POLYNOMIAL EQUATIONS OF IMMersed SURFACES

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If $V$ is a nonsingular real algebraic set we say $H_i(V; \mathbb{Z}_2)$ is algebraic if it is generated by nonsingular algebraic subsets of $V$.

Let $V^3$ be a 3-dimensional nonsingular real algebraic set. Then, we prove that any immersed surface in $V^3$ can be isotoped to an algebraic subset if and only if $H_i(V; \mathbb{Z}_2)$ $i = 1, 2$ are algebraic. This isotopy above carries the natural stratification of the immersed surface to the algebraic stratification of the algebraic set. Along the way we prove that if $V$ is any nonsingular algebraic set then any simple closed curve in $V$ is $\epsilon$-isotopic to a nonsingular algebraic curve if and only if $H_1(V; \mathbb{Z}_2)$ is algebraic.

Let $V^3$ be a 3-dimensional nonsingular real algebraic set. We call a homology group of $V$ algebraic if it is generated by nonsingular algebraic subsets. In this paper we prove:

**THEOREM.** The following are equivalent:

(a) If $f: M^2 \hookrightarrow V^3$ is any immersion of a closed smooth surface in general position, then $f(M^2)$ is isotopic to an algebraic subset $Z$ of $V^3$ by an arbitrarily small isotopy. This isotopy carries the natural stratification of $f(M^2)$ to the algebraic stratification of $Z$. 

(b) $H_1(V; \mathbb{Z}_2)$ and $H_2(V; \mathbb{Z}_2)$ are algebraic.

To be more precise for $i = 1, 2$ let $AH_i(V^3; \mathbb{Z}_2)$ be the subgroup of $H_i(V^3; \mathbb{Z}_2)$ generated by nonsingular algebraic subsets. Then $H_i(V; \mathbb{Z}_2)$ is algebraic if it is equal to $AH_i(V; \mathbb{Z}_2)$. In particular zero homology groups are algebraic. We will refer to elements of $AH_i(V^3; \mathbb{Z}_2)$ as algebraic homology classes. This definition is consistent with the conventions of [AK].

In case $f$ is an imbedding this theorem reduces to a special case of Proposition 1 below, which is Theorem 4.1 and Remark 4.2 of [AK]. Recall, if $W^n$ is a nonsingular algebraic set of dimension $n$, then $AH_{n-1}(W; \mathbb{Z}_2)$ is the subgroup of $H_{n-1}(W; \mathbb{Z}_2)$ generated by nonsingular algebraic subsets. Also if $M \subset W$ is a closed submanifold, denote the
$\mathbb{Z}_2$-homology class in $W$ induced by the fundamental class of $M$ by $[M]_2$. Then

**Proposition 1.** A codimension one closed smooth submanifold $M$ of $W$ is $\varepsilon$-isotopic to a nonsingular real algebraic subset if and only if $[M]_2 \in AH_{n-1}(W; \mathbb{Z}_2)$. Furthermore, this isotopy can fix any smooth submanifold $L$ of $M$ which is already a nonsingular algebraic set.

**Remark.** Proposition 1 remains true if $L$ is a union of nonsingular algebraic sets in $M$ ([T]).

We first prove a codimension two version of this proposition for $V^3$, which is an interesting result in itself.

**Proposition 2.** A simple closed curve $C \subset V^3$ is $\varepsilon$-isotopic to a nonsingular algebraic curve if and only if $[C]_2 \in AH_1(V; \mathbb{Z}_2)$. Furthermore this isotopy can fix any collection of points in $C$.

**Remark.** This proposition remains true if $V^3$ is replaced by a nonsingular algebraic set of any dimension. The proof is essentially the same.

**Lemma 3.** Let $C \subset V^3$ be a nonsingular algebraic curve and $L \subset V^3$ be a smooth manifold. Then $C$ can be moved by an $\varepsilon$-isotopy to a nonsingular algebraic curve $C'$ which is transversal to $L$.

**Proof.** Let $F^2$ be the boundary of a small closed tubular neighborhood of $C$ in $V$. $F$ is a circle bundle over $C$ and hence has a section, so after a small isotopy of $F$ we can assume that $C \subset F$. Since $F$ is null homologous, by Proposition 1, it is $\varepsilon$-isotopic to a nonsingular algebraic surface $Z$ with $C \subset Z$. By the terminology of [AKJ] $C$ is a stable algebraic set. Stable algebraic sets have the required property (Proposition 4.3 of [AKJ]).

**Lemma 4.** If $V^3$ is orientable and $F^2 \subset V^3$ is a compact orientable surface with $\partial F^2 = C \cup A$ where $A$ is a nonsingular algebraic curve, then $C$ is $\varepsilon$-isotopic to a nonsingular algebraic curve.

**Proof.** Since $V$ is orientable $F$ has a trivial normal bundle in $V$. Let $F' = \partial(F \times I) \subset V^3$ corners smoothed, and $C \cup A = \partial(F \times 0) \subset F'$. $C \cup A$ separates $F'$. Since $[F']_2 = 0$ by Proposition 1 $F'$ is $\varepsilon$-isotopic to a nonsingular algebraic surface $Z$ with $A \subset Z$. After a small isotopy of $C$
we can assume $C \subset Z$. Then $C \cup A$ separates $Z$; this means $[C]_2 = [A]_2 \in \text{AH}_1(Z; \mathbb{Z}_2)$. Hence by Proposition 1 $C$ is $\epsilon$-isotopic to a nonsingular algebraic curve $C^*$ in $Z$. $C^*$ is the required algebraic curve.

**Remark.** We can assume that the isotopy $C \rightarrow C^*$ fixes any finite number of points of $C$. This is because by Proposition 1 we can arrange that $Z$ and $C^*$ fix these points.

**Lemma 5.** If $S \subset V^3$ is an orientable surface and 

$$i_*: H_1(V - S; \mathbb{Z}_2) \rightarrow H_1(V; \mathbb{Z}_2)$$

is the map induced by the inclusion, then $\text{ker}(i_*) \subset \text{AH}_1(V - S; \mathbb{Z}_2)$.

**Proof.** From the homology exact sequence

$$H_2(V, V - S; \mathbb{Z}_2) \xrightarrow{\partial} H_1(V - S; \mathbb{Z}_2) \xrightarrow{i_*} H_1(V; \mathbb{Z}_2) \xrightarrow{\text{im}(\partial)} \text{ker}(i_*)$$

Also we have isomorphisms

$$H_2(V, V - S; \mathbb{Z}_2) \xrightarrow{\text{excision}} H_2(N, \partial N; \mathbb{Z}_2) \xrightarrow{\text{Thom}} H_1(S; \mathbb{Z}_2)$$

where $N$ is a small closed tubular neighborhood of $S$ in $V$. In particular $N$ is an $I$-bundle over $S$, and $\partial N$ is an $I^*$-bundle over $S$ ($I = S^0$). From the above isomorphism we see that elements of $\text{im}(\partial)$ are represented by the induced $I^*$-bundles $\tilde{\gamma}$ over the curves $\gamma$ of $S$.

Let $E$ be a small closed tubular neighborhood of $\gamma$ in $S$, since $S$ orientable $E \approx \gamma \times I$. Let $E'$ be the induced $I$-bundle over $E$. Let $F^2 = \partial E'$. Clearly $F^2$ is a null homologous surface in $V$ containing $\tilde{\gamma}$. Furthermore $\tilde{\gamma}$ separates $F^2$. By Proposition 1 $F^2$ can be $\epsilon$-isotoped to a
nonsingular algebraic surface $Z$. After a small isotopy of $\tilde{\gamma}$ we can assume that $\tilde{\gamma} \subset Z$. Since $\tilde{\gamma}$ separates $Z$, by Proposition 1 $\tilde{\gamma}$ is $\varepsilon$-isotopic to a nonsingular algebraic curve $\gamma^*$ in $Z$. By construction $\gamma^* \subset V - S$ and $[\gamma]_2 = [\gamma^*]_2 \in AH_1(V - S; \mathbb{Z}_2)$. \hfill $\square$

**Lemma 6.** Every element of $AH_1(V; \mathbb{Z}_2)$ can be represented by a connected nonsingular algebraic curve.

**Proof.** Let $\alpha \in AH_1(V; \mathbb{Z}_2)$ then $\alpha$ is represented by a union of nonsingular algebraic curves $C = C_1 \cup \cdots \cup C_k$. By Lemma 3 we can assume that they are disjoint. Let $S$ be the boundary of a closed tubular neighborhood of $C$. Since the normal bundle of $C$ has nowhere zero section, after an $\varepsilon$-isotopy of $S$ we can assume that $C \subset S$. Then by tubing the components of $S$ we get a connected surface $S'$ with $C \subset S'$. Let $C'_i$ be $\varepsilon$-isotopic copies of $C_i$ on $S'$ which are in general position with $C_i$. Connect $C'_i$, $i = 1, \ldots, k$, by tubes in $S'$ to get a connected curve $C' = C'_1 \# \cdots \# C'_k$ such that $C'$ is homologous to $C$ in $S'$.

By construction $[S']_2 = 0$ in $H_2(V; \mathbb{Z}_2)$, so by Proposition 1 we can $\varepsilon$-isotop $S'$ to a nonsingular algebraic surface $Z$ with $C \subset Z$. Continue to denote the isotopic copy of $C'$ in $Z$ by $C'$. Again since $[C']_2 = [C]_2 \in AH_1(Z; \mathbb{Z}_2)$ by Proposition 1, $C'$ is $\varepsilon$-isotopic to a nonsingular algebraic curve $C^*$ in $Z$. $C^*$ is connected and $\alpha = [C]_2 = [C^*]_2 \in AH_1(V; \mathbb{Z}_2)$. \hfill $\square$

**Proof of Proposition 2.** We will prove this in three steps,

**Case 1.** $V^3$ is orientable.

Let $c = [C] \in H_1(V; \mathbb{Z})$. Since $[C]_2$ is algebraic there is a nonsingular algebraic curve $A \subset V$ such that $[C] = [A] + 2b$ for some $b \in H_1(V; \mathbb{Z})$. This means if $B \subset V$ is a simple closed curve with $b = [B]$, then $A \cup 2B \cup C$ bounds an orientable surface. Here $2B$ denotes the link $B \cup B'$ where $B'$ is a parallel copy of $B$, so $2B$ is a boundary of an orientable surface $B \times I$ in $V$. By Lemma 4 we can assume that $2B$ is a nonsingular algebraic curve. Again by Lemma 4 $C$ is $\varepsilon$-isotopic to a nonsingular
algebraic curve. By the Remark following Lemma 4 we can assume that this isotopy fixes any finite number of points of $C$.

**Case 2.** $[C]_2 = 0$ in $H_1(V; \mathbb{Z}_2)$

Let $S \subset V$ be a surface representing the dual of the first Steifel-Whitney class $w_1(V)$ of $V$. We can assume that $C \cap S = \emptyset$. This is because by homological reasons $C \cap S$ must be an even number of points, and we can modify $S$ as in the picture below without affecting its homology class.

![Diagram](image)

Hence $C \subset V - S$, and by assumption $[C]_2 \in \ker(i_*)$ where

$$i_* : H_2(V - S; \mathbb{Z}_2) \to H_2(V; \mathbb{Z}_2)$$

is the induced map by inclusion. Since $[S]_2 = w_1(V)$, $S$ is orientable (exercise), so by Lemma 5 $[C]_2 \in AH_1(V - S; \mathbb{Z}_2)$. Since $V - S$ is orientable, by Case 1 $C$ is $\varepsilon$-isotopic to a nonsingular algebraic curve in $V - S$, fixing any finite number of points of $C$.

**Case 3. The general case.**

We choose a connected nonsingular algebraic curve $D$ disjoint from $C$ so that $[C]_2 = [D]_2$. Let $S$ be the boundary of a closed tubular neighborhood of $C \cup D$. As in the proof of Lemma 6 after a small isotopy of $S$ we can assume that $C \cup D \subset S$, and let $S'$ be the connected surface obtained by tubing the two components of $S$. By construction $C \cup D \subset S'$. Let $C'$ and $D'$ be $\varepsilon$-isotopic transverse copies of $C$ and $D$ in $S'$. Then by tubing $C'$ and $D'$ in $S'$ we get a curve $E = C' \# D'$ as in the picture.
By construction we have
(a) \([S']^2 = 0\) in \(H_2(V; \mathbb{Z}_2)\)
(b) \([E]^2 = [C \cup D]_2\) in \(H_1(S'; \mathbb{Z}_2)\)
(c) \([E]^2 = 0\) in \(H_1(V; \mathbb{Z}_2)\)

By Case 2 \(E\) is \(\varepsilon\)-isotopic to a nonsingular algebraic curve \(E^*\) in \(V\) fixing the points \(E \cap (C \cup D)\). After an \(\varepsilon\)-isotopy of \(S'\) we may assume \(C \cup D \cup E^* \subset S'\). By Proposition 1 (and by the remark following it) we can \(\varepsilon\)-isotop \(S'\) to a nonsingular algebraic surface \(Z\) with \(D \cup E^* \subset Z\). Let \(C'\) be the corresponding \(\varepsilon\)-isotopic copy of \(C\) in \(Z\). Since \([C']_2 = [D \cup E^*]_2 \in \text{AH}_1(Z; \mathbb{Z}_2)\) by Proposition 1. \(C'\) is \(\varepsilon\)-isotopic to a nonsingular algebraic curve \(C^*\) in \(Z\). Furthermore given any finite number of points on \(C_1\) by Proposition 1 we can require that all these isotopies fix these points.

Proof of the Theorem. First we show \((b) \Rightarrow (a)\). For every \(y \in \text{f}(M^2)\) consider \(n(y) = \max\{n \mid \text{there are } n \text{ distinct points } x_1, \ldots, x_n \in M \text{ with } f(x_i) = y \text{ for } i = 1, 2, \ldots, n\}\) = the cardinality of \(f^{-1}(y)\). \(f(M)\) is a stratified set with strata \(\{L_i\}_{i=1}^{3}\) where \(L_i\) are the \(i\)-fold point sets, \(L_i = \{y \in \text{f}(M) \mid n(y) = i\}\). Call \(d(f) = \max\{i \mid L_i \neq \emptyset\}\), then \(d(f) \leq 3\) and if \(d(f) = 3, L_3\) is a collection of points (the triple points). Let \(M_3 = f^{-1}(L_3)\). By ([AK1], Lemma 2.3) there is a unique immersion \(f'\) with \(d(f') = 2\) making the following commute

\[
\begin{array}{ccc}
M' &=& B(M, M_3) \\ \downarrow p' & \\ M & \mapleft{f} & V
\end{array}
\begin{array}{ccc}
& \Rightarrow & \\
& \downarrow \pi' & \\
B(V, L_3) &=& V'
\end{array}
\]

where the vertical maps are the blowing up maps along the centers \(M_3, L_3\). Since the points are algebraic, we can assume that \(V' \maprightarrow V\) is the algebraic blow up of \(V\) along \(L_3\).

Since \(d(f') = 2\) the 2-fold point set \(L_2 \subset V'\) of the map \(f'\) is a smooth manifold (i.e., collection of smooth circles). Let \(M_2 = (f')^{-1}L_2\). Once again by [AK1] there is a unique immersion \(f''\) with \(d(f'') = 1\) (i.e., it is an imbedding) making the following commute

\[
\begin{array}{ccc}
M'' &=& B(M', M_2) \\ \downarrow p'' & \\ M' & \mapleft{f'} & V'
\end{array}
\begin{array}{ccc}
& \Rightarrow & \\
& \downarrow \pi'' & \\
B(V', L_2) &=& V''
\end{array}
\]
where the vertical maps are the blowing up maps. In particular $M'' = M'$
and $p'' = \text{identity}$, since $M_2 \subset M'$ is codimension one.

$V' = V \# \mathbb{R}P^3$ so $H_i(V') = H_i(V) \oplus H_i(\# \mathbb{R}P^3)$ for $i = 1, 2$; in
particular $H_i(V'; \mathbb{Z}_2)$ and $H_2(V'; \mathbb{Z}_2)$ are algebraic. By Proposition 2 the
curve $L_2$ is $\varepsilon$-isotopic to a nonsingular algebraic set. We can change $f(M)$
by a small isotopy in $V$ keeping $L_3$ fixed so that the corresponding
double point set $L_2$ in $V'$ is this nonsingular algebraic set. Therefore we
can take $\pi''$ to be the algebraic blow up along $L_2$, in particular $V''$ is a
nonsingular algebraic set.

We claim that $H_2(V''; \mathbb{Z}_2)$ is algebraic. This can be seen by the
homology exact sequences

$$
\cdots \to H_2(C'') \xrightarrow{i_*} H_2(V'') \to H_2(V'', C'') \to \cdots
$$

$$
\cdots \to H_2(C') \xrightarrow{\pi_*} H_2(V') \to H_2(V', C') \to \cdots
$$

where all the homology groups have coefficient $\mathbb{Z}_2$, and $C', C''$ are closed
tubular neighborhoods of $L_2$, $(\pi'')^{-1}(L_2)$ respectively. Since $\pi''$ is degree
1 $\pi''$ is onto, and by the above diagram $\ker \pi'' = \text{im}(i_*)$ where $i$ is the
inclusion $C'' \hookrightarrow V''$. So $H_2(V''; \mathbb{Z}_2)$ is generated by the nonsingular
algebraic sets $(\pi'')^{-1}(L_2)$, and $(\pi'')^{-1}(S_i)$ where $S_i$ are surfaces in $V'$. By
Proposition 1 we can assume $S_i$ are nonsingular algebraic surfaces. By
([AK_1] Proposition 4.3) we can assume $S_i$ are transverse to $L_2$. Hence
$H_2(V''; \mathbb{Z}_2)$ is generated by nonsingular algebraic sets.

By Proposition 1 we can $\varepsilon$-isotop the smooth submanifold $f''(M'')$
to a nonsingular algebraic subset $Q$ of $V''$ by a smooth isotopy. By
([AK_1] Lemma 2.5) $\pi' \circ \pi''(Q)$ is an algebraic set. $\pi' \circ \pi''(Q)$ is isotopic
to $f(M)$ by a small isotopy. More precisely, the last remark can be seen
by applying ([AK_2] Proposition 5.5). Namely [AK_2] gives an isotopy $h_i$: $V'' \to V''$ such that

1. $h_0 = \text{Id}$,
2. $h_1(f''(M'')) = Q$,
3. $h_i^{-1}(\pi''(x)) = \pi^{-1}(x)$ for all $x \in L \subset V$, where $\pi = \pi' \circ \pi''$, $L = L_3 \cup \pi'(L_2)$.

Then we can define an isotopy

$$
g_i : V \to V \text{ by } g_i(x) = \pi h_i(y) \text{ for } \begin{cases} y = \pi^{-1}(x), & \text{if } x \notin L, \\
y \in \pi^{-1}(x), & \text{if } x \in L. \end{cases}
$$
(Notice $\pi$ is a diffeomorphism over the complement of $L$.) $g_t$ gives an isotopy of $f(M)$ to $\pi(Q)$ fixing $L$ pointwise. Also $g_t$ is smooth in the complement of $L$.

![Diagram](image)

It remains to show (a) $\implies$ (b). Clearly (a) implies $H_2(V; \mathbb{Z}_2)$ algebraic. To see $H_1(V; \mathbb{Z}_2)$ algebraic we write every simple closed curve $C \subset V^3$ as the double point of an immersion. $C$ has a normal bundle $C \times D^2 \subset V$. Then $C \times X \subset V$ where $X$ is the figure eight, so $C \times X = f(S^1 \times S^1)$ where $f: S^1 \times S^1 \to V$ is the obvious immersion. Hence by (a) $f(S^1 \times S^1)$ can be made algebraic and $C$ is the singular set of this algebraic set.

$\square$

*Note added in proof.* After writing this paper we have been informed by W. Kucharz that he had proved a special case of Proposition 2 when $V$ is orientable in “Topology of Real Algebraic Threefolds” Duke Math. Journal, vol. 53, No. 4, Dec. 1986.

**References**


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