AN EXOTIC 4-MANIFOLD

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In [1] we have constructed a fake smooth structure on a contractible 4-manifold $W^4$ relative to boundary. This is a smooth manifold $V$ with $\partial V = \partial W$ such that the identity map $\partial V \to \partial W$ extends to a homeomorphism but not to a diffeomorphism $V \to W$. This is a relative result in the sense that $V$ itself is diffeomorphic to $W$, even though no such diffeomorphism can extend the identity map on the boundary. Here we strengthen this result by dropping the boundary hypothesis at the expense of slightly enlarging $W$: We construct two compact smooth 4-manifolds $Q_1, Q_2$ which are homeomorphic but not diffeomorphic to each other. In particular no diffeomorphism $\partial Q_1 \to \partial Q_2$ can extend to a diffeomorphism $Q_1 \to Q_2$.

Let $Q_i^n, i = 1, 2,$ be the 4-manifolds obtained by attaching 2-handles to $B^4$ along knots $K_i, i = 1, 2,$ with +1-framing (see Figures 1 and 2). Clearly $Q_1$ and $Q_2$ are homotopy equivalent to $CP_0^2 = CP^2 - \text{int}(B^4)$, and it will be shown that $\partial Q_1 = \partial Q_2$.

**Theorem 1.** $Q_1$ and $Q_2$ are homeomorphic but not diffeomorphic to each other. In fact, even their interiors are not diffeomorphic to each other.

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The fact that they are homeomorphic to each other follows from [2]. The proof of Theorem 1 uses the method of [1]; as a by-product we will get the following result: Let $Q_3$ be the 4-manifold obtained by attaching a 2-handle to $B^4$ along the knot $K_3$ with $+1$-framing (Figure 3).

![Figure 3](image)

**Theorem 2.** There is a diffeomorphism $f: \partial Q_3 \to \partial Q_3$ which extends to a self-homeomorphism of $Q_3$, but $f$ cannot extend to a self-diffeomorphism of $Q_3$.

It is an easy exercise to show that $f$ extends to a homotopy equivalence, hence by [2] it extends to a homeomorphism. The existence of $Q_2$ is already contained in [1] as $W_1 \# \mathbb{C}P^2$, but not as $B^4$ with a single 2-handle. We use the usual conventions: $\approx$ for homotopy equivalence and $\approx$ for diffeomorphism. For every oriented manifold $M$, we denote the oppositely oriented manifold by $-M$. Also we denote $-\mathbb{C}P^2$ by $\overline{\mathbb{C}P^2}$.

In [1] we constructed a 1-connected compact smooth 4-manifold $M_1$ with $\partial M_1 = \partial Q_1$. $M_1$ has the properties: $M_1$ is even with signature 16 and has second betti number $b_2(M) = 22$. If $V$ is any smooth contractible manifold with $\partial M_1 = \partial V$, then if we call $\widetilde{M} = M_1 \cup \partial (-Q_1)$ and $M' = M_1 \cup \partial (-V)$ we have:

1. $\widetilde{M} \approx (3\mathbb{C}P^2) \# (20\overline{\mathbb{C}P^2})$,
2. $(M' \# \overline{\mathbb{C}P^2}) \# (k\overline{\mathbb{C}P^2}) \neq \widetilde{M} \# (k\overline{\mathbb{C}P^2})$, $k = 0, 1, 2, \ldots$.

Furthermore $\widetilde{M}$ is obtained from $M' \# \overline{\mathbb{C}P^2}$ (for some choice of $V$) by removing a contractible manifold $W$ and regluing with a diffeomorphism $f: \partial W \to \partial W$ as described in [1]. That is, for some smooth $N$ with $\partial N = \partial W$ we have:

1. $M' \# \overline{\mathbb{C}P^2} = N \cup \partial (-W)$,
2. $\widetilde{M} = N \cup f (-W)$.

Let $W_k$ be the contractible manifold of Figure 4. By [1] $\partial M_1 = \partial W_1$. We claim that $W_k \# \mathbb{C}P^2 \approx N_k$, where $N_k$ is the manifold of Figure 8.
This can be seen as follows: Figure 5 is $W_k \# \mathbb{C}P^2$, by a handle slide and an isotopy we obtain Figures 6 and 7. After cancelling the 1 and 2 handle pair in Figure 7 we obtain Figure 8 ($-k + 5$ in the figure indicates that many full twists across the two strands). Since $N_1 = Q_2$, $N_0 = Q_3$, and $W_0 = W$ of [1] we have

(a) $W_1 \# \mathbb{C}P^2 = Q_2$,
(b) $W \# \mathbb{C}P^2 = Q_3$. 

\[ \begin{align*} 
\text{Figure 4} & \quad \text{Figure 5} \\
\text{Figure 6} & \quad \text{Figure 7} \\
\text{Figure 8} & 
\end{align*} \]
Proof of Theorem 1. If \( \text{Int}(Q_1) \approx \text{Int}(Q_2) \), then by (a) \( \text{Int}(Q_1) \approx \text{Int}(W'_1 \# CP^2) \). Then \( Q_1 \) would have a smoothly imbedded \( S^2 \hookrightarrow Q_1 \) with self-intersection +1. This would imply \( Q_1 \approx W' \# CP^2 \) for some contractible \( W' \) with \( \partial W' = \partial Q_1 \). Hence

\[
\widetilde{M} \approx M_1 \cup_{\partial} (-W' \# CP^2) \approx M' \# CP^2,
\]

where \( M' = M_1 \cup_{\partial} (-W') \). This contradicts (2). q.e.d.

Proof of Theorem 2. By (b) \( W \# CP^2 = Q_3 \), hence by (i) and (ii) \( M' \# CP^2 \# CP^2 = N \cup_{\partial} (-W \# CP^2) \approx N \cup_{\partial} (-Q_3) \), and \( \widetilde{M} \# CP^2 = N \cup_f (-W \# CP^2) = N \cup_f (-Q_3) \). So if \( f: \partial(-Q_3) \to \partial(-Q_3) \) extended to a diffeomorphism of \( -Q_3 \), \( M' \# (2CP^2) \) would be diffeomorphic to \( \widetilde{M} \# CP^2 \) contradicting (2). q.e.d.

Remark. \( Q_1 \) is obtained from \( Q_2 \) by removing the contractible manifold \( W \) from the interior and glueing it back by a diffeomorphism (i.e., Gluck contraction to \( W \)). That is, we can write \( Q_1 = P \cup_{\partial} W \) and \( Q_2 = P \cup_f W \) for some smooth \( P \) with \( \partial P = \partial Q_1 \sqcup \partial W \).

This can be seen as follows. \( Q_1 \) (Figure 9) is diffeomorphic to Figure 10, and \( Q_2 \) (Figure 12) is diffeomorphic to Figure 11. There is a diffeomorphism \( F \) between the boundaries of Figures 10 and 11, induced by the obvious involution \( f \) (of [1]). \( F \) carries the loop \( \gamma \) of Figure 10 to the loop \( F(\gamma) \) of Figure 11. \( \gamma \) and \( F(\gamma) \) bound obvious discs \( D_1, D_2 \) in \( Q_1 \) (Figure 10) and \( Q_2 \) (Figure 11), respectively. We can extend \( F \) across these discs:
Let \((D_i \times B^2, \partial D_i \times B^2) \hookrightarrow (Q_i, \partial Q_i), \ i = 1, 2,\) be the tubular neighborhoods of these discs. Then obviously \(Q_i - D_i \times B^2 \approx W\) for \(i = 1, 2,\) and \(F\) induces \(f: \partial W \to \partial W.\) q.e.d.

\begin{align*}
\text{Figure 9} & \quad \approx \quad \text{Figure 10} \\
\text{Figure 11} & \quad = \quad \text{Figure 12}
\end{align*}

References
