent to $(V', g)$ otherwise $f(M)$ would be an algebraic subset of $V'$, i.e. $(M, f)$ would be algebraic. So we have shown that $S_{\text{Alg}}(V) \neq \{0\}$.

Given a closed smooth submanifold $M''$ of $\mathbb{R}^n$ imbedded with a trivial normal bundle, it is known by Seifert [S] that $M$ is isotopic to a component of a complete intersection. One can ask whether the extra components are necessary. Here we give an obstruction obtained as a result of a discussion with E. Rees and L. Taylor: Let $\mathcal{F}$ be a trivialization of the normal bundle. Let $\beta(M, \mathcal{F}) \in \pi_n(S^{n-m})$ be the associated element of the framed cobordism group (obtained by Thom-Pontryagin construction). If, in particular $M''$ is isotopic to a complete intersection, i.e. if there is a polynomial map $f: \mathbb{R}^n \to \mathbb{R}^{n-m}$ with

(i) $f^{-1}(0) = M$
(ii) $\text{rank}(df)|_M = n - m$

then $f$ gives a trivialization of the normal bundle of $M$. After perhaps multiplying $f$ by a suitably high power $(1 + |x|^2)^N$ we may assume $f$ is proper, so $\beta(M, f)$ is represented by the one point compactification of $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$. Therefore we can assume that $f$ takes $S^{n-1}$ to $S^{n-m-1}$. Hence $\beta(M, f)$ lies in the image of the suspension map $\pi_{n-1}(S^{n-m-1}) \xrightarrow{\Sigma} \pi_n(S^{n-m})$, i.e. desuspends. So any closed submanifold $M'' \subset \mathbb{R}^n$ imbedded with a trivial normal bundle, with the property that none of $\beta(M, \mathcal{F})$ desuspends for any $\mathcal{F}$, cannot be isotopic to a complete intersection. For example there is an exotic sphere $M^{16} \subset \mathbb{R}^{30}$ with this property [HLS] (this example along with infinite number of other exotic spheres which are constructed by Mahowald [Ma] was communicated to us by R. Shultz).

The above discussion proves the following:
Proposition 5.7. Let $M'' \subset \mathbb{R}''$ be a closed smooth submanifold imbedded with a trivial normal bundle and let $M$ be isotopic to a complete intersection. Then the normal bundle of $M$ has a trivialization $\mathcal{F}$ such that $\alpha(M, \mathcal{F})$ is in the image of the suspension map $\Sigma: \pi_{n-1}(S^{n-m-1}) \to \pi_n(S^{n-m})$.

Theorem 5.8. There are closed smooth submanifolds $M \subset \mathbb{R}''$ with trivial normal bundle so that $M$ is not isotopic to a complete intersection in $\mathbb{R}''$.

6. Applications to $H_*(V; \mathbb{Z}/2\mathbb{Z})$. Let $V$ be a compact nonsingular algebraic set. Recall, $H_k^A(V; \mathbb{Z}/2\mathbb{Z})$ is the subgroup of $H_k(V; \mathbb{Z}/2\mathbb{Z})$ generated by algebraic cycles; $\theta$ is algebraic if there is $M^k \to V$ where $M$ is a compact nonsingular algebraic set and $f$ is a polynomial with $f_*[M] = \theta$ (see Section 4). Without loss of generality we will assume that $f_*[Z] \neq 0$ for each irreducible component $Z$ of $M$. If $Y$ is the Zariski closure of $f(M)$ in $V$ then $\dim(Y) = k$, and by Lemma 5.1 $\dim(Y - f(M)) < \dim f(M)$ (observe $f$ is degree 1 onto its image). Hence the fundamental homology class $[Y]$ of the algebraic subset $Y$ represents $\theta$. Conversely by resolution of singularities for any $k$-dimensional algebraic subset $Y$ of $V$ we can find a nonsingular $M$ and an algebraic multiblowup $M \to Y$ so that $(M, \text{inclusion} \circ \pi)$ represents $[Y]$. Therefore $H_k^A(V; \mathbb{Z}/2\mathbb{Z})$ is also the subgroup generated by $k$-dimensional algebraic subsets of $V$. Recall $AH_k(V; \mathbb{Z}/2\mathbb{Z})$ is the subgroup of $H_k(V; \mathbb{Z}/2\mathbb{Z})$ generated by $k$-dimensional stable algebraic subsets of $V$. Let $H_k^{\text{imb}}(V; \mathbb{Z}/2\mathbb{Z})$ to be the subgroup generated by imbedded smooth closed submanifolds of $V$. In particular every element of this group is represented by an immersed submanifold of $V$. All cycles in $H_k^{\text{imb}}(V; \mathbb{Z}/2\mathbb{Z})$ come from algebraic cycles. This means that there is a nonsingular algebraic set $V'$ and a diffeomorphism $\varphi: V \to V'$ such that $\varphi_*H_k^{\text{imb}}(V; \mathbb{Z}/2\mathbb{Z}) \subset H_k^A(V; \mathbb{Z}/2\mathbb{Z})$; this follows from [AK_2]. We have the inclusions $H_k^A(V; \mathbb{Z}/2\mathbb{Z}) \supseteq AH_k(V; \mathbb{Z}/2\mathbb{Z}) \subseteq H_k^{\text{imb}}(V; \mathbb{Z}/2\mathbb{Z})$. For convenience we define $H_k^{\text{imb}}(Z; \mathbb{Z}/2\mathbb{Z}), H_k^A(V; \mathbb{Z}/2\mathbb{Z}), AH_k(V; \mathbb{Z}/2\mathbb{Z})$ to be the Poincare' duals of $H_k^{\text{imb}}(V; \mathbb{Z}/2\mathbb{Z}), H_k^A(V; \mathbb{Z}/2\mathbb{Z}), AH_k(V; \mathbb{Z}/2\mathbb{Z})$ respectively. These subgroups behave nicely under rational maps. In particular, we have the following theorem.

Theorem 6.1. Let $f: V \to W$ be an entire rational function. Then

(a) $f_*(H_k^A(V; \mathbb{Z}/2\mathbb{Z})) \subset H_k^A(W; \mathbb{Z}/2\mathbb{Z})$
(b) $f^*(H_k^A(W; \mathbb{Z}/2\mathbb{Z})) \subset H_k^A(V; \mathbb{Z}/2\mathbb{Z})$
(c) $f^*(AH^*(W; \mathbb{Z}/2\mathbb{Z})) \subset AH^*(V; \mathbb{Z}/2\mathbb{Z})$
Proof. The proof of a) is trivial. To prove b), pick a rational function $g: X \to W$ from a nonsingular algebraic set $X$. We must show $f^*D^{-1}g_*[X] \subseteq H^k_*(V; \mathbb{Z}/2\mathbb{Z})$. We may homotop $g$ to a smooth function $g': X \to W$ which is transverse to $f$ (i.e. $g' \times f: X \times V \to W \times W$ is transverse to the diagonal). But then Proposition 2.3 of [AK3] implies we may find a nonsingular algebraic set $X'$ and a rational function $g'': X' \to W$ and a diffeomorphism $h: X' \to X$ so that $g''$ approximates $gh$. In particular, $g''$ is transverse to $f$. But then the pullback $Z = \{ (x, v) \in X' \times V | f(v) = g''(x) \}$ is a nonsingular algebraic set so that if $\pi: Z \to V$ is induced by projection $X' \times V \to V$, then $D^{-1}\pi_*[Z] = f^*D^{-1}g''_*[X'] = f^*D^{-1}g_*[X]$. To prove c), pick $W_k \subseteq W_{k+1} \subseteq \cdots \subseteq W_n = W$ so that each $W_i$ is nonsingular and dim $W_i = i$. Then by the proof of Proposition 4.3 we may isotop each $W_i$ to a nonsingular algebraic set $W'_i$ so that $W'_k \subseteq W'_{k+1} \subseteq \cdots \subseteq W'_n$ and so that each $W'_i$ is transverse to $f$. Now $f^{-1}(W'_k)$ is a stable algebraic subset of $V$ since each $f^{-1}(W'_i)$ is a nonsingular dimension $V - n + i$ dimensional algebraic set by Lemma 1.4 of [AK1]. But $Df^*D^{-1}[W_k] = Df^*D^{-1}[W'_k] = [f^{-1}(W'_k)]$, so c) is proven.

We presume that the kind of transversality used above is actually true in the smooth case (although it is listed as a triple starred unknown problem in Hirsch’s Differential Topology). However it is clearly true in the algebraic case since we may stratify $V$ and $W$ so that $f$ submerses each stratum of $V$ to a stratum of $W$. After refining the stratification of $W$ to be Whitney, one only needs to make $g$ or the $W_i$’s transverse to this stratification.

**Theorem 6.2.** If $V$ is a nonsingular algebraic set and $\theta \in H_k(V; \mathbb{Z}/2\mathbb{Z})$ is a nontrivial class, then there exists an algebraic multiblow-up $\pi: \tilde{V} \to V$ of $V$ along centers of dimension less than $k$ and a $k$-dimensional nonsingular algebraic subset $Z^k$ of $\tilde{V}$ and a union of connected components $Z^k_0$ of $Z^k$ such that $\pi|_{Z^k_0}: Z^k_0 \to V$ represents $\theta$.

**Proof.** By Steenrod representability [T] there is a map $f: N^k \to V$ where $N^k$ is a smooth manifold and $f_*[N^k] = \theta$. By transversality, we may as well assume $f$ is one to one almost everywhere. We may also assume that $f_*[N_i] \neq 0$ for each component $N_i$ of $N^k$. By Lemma 2.9 of [AK2] we can find an algebraic set $Q$, a polynomial $g: Q \to V$ and a union of nonsingular connect components $Q_0$ of $Q$ so that $Q_0$ is diffeomorphic to $N$, $g_*[Q_0] = \theta$ and $g|_{Q_0}$ is one to one almost everywhere. Hence, $g(Q_0)$ is a semialgebraic set of dimension $k$. Let $Y$ be the Zariski closure of $g(Q_0)$. Then dim $Y = k$. 


By resolution of singularities \([H]\) there exists an algebraic multiblowup \(\pi: \tilde{V} \to V\) such that the strict preimage \(Z\) of \(Y\) is a \(k\) dimensional nonsingular algebraic set. By \([H]\) there is an algebraic multiblowup \(\rho: \tilde{Q} \to Q\) and an entire rational function \(\tilde{g}: \tilde{Q} \to \tilde{V}\) so that the centers of \(\rho\) have dimension less than \(k\) and the following diagram commutes.

\[
\begin{array}{ccc}
\tilde{Q} & \xrightarrow{\tilde{g}} & \tilde{V} \\
\downarrow{\rho} & & \downarrow{\pi} \\
Q & \xrightarrow{g} & V
\end{array}
\]

Let \(\tilde{Q}_0 = \rho^{-1}(Q_0)\) and \(Z_0 = \tilde{g}(\tilde{Q}_0)\). We claim that \(Z_0\) is a union of connected components of \(Z\) and \(\pi_\ast[Z_0] = \theta\). To see this, note that \(\tilde{g}: \tilde{Q}_0 \to Z\) is one to one almost everywhere, so \(\tilde{g}_\ast[\tilde{Q}_0] = [\tilde{g}(\tilde{Q}_0)] = [Z_0]\). Hence, \(\pi_\ast[Z_0] = \pi_\ast\tilde{g}_\ast[\tilde{Q}_0] = g_\ast\rho_\ast[\tilde{Q}_0] = g_\ast[Q_0] = \theta\). Although \(Z_0\) is in fact a union of connected components of \(Z\), we do not need this to get the theorem, if \(Z'\) is a component of \(Z_0\) which is not a component of \(Z\), we must have \(\pi_\ast[Z'] = 0\). Hence, we can get the theorem by just deleting those components of \(Z_0\) which are not also components of \(Z\). \(\square\)

**Remark 6.3.** If \(\theta \in H^k_\text{et}(V; \mathbb{Z}/2\mathbb{Z})\) then we can take \(Z = Z_0\) in the conclusion of Theorem 6.2. To see this observe that in the proof we would have \(Q_0 = Q\); and \([Y] = \theta\) by the remarks of the first paragraph of this section.

We must now generalize the notion of algebraic multiblowup slightly. An *uzunblowup* is a composition of rational functions \(V_s \xrightarrow{\pi_i} V_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_1} V_0\) so that each \(\pi_i\) is either a blowup with nonsingular center or a diffeomorphism. In particular, an uzunblowup is a smooth multiblowup. In [AK6] we prove the following theorem, using the ideas in this paper.

**Theorem 6.4.** For any compact nonsingular algebraic set \(V\) there exists an uzunblowup \(\pi: \tilde{V} \to V\) along centers of dimension less than \(k\) such that we have:

\[
H_i(\tilde{V}; \mathbb{Z}/2\mathbb{Z}) = H^{\text{imb}}_i(\tilde{V}; \mathbb{Z}/2\mathbb{Z}) \quad \text{if} \quad i \leq k.
\]

Since every closed smooth manifold is diffeomorphic to a nonsingular algebraic set Theorems 6.2 and 6.4 apply to smooth manifolds. Because of injectivity of \(\pi_\ast\) (Lemma 6.5) Theorem 6.4 has useful applications to topology (Corollaries 6.7 and 6.8). We first need to discuss homological proper-
ties of multiblowups. In the following discussion all lower (upper) stars on maps indicate the induced maps on $\mathbb{Z}/2\mathbb{Z}$ homology (cohomology).

Let $\tilde{V} \xrightarrow{\pi} V$ be an algebraic multiblowup of a nonsingular algebraic set $V$. $\pi$ is a composition of blowup maps:

$$
\tilde{V} = \tilde{V}_s \xrightarrow{\pi_3} \tilde{V}_{s-1} \xrightarrow{} \cdots \xrightarrow{\pi_1} \tilde{V}_0 = V
$$

with centers $L_i \subset V_i$, and $V_{i+1} = B(V_i, L_i)$. Then each $P_i = \pi_i^{-1}(L_{i-1})$ is a codimension one nonsingular algebraic subset of $V_i$. By using Theorem 4.1 we can make $P_i$ transverse to $L_i$ by a small isotopy; then we get a codimension one nonsingular algebraic subset $P_i(i + 1) \subset V_{i+1}$ which is the strict preimage of the isotoped $P_i$. By iterating this process for any $j > i$ we get a codimension one nonsingular algebraic subset $P_i(j) \subset V_j$ which is the strict preimage of $P_i(j - 1)$ under $\pi_j$. Make the convention that $P_i(i) = P_i$.

Let $P_i(j) \xrightarrow{\gamma_{ij}} V_j$ be the inclusion. The following lemma gives some useful algebraic topological properties of $(\tilde{V}, \pi)$.

**Lemma 6.5.**

(a) $\pi^*$ is injection, and $\pi_*$ is surjection

(b) $\pi^*H^r_{\text{imm}}(V; \mathbb{Z}/2\mathbb{Z}) \subseteq H^r_{\text{imm}}(\tilde{V}; \mathbb{Z}/2\mathbb{Z})$

(c) $\pi^*\text{AH}^r(V; \mathbb{Z}/2\mathbb{Z}) \subseteq \text{AH}^r(\tilde{V}; \mathbb{Z}/2\mathbb{Z})$

(c) Kernel of $\pi_*$ is generated by the elements of image of $(\gamma_{i*})_*$, with $i = 1, \ldots, s$

**Proof.** (a) $\pi$ is a degree 1 map so the following diagram commutes, this in turn implies the result

$$
\begin{array}{ccc}
H^r(\tilde{V}; \mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\pi^*} & H^r(V; \mathbb{Z}/2\mathbb{Z}) \\
D \downarrow & & \downarrow D \\
H_k(\tilde{V}; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\pi^*} & H_k(V; \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

(b) It is enough to prove this for each $\pi_i$. Let $\theta \in H^k_{\text{imm}}(V_{i-1}; \mathbb{Z}/2\mathbb{Z})$ then $\theta = \Sigma_{\alpha} M^k_{\alpha}$ where each $M^k_{\alpha} \xrightarrow{} V_{i-1}$ is imbedded smooth submanifold. Make each $M^k_{\alpha}$ transverse to $L_{i-1}$ take $\tilde{M}_{\alpha} = \pi_i^{-1}(M_{\alpha})$ where $\tilde{M}_{\alpha}$ are the strict preimages. Then $\pi_i^{-1}(\cup M_{\alpha}) = \cup \tilde{M}_{\alpha}$ represents $D\pi_i^*D^{-1}(\theta)$ hence the result follows. The proof of the second part is similar; in this case we assume that each $M^k_{\alpha}$ is a stable algebraic subset. By using Proposition 4.3 we can make each $M_{\alpha}$ transverse to $L_{i-1}$ and in fact we can make $\tilde{M}_{\alpha}$ stable.
(c) We prove this by induction on \( s \). For \( s = 1 \) \( \pi_1 \) is a homeomorphism in the complement of \( P_1 \). Let \( U \) be a tubular neighborhood of \( P_1 \) in \( V_1 \). Then the commutative diagram

\[
\begin{array}{ccc}
H_k(P_1) & \approx & H_k(U) \xrightarrow{\gamma_{11}} H_k(V_1) \xrightarrow{\pi_1} H_k(V_1, U) \\
\downarrow & & \downarrow \quad \text{(from excision)} \\
H_k(\pi_1(U)) & \rightarrow & H_k(V) \rightarrow H_k(V; \pi_1(U))
\end{array}
\]

gives \( \ker(\pi_1)_* \subset \im(\gamma_{11})_* \). Hence (c) holds for \( s = 1 \). Now let \( \theta \in \ker(\pi_*) \). Let \( \pi = \pi' \circ \pi_s \) so \( (\pi_s)_*(\theta) \in \ker(\pi'_*) \). Then by induction

\[
(\pi_s)_*(\theta) = \sum_{i=1}^{s-1} (\gamma_{i,s-1})_*(\theta_{i,s-1}), \quad \text{for some } \theta_{i,s-1}.
\]

We have commutative diagrams

\[
\begin{array}{ccc}
P_i(s) & \xrightarrow{\gamma_{i,s}} & V_s \\
\downarrow & & \downarrow \pi_s \\
P_i'(s - 1) & \xrightarrow{\gamma_{i,s-1}} & V_{s-1} \\
\uparrow & & \uparrow \\
P_i(s - 1) & \xrightarrow{\gamma_{i,s-1}} & V_{s-1}
\end{array}
\]

where the bottom square comes from isotoping \( P_i(s - 1) \) to \( P_i'(s - 1) \) which is transverse to \( L_{s-1} \). Hence, on homology we get a commutative diagram

\[
\begin{array}{ccc}
H_*(P_i(s); \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\gamma_{i,s}} & H_*(V_s; \mathbb{Z}/2\mathbb{Z}) \\
\downarrow \rho_i_* & & \downarrow \pi_* \\
H_*(P_i(s - 1); \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\gamma_{i,s-1}} & H_*(V_{s-1}; \mathbb{Z}/2\mathbb{Z})
\end{array}
\]

where \( \rho_i \) is the composition \( P_i(s) \to P_i'(s - 1) \to P_i(s - 1) \). Since \( \rho_i \) is degree 1 we know that \( \rho_i_* \) is onto so we can find \( \theta_{i,s} \in H_k(P_i(s); \mathbb{Z}/2\mathbb{Z}) \) with \( (\rho_i)_*(\theta_{i,s}) = \theta_{i,s-1} \) and thus \( (\pi_s)_*(\gamma_{i,s})_*(\theta_{i,s}) = (\gamma_{i,s-1})_*(\theta_{i,s-1}) \). Then we have

\[
(\pi_s)_*(\theta) = (\pi_s)_* \left( \sum_{i=1}^{s-1} (\gamma_{i,s})_*(\theta_{i,s}) \right), \quad \text{hence}
\]
\[ \theta = \sum_{i=1}^{s-1} (\gamma_{is})_\ast(\theta_{i,s}) \in \text{kernel}(\pi_i)_\ast \]

Since kernel(\pi_i)_\ast \subset \text{image}(\gamma_{s,s})_\ast \text{ (from the first case) } \theta = \Sigma_{i=1}^{s} (\gamma_{is})_\ast(\theta_{i,s}) \text{, for some } \theta_{s,s}. \]

Let \( V \) be a compact nonsingular algebraic set the following says that \( H^s_A(V; \mathbb{Z}/2\mathbb{Z}) \) is closed under cohomology operations. It includes real algebraic version of Chow’s moving lemma \([K_3]\) and Kawai’s theorem \([Ka]\).

**Theorem 6.6.** \( H^s_A(V; \mathbb{Z}/2\mathbb{Z}) \) is closed under cup products and Steenrod squaring operations.

**Proof.** Pick \( \theta, \eta \in H^s_A(V; \mathbb{Z}/2\mathbb{Z}) \). Represent \( D(\theta) \) by an algebraic subset \( Y \) of \( V \). Represent \( D(\eta) \) by a nonsingular algebraic set \( X \subset V \times \mathbb{R}^k \).

By Proposition 2.3 of \([AK1]\) we may assume that \( X \) is transverse to \( Y \times \mathbb{R}^k \) (i.e. transverse to each stratum in a Whitney stratification of \( Y \times \mathbb{R}^k \)). Then \( D(\eta \cup \theta) \) is represented by \( X \cap (Y \times \mathbb{R}^k) \). By \([H]\) we may blow up \( X \cap (Y \times \mathbb{R}^k) \) to make it nonsingular, hence \( \eta \cup \theta \in H^s_A(V; \mathbb{Z}/2\mathbb{Z}) \). Independent proofs of this cup product result were given by Bennedetti-Tognoli and Shiota (with some restrictions). Now we will do the Steenrod squares. We thank Brumfiel for pointing out the following construction of Steenrod squares to us. It lends itself nicely to algebraization.

Let \( X \) be an \( n \) dimensional manifold and pick \( \alpha \in H^q(X; \mathbb{Z}/2\mathbb{Z}) \). Pick a map \( f: M \rightarrow X \) from a compact \( n - q \) manifold \( M \) so that \( f_\ast[M] = D\alpha \).

Let \( S^i \) be the \( i \)-dimensional sphere. Let \( M' = M \times M \times S^i/(x, y, z) \sim (y, x, -z) \) and \( X' = X \times X \times S^i/(x, y, z) \sim (y, x, -z) \). Both \( M' \) and \( X' \) are manifolds since they are orbit spaces of free \( \mathbb{Z}/2\mathbb{Z} \) actions. We have a map \( f': M' \rightarrow X' \) given by \( f'(x, y, z) = (f(x), f(y), z) \). Let \( \mathbb{R}P^i \) be real projective \( i \) space, \( \mathbb{R}P^i = S^i/x \sim x \). There is an imbedding \( g: X \times \mathbb{R}P^i \rightarrow X' \) given by \( g(x, y) = (x, x, y) \). Then \( Sq^i\alpha = D^{-1} \pi_\ast Dg_\ast D^{-1}(f'_\ast[M']) \) where \( \pi: X \times \mathbb{R}P^i \rightarrow X \) is projection. (See Proposition 4.2 of \([BRS]\).)

Let us now make this construction algebraic. First we will find nonsingular algebraic sets \( Z_{ni} \) and diffeomorphisms \( \varphi_{ni}: Z_{ni} \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times S^i/(x, y, z) \sim (y, x, -z) \) so that for any nonsingular algebraic set \( Y \subset \mathbb{R}^n \), \( \varphi_{ni}^{-1}(Y \times Y \times S^i/\sim) \) is a nonsingular algebraic set. Let

\[
Z_{ni}' = \mathbb{R}^n \times (\mathbb{R}P^{n+i} - \mathbb{R}P^{n-1})
\]

\[= \{(w_1, \ldots, w_n, [u_0: \cdots: u_{n+i}])|u_j \neq 0 \text{ some } j = 0, \ldots, i\}.\]
We have a diffeomorphism \( \varphi_{ni}' : Z_{ni}' \to \mathbb{R}^n \times \mathbb{R}^n \times S^i / \sim \) given by \( \varphi_{ni}'(w_1, \ldots, w_n, [u_0 : \cdots : u_{n+i}]) = (x, y, z) \) where \( z_j = u_j(u_0^2 + \cdots + u_i^2)^{-1/2}, \) \( x_j = (w_j + u_{j+i}(u_0^2 + \cdots + u_i^2)^{-1/2})/2, \) \( y_j = w_j - x_j. \) Furthermore, it is easy to check that if \( W \subset \mathbb{R}^n \times \mathbb{R}^n \times S^i \) is a real algebraic set invariant under the involution \( (x, y, z) \to (y, x, -z) \) then \( \varphi_{ni}'^{-1}(W / \sim) \) is a (quasiprojective) algebraic subset of \( Z_{ni}' \) with singularities \( \varphi_{ni}'^{-1}(\text{Sing } W / \sim). \)

We now can make \( Z_{ni}' \) affine by imbedding it as a Zariski open subset of \( \mathbb{R}^n \times G_{n+i+1,1} \), and then sending its complement in \( \mathbb{R}^n \times G_{n+i+1,1} \) off to infinity. For instance we may take \( Z_{ni} \) to be \( \{ (w, L) | w \in \mathbb{R}^n, L \text{ is a symmetric } (n + i + 1) \times (n + i + 1) \text{ matrix}, \Sigma_{j=0} L_{jj} = 1 \text{ and } L_{ab} L_{cd} = L_{ad} L_{bc} \text{ for all } a, b, c, d \}. \) A birational diffeomorphism \( \psi : Z_{ni}' \to Z_{ni} \) may be defined by \( \psi(w_1, \ldots, w_n, [u_0 : \cdots : u_{n+i}]) = ((w_1, \ldots, w_n), L) \) where \( L_{ab} = u_a u_b / (u_0^2 + \cdots + u_i^2) \). Then \( \varphi_{ni} = \varphi_{ni}' \psi^{-1} \) has the required properties.

Now suppose \( M \subset \mathbb{R}^n \) and \( X \subset \mathbb{R}^k \) are nonsingular real algebraic sets and \( f : M \to X \) is a rational function. Let \( M'' = \varphi_{mi}^{-1}(M \times M \times S^i / \sim) \) and \( X'' = \varphi_{ki}^{-1}(X \times X \times S^i / \sim). \) Then \( M'' \) and \( X'' \) are nonsingular algebraic sets by the above discussion. Furthermore there is a rational function \( f'' : M'' \to X'' \) so that \( \varphi_{ki} f'' \varphi_{mi}^{-1}(x, y, z) = (f(x), f(y), z). \)

We also have a rational imbedding \( \theta : X \times \mathbb{R}^p \to X'' \) so that \( \varphi_{ki} \theta = g. \) But

\[
DSq^i Df_*^{-1}[M] = \pi_\ast Dg_* D^{-1} f_*^{-1}[M']
= \pi_\ast D\theta_* D^{-1} f_*^{-1}[M''].
\]

So \( DSq^i Df_*^{-1}[M] \in H^k_\mathbb{A}(X; \mathbb{Z} / 2\mathbb{Z}) \) by Theorem 6.1.

Hence \( Sq^i(H^k_\mathbb{A}(X; \mathbb{Z} / 2\mathbb{Z})) \subset H^k_\mathbb{A}(X; \mathbb{Z} / 2\mathbb{Z}). \)

For any given \( \theta \in H_k(V; \mathbb{Z} / 2\mathbb{Z}) \) where \( V \) is a closed smooth manifold we have maps

\[
MO(r) \xrightarrow{p_2} \Omega'' \Sigma'' MO(r) \xrightarrow{p_1} V \xrightarrow{f} K(\mathbb{Z} / 2\mathbb{Z}, r)
\]
Where $MO(r)$ is the Thom space $[T]$ of the universal $R^k$-bundle, $\Omega^n\Sigma^n$ is the operation of taking suspension $n$-times followed by loops $n$-times. $p = p_1 \circ p_2$, $n$ is an arbitrarily large number, $p_2$ is induced by the identity map, $p_1$ is induced by the Thom class (also recall $K(\mathbb{Z}/2\mathbb{Z}, r) = \Omega^nK(\mathbb{Z}/2\mathbb{Z}, r + n)$). Finally $f$ is the classifying map for $\theta$, i.e. $f^*(i) = D^{-1}(\theta)$ where $i$ is the fundamental $r$-th cohomology class.

It is well known that lifting $f$ to $\Omega^n\Sigma^nMO(r)$ (or to $MO(r)$) is equivalent to representing $\theta$ by an immersed (or imbedded) submanifold of $V$. One can easily find examples $(V, \theta)$ such that $\theta$ cannot be represented by an immersed submanifold. This is because $p_1$ is not injective in $\mathbb{Z}$-cohomology ($W.$ Browder). Also $p$ is not injective in $\mathbb{Z}/2\mathbb{Z}$-cohomology (Thom). But surprisingly the following can be shown by using Theorem 6.3.

**Corollary 6.7.** The map $\Omega^n\Sigma^nMO(r) \xrightarrow{\pi \circ f} K(\mathbb{Z}/2\mathbb{Z}, r)$ is an injection in $\mathbb{Z}/2\mathbb{Z}$-cohomology.

**Proof.** Let $V_r$ be the boundary of the tubular neighborhood of some large skeleton of $K(\mathbb{Z}/2\mathbb{Z}, r)$ in some $R^m$. Hence there is a map $f: V_r \to K(\mathbb{Z}/2\mathbb{Z}, r)$ with $f^*$ isomorphism for all values of $*$ less than some arbitrarily large number. Let $\gamma \in \text{kernel}(p_1^*)$, by Theorem 6.4 there is a multiblow-up $\tilde{V}_r \to V_r$ with $p_1^*(f^*(i)) \in H^*_\text{imb}(\tilde{V}_r; \mathbb{Z}/2\mathbb{Z})$ where $i$ is the fundamental class. In particular $\pi^*f^*(i)$ is an immersed class. Therefore there is a map $g$ making the following commute

\[
\begin{array}{ccc}
\tilde{V}_r & \xrightarrow{\pi} & \Omega^n\Sigma^n MO(r) \\
\xrightarrow{\gamma} & & \xrightarrow{p_1} \\
V_r & \xrightarrow{f} & K(\mathbb{Z}/2\mathbb{Z}, r)
\end{array}
\]

Hence $\pi^*(f^*(\gamma)) = g^*(p_1^*(\gamma)) = 0$. But if $\gamma \neq 0$ then $f^*(\gamma) \neq 0$ hence $\pi^*(f^*(\gamma)) \neq 0$ by (a) of Lemma 6.5. Contradiction. $\square$

**Corollary 6.8.** There exists a closed smooth manifold $M$ and $\theta \in H^*_\text{imb}(M; \mathbb{Z}/2\mathbb{Z})$ such that $\theta$ cannot be represented by an imbedded class.

**Proof.** Using the notation of Corollary 6.7 let $f: V_2 \to K(\mathbb{Z}/2\mathbb{Z}, 2)$ as above. Pick $T(i) = (Sq^2Sq^1i) \cup i^2 + (Sq^1i)^3 + (Sq^1i) \cup i^3$ where $i$ is the fundamental class. Then $T(i) \neq 0$ and $T(i) \in \text{kernel}(p^*)$, where $p$ is the map $MO(2) \to K(\mathbb{Z}/2\mathbb{Z}, 2)$ discussed before [T]. By using Theorem 6.4 pick a multiblowup $\tilde{V}_2 \to V_2$ with $\pi^*f^*(i) \in H^2_\text{imb}(\tilde{V}_2; \mathbb{Z}/2\mathbb{Z})$. We claim $(M, \theta) = (\tilde{V}_2, D\pi^*f^*(i))$ is a required example with $k = \dim(V) - 2$,
since if $\theta$ were an imbedded class we would have a map $g$ making the following commute

$$
\begin{array}{ccc}
\tilde{V}_2 & \xrightarrow{g} & MO(2) \\
\downarrow \pi & & \downarrow p \\
V_2 & \xrightarrow{f} & K(\mathbb{Z}/2\mathbb{Z}, 2)
\end{array}
$$

Then $\pi*f*T(i) = g*p*T(i) = 0$, then since $f*$ is an isomorphism and $\pi*$ is an injection $T(i) = 0$. Contradiction.

W. Browder has recently found a purely algebraic topological proof of Corollary 6.7.

Finally we want to show that often $H^A_*(V; \mathbb{Z}/2\mathbb{Z})$ does not coincide with the ordinary homology, for example:

**Theorem 6.9.** For any $n \geq 3$, there exists a connected nonsingular algebraic set $V''$ of dimension $n$ such that $H^A_k(V; \mathbb{Z}/2\mathbb{Z}) \neq H_k(V; \mathbb{Z}/2\mathbb{Z})$ for all $k = 2, 3, \ldots, n - 1$.

**Proof.** Let $V_1$ be any closed smooth manifold of dimension $n$, which contains two disjointly imbedded circles with trivial normal bundles $S_1 \cup S_2 \hookrightarrow V_1$ such that the inclusion $H_1(S_1 \cup S_2; \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H_1(V; \mathbb{Z}/2\mathbb{Z})$ is an injection. Then an open tubular neighborhood of $S_1 \cup S_2$ in $V_1$ is diffeomorphic to an open neighborhood $U$ of $X$ in $\mathbb{R}^n$, where $X$ is an irreducible algebraic variety diffeomorphic to $S_1 \cup S_2$. For example $X = V(x_1^4 + (x_1^2 - 1)(x_1^2 - 4), x_3, \ldots, x_n)$ in $\mathbb{R}^n$. Extend $U$ to an imbedding $V_1$ into $\mathbb{R}^n \times \mathbb{R}^l$ for some big $l$ ($\mathbb{R}^n$ is identified with $\mathbb{R}^n \times 0$). By relative Nash theorem (Proposition 2.8 of [AK1]) approximate $V_1$ by a nonsingular algebraic set $V_2$ keeping $X$ fixed (i.e. $X \subset V_2$). Let $V$ be the algebraic blow up $B(V_2, X) \rightarrow V_2$. Write $X = X_1 \cup X_2$ where $X_i$ are components of $X$, also continue to denote an open tubular neighborhood of $X$ in $V_2$ by $U$.

For each $j = 1, 2, \ldots, n - 1$ pick a trivial $\mathbb{R}^j$ subbundle of the normal bundle of $X_j$ in $V_2$. Let $L_j \subset \pi^{-1}(X_j)$ be the set of directions in this $\mathbb{R}^j$ subbundle, so $L_j \approx X_1 \times \mathbb{R}P^{j-1}$. We claim that none of the homology classes $\gamma_j = [L_j] \in H_j(V; \mathbb{Z}/2\mathbb{Z})$ are algebraic, $j = 2, \ldots, n - 1$.

Suppose $\gamma_j$ were represented by an algebraic set $Y_j \subset V \times \mathbb{R}^k$. Then by Proposition 2.3 of [AK3] we may as well assume that $Y_j$ is transverse to $\pi^{-1}(X) \times \mathbb{R}^k$. But then $y_{j-1} = \pi^{-1}(X) \times \mathbb{R}^k \cap Y_j$ is an algebraic set representing $\gamma_{j-1}$, since
\[ D^{-1}(\gamma_{j-1}) = D^{-1}(\gamma_j) \cup D^{-1}(\pi^{-1}(X_1)) \]
\[ = D^{-1}(\gamma_j) \cup D^{-1}(\pi^{-1}(X)) \]
\[ = D^{-1}(\rho_*[Y_j]) \cup D^{-1}(\pi^{-1}(X)) \]
\[ = D^{-1}(\rho_*[Y_j \cap \pi^{-1}(X) \times \mathbb{R}^k]). \]

where \( \rho: Y_j \to V \) is induced by projection. Continuing in this manner, we obtain an algebraic set \( Y_2 \subset V \times \mathbb{R}^k \) which represents \( \gamma_2 \) and is transverse to \( \pi^{-1}(X) \times \mathbb{R}^k \). Then \( Y_1 = Y_2 \cap \pi^{-1}(X) \times \mathbb{R}^k \) represents \( \gamma_1 \) and since \( \pi_*\gamma_1 = [X_1] \) we know that \( \pi_*\rho_*([Y_1]) = [X_1] \) and also \( \pi\rho(Y_1) \subset X \). But then \( \rho^{-1}\pi^{-1}(x) \) is an odd number of points for the generic point \( x \in X_1 \) and it is an even number of points for the generic point \( x \in X_2 \) which contradicts Lemma 5.2.

**Corollary 6.10.** There are nonsingular algebraic sets \( V \) and vector bundles \( E \) over \( V \) such that \( E \) cannot be made a nonsingular algebraic set containing \( V \). Furthermore \( E \) can be chosen to be orientable if one wishes.

**Proof.** By applying Theorem 6.9 to \( V_1 = S^1 \times S^{n'} \# S^1 \times S^{n''} \) we get a nonsingular algebraic set \( V \approx S^1 \times \mathbb{R}P^{n''} \# S^1 \times \mathbb{R}P^{n'} \). Let \( E_1 \) and \( E_2 \) be the normal bundles of \( S^1 \times \mathbb{R}P^{n''} \subset S^1 \times \mathbb{R}P^{n'+2} \) and \( S^1 \times \mathbb{R}P^{n''} \subset S^1 \times \mathbb{R}P^{n} \times \mathbb{R}^2 \) respectively. Let \( E \) be the induced orientable bundle over \( V \), i.e. \( E \) restricts to \( E_1 \) and \( E_2 \) on the first and second factors of the connected sum. Then the imbedded copy of \( L \) of \( S^1 \times \mathbb{R}P^{n'-2} \) in the first factor of \( V \) represents the dual of the second Steifel-Whitney class \( w_2 \in H^2(V; \mathbb{Z}_2) \) of the bundle \( E \to V \). But by Theorem 6.9 the class \([L]\) cannot be algebraic. Therefore \( E \) cannot be made a nonsingular algebraic set containing \( V \), otherwise one can get a rational function \( V \not\to \mathbb{G}_{1,2} \) into the Grassmanians classifying the normal map \( V \subset E ([AK_2] \text{ Lemma 2.7}) \), then the dual of \( w_2 \) would be represented by an algebraic set \( g^{-1} \) (Schubert Cycle representing dual of \( w_2 \)). \( \square \)

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