The topology of real algebraic sets with isolated singularities

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In this paper we give a topological classification of real algebraic sets with isolated singularities, showing that they are exactly smooth closed manifolds with smooth subpolyhedra crushed to points.

The question of which topological spaces are homeomorphic to real algebraic sets (solutions of polynomial equations in Euclidean space) has been long studied. In 1936 Seifert showed that any smooth compact stably parallelizable manifold is diffeomorphic to a component of an algebraic set [12] and in 1952 Nash extended this result to all smooth compact manifolds [11]. In 1973 Tognoli showed that any smooth compact manifold is diffeomorphic to a nonsingular algebraic set [13], so at least compact nonsingular algebraic sets are classified. Little has been done with singular algebraic sets however, since the transversality arguments used by Seifert, Nash and Tognoli no longer apply except in some special cases. One could use stability of singularities such as Kuiper [7] and Akbulut [1] used to show certain nonsmoothable PL manifolds are algebraic sets or one could use the projective version of Seifert-Nash-Tognoli as King did [6], but one could still not hope these techniques would allow even a characterization of isolated singularities. To get around this problem we take a cue from Hironaka’s resolution of singularities [4]. The idea is to take a ‘topological resolution’ of a space if it exists. We can apply transversality techniques (Seifert-Nash-Tognoli) to the resolved space and then blow down algebraically and end up with the original space as an algebraic set. It seems likely that this technique allows one to classify all algebraic sets but in any case, we show that it classifies all algebraic sets with isolated singularities. In future papers we will use this technique to show, for instance, that all compact PL manifolds are homeomorphic to real algebraic sets [17] and that 2-dimensional real algebraic sets are topologically characterized as polyhedra satisfying Sullivan’s even local Euler characteristic condition [16].

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This paper gives details of the proof which was sketched in our announcement [2]. A version of the proof is discussed in [19].

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1. Remarks on algebraic sets

Definition. If $k$ is a field, a $k$ algebraic set is a set $V$ of the form $V(I) = \{ x \in k^n \mid p(x) = 0 \text{ for all } p \in I \}$ where $I$ is a set of polynomial functions from $k^n$ to $k$.

In this paper ‘algebraic set’ will always mean ‘real algebraic set.’ We will only occasionally refer to complex algebraic sets in the middle of some proof and we never mention algebraic sets for other fields. A good reference for the basic properties of real algebraic sets is [14].

If $V$ is a $k$ algebraic set, $\mathcal{I}(V)$ denotes the ideal of polynomials $p$ so that $p(V) = 0$.

Note that if $V$ is a real algebraic set then $V = p^{-1}(0)$ for some polynomial $p$. (We may take $p$ to be the sum of squares of generators of the ideal $\mathcal{I}(V)$. $\mathcal{I}(V)$ is always finitely generated.)

We denote $RP^* = R^{*+1} - 0/\sim$ where $x \sim y$ if $x = \lambda y$ for some $\lambda \in R - 0$. A point in $RP^*$ is denoted $(x_0; x_1; \cdots; x_n)$ if it is the equivalence class containing $(x_0, x_1, \cdots, x_n)$. A projective algebraic set $V \subset RP^*$ is a set of the form

$$V(h) = \{(x_0; x_1; \cdots; x_n) \mid h(x_0, x_1, \cdots, x_n) = 0\}$$

where $h$ is a homogeneous polynomial (i.e., $h(\lambda x) = \lambda^d h(x)$ for all $x$ and $\lambda$).

Given an algebraic set $V \subset R^n$, there is an associated complex algebraic set $V_c \subset C^n$ where $V_c$ is the smallest complex algebraic set containing $V$. Also $V_c = V(\mathcal{I}_c(V))$ where $\mathcal{I}_c(V)$ is the ideal of complex polynomials generated by $\mathcal{I}(V)$ where the real polynomials in $\mathcal{I}(V)$ are thought of as complex polynomials [14].

Definition. A point $x$ in an algebraic set $V \subset R^k$ is called nonsingular of dimension $d$ in $V$ if there are polynomials $p_i \in \mathcal{I}(V)$, $i = 1, \cdots, k - d$, and a neighborhood $U$ of $x$ in $R^k$ so that

i) $V \cap U = U \cap \bigcap_{i=d}^{k-1} p_i^{-1}(0)$,

ii) the gradients $(\nabla p_i)_x, i = 1, \cdots, k - d$, are linearly independent.

Definition. For an algebraic set $V$,

$$\dim V = \max \{d \mid \text{there is an } x \in V \text{ which is nonsingular of dimension } d\}.$$
Definition. For an algebraic set $V$,

\[ \text{Nonsing } V = \{ x \in V | x \text{ is nonsingular of dimension dim } V \} \]

and

\[ \text{Sing } V = V - \text{Nonsing } V. \]

Some facts are that Sing $V$ is an algebraic set and dim Sing $V < \text{dim } V$. Also, Nonsing $V$ is a smooth manifold of dimension dim $V$. Notice that if $x$ is nonsingular of some dimension in $V$ then $x$ is not necessarily in Nonsing $V$.

We use a number of elementary properties: union, intersection and cartesian product of algebraic sets are algebraic sets

\begin{align*}
    p^{-1}(0) \cap q^{-1}(0) &= (p^* + q^*)^{-1}(0), \\
    p^{-1}(0) \cup q^{-1}(0) &= (pq)^{-1}(0), \\
    p^{-1}(0) \times q^{-1}(0) &= r^{-1}(0)
\end{align*}

where $r(x, y) = p^*(x) + q^*(y)$. Disjoint union of $U$ and $V$ is $U \times 0 \cup V \times 1$ which is an algebraic set.

We will often identify an algebraic set $V \subset \mathbb{R}^n$ with $V \times 0 \subset \mathbb{R}^n \times \mathbb{R}^k$ or $0 \times V \subset \mathbb{R}^k \times \mathbb{R}^*.

Given a polynomial $p: \mathbb{R}^n \to \mathbb{R}$ of degree $d$ we may associate a homogeneous polynomial $p^*: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which we call the homogenization of $p$. This $p^*$ is defined by $p^*(t, x) = t^d p(x/t) = \sum_{i=0}^d t^{d-i} p_i(x)$ where $p_i$ is the sum of the monomials of $p$ of degree $i$. Hence if $\theta: \mathbb{R}^n \to RP^*$ is the imbedding $\theta(x_1, \ldots, x_n) = (1: x_1: \ldots: x_n)$ and $V(p^*) \subset RP^*$ is the projective algebraic set given by the zeros $p^*$ we have $V(p^*) \cap \theta(R^n) = \theta(p^{-1}(0))$. We call $p$ overt if $V(p^*) = \theta(p^{-1}(0))$ (thus $p$ is not "hiding" any zeros at infinity). Notice that $RP^* - \theta(R^n) = \{(0: x_1: \ldots: x_n) \in RP^* \}$ so $p$ is overt if and only if $p^{-1}(0)$ is either empty or 0.

For example $p(x, y) = x^2 + y^2 - x + 2y$ is overt but $q(x, y) = x^2 + y^3 - x + 2y$ is not overt since $q_i(x, y) = y^i$.

Definition. An algebraic set $V$ is called projectively closed if $V = p^{-1}(0)$ for an overt polynomial $p$.

From the above discussion we see that $V$ is projectively closed if and only if $\theta(V)$ is a projective algebraic subset of $RP^n$.

Unions, intersections and cartesian products of projectively closed algebraic sets are projectively closed. Also, an algebraic subset of a projectively closed algebraic set is projectively closed.

Definition. Let $V$ and $W$ be algebraic sets and let $U$ be an algebraic subset of $V$. Suppose $W \subset \mathbb{R}^n$. Then a function $f: V - U \to W$ is called a rational function if there are polynomials $p: V \to \mathbb{R}^n$ and $q: V \to \mathbb{R}$ so that
$q^{-1}(0) \subset U$ and $f(x) = p(x)/q(x)$ for all $x \in V - U$. If $U$ is empty then we say $f$ is an entire rational function.

**Lemma 1.3.** Let $U, V, W$ and $X$ be algebraic sets with $X \subset V$ and $U \subset W$. Suppose that $f: V - X \rightarrow W$ is a rational function. Then $X \cup f^{-1}(U)$ is an algebraic set. In particular, if $f$ is entire then $f^{-1}(U)$ is an algebraic set.

**Proof.** Let $f = p/q$, $V = r^{-1}(0)$ and $U = s^{-1}(0)$ where $p, q, r$ and $s$ are polynomials and $q^{-1}(0) \subset X$. Let $s^*$ be the homogenization of $s$. Consider the polynomial $t(x) = r^*(x) + (s^*(q(x), p(x)))^2$. Then $t^{-1}(0) = f^{-1}(U) \cup q^{-1}(0)$ since $s^*(q(x), p(x)) = 0$ implies $q(x) = 0$ or $s(p(x)/q(x)) = 0$. So $X \cup t^{-1}(0) = X \cup f^{-1}(U)$ is an algebraic set.

**Lemma 1.4.** Under the same hypotheses as Lemma 1.3 suppose $z \in f^{-1}(U)$ is nonsingular of dimension $r$ in $V$ and $f(z)$ is nonsingular of dimension $s$ in $U$ and nonsingular of dimension $t$ in $W$, and $f$ is transverse to $U$ at $z$. Then $z$ is nonsingular of dimension $r + s - t$ in $f^{-1}(U)$.

**Proof.** Let $W \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$. Then there are polynomials $p_i: (\mathbb{R}^n, U) \rightarrow (\mathbb{R}, 0)$, $i = 1, \ldots, n - s$ so that $p_i^{-1}(0) \supset W$, $i > t - s$ and the gradients $\nabla p_i(f(z))$ are linearly independent. There are also polynomials $q_i: (\mathbb{R}^m, V) \rightarrow (\mathbb{R}, 0)$, $i = 1, \ldots, m - r$ so that the $\nabla q_i(x)$ are linearly independent. Let $f = p/q$ with $p$ and $q$ polynomials and pick $d > \text{deg}p_i$ for all $i$. Define $q_i(x) = q^d(x)p_i(x) + s^i(x)f(x)$ for $m - r < i \leq m - r + t - s$. Then the $q_i$, $i = 1, \ldots, m - r + t - s$ are all polynomials and since $f$ is transverse to $U$, the gradients $\nabla q_i(z)$ are linearly independent. Also if $x \in \bigcap_{i=r+1}^{n-s} q_i^{-1}(0)$ and $x$ is near $z$, then $f(x)$ is near $f(z)$, so since $f(x) \in W$ and $p_i f(x) = 0$, $i = 1, \ldots, t - s$ then $f(x) \in U$. Since $q_f(f^{-1}(U)) = 0$ this shows $z$ is nonsingular of dimension $m - (m - r + t - s)$ in $f^{-1}(U)$.

**Lemma 1.5.** Let $X$ be an algebraic set and suppose $X_i \subset X$, $i = 1, \ldots, k$ are pairwise disjoint algebraic sets and $u_i: X_i \rightarrow \mathbb{R}^n$ are entire rational functions. Then there is an entire rational function $u: X \rightarrow \mathbb{R}^n$ so that $u|_{X_i} = u_i$ for each $i = 1, \ldots, k$.

**Proof.** It is sufficient to prove the case $k = 2$. Pick polynomials $p_i: X \rightarrow \mathbb{R}$ so that $p_i^{-1}(0) = X_i$. Let $u_i(x) = q_i(x)/r_i(x)$ for polynomials $q_i: X \rightarrow \mathbb{R}$ and $r_i: X \rightarrow \mathbb{R}$ so that $r_i^{-1}(0) \cap X_i$ is empty. Then we may define $u$ by

$$
u(x) = r_i(x)q_i(x)p_i^2(x)/[q_i(x) + p_i(x)]^2$$

+ $r_i(x)q_i(x)p_i^2(x)/[q_i(x) + p_i(x)](r_i(x) + p_i(x))].$

**Lemma 1.6.** Suppose $V \subset \mathbb{R}^n$ is an algebraic set and $W \subset \text{Nonsing} V$ is an algebraic subset with $\dim W = \dim V$. Then $V - W \subset \mathbb{R}^n$ is an algebraic set. Also $\text{Sing} (V - W) = \text{Sing} V$ unless $W = \text{Nonsing} V$. 


Proof. We will temporarily talk about complex algebraic sets. Let $V_c \subset \mathbb{C}^n$ and $W_c \subset \mathbb{C}^n$ be the complexifications of $V$ and $W$. Let $U$ be an irreducible algebraic set in $V$ so that $U - W$ is nonempty. Let $U_c$ be the complexification of $U$. Then $U_c$ is irreducible by [14]. Nonsingularity in $V$ and $W$ implies that $V_c$ is a 2 dim $V$ dimensional manifold near $W$ and $W_c$ is a 2 dim $V$ dimensional manifold near $W$ so there is a neighborhood of $W$ in $V_c$ which is contained in $W_c$. But the nonsingular points of a complex variety are dense, so if $U \cap W$ is nonempty we must have $\dim(U_c \cap W_c) = \dim(U_c \cap V_c) = \dim(U_c)$ (since $U_c \subset V_c$). This contradicts Lemma 2 of [14], so $U \cap W$ is empty. Hence $V - W$ is the union of all the irreducible components of $V$ which are not contained in $W$. This is an algebraic set. It is immediate from the definition that $\text{Nonsing} (V - W) \supset (\text{Nonsing} V) - W$. On the other hand, if $x \in \text{Nonsing}(V - W)$ and $p_i: (\mathbb{R}^n, V - W) \to (\mathbb{R}, 0)$, $i = 1, \cdots, n - \dim V$, are polynomials whose gradients at $x$ are linearly independent and $q: R^n \to R$ is any polynomial with $q^{-1}(0) = W$, then the gradients of $qq_i: (\mathbb{R}^n, V) \to (\mathbb{R}, 0)$ are linearly independent at $x$ so $x \in \text{Nonsing} V$ if $W \neq \text{Nonsing} V$.

The reader should be aware that the requirement $W \subset \text{Nonsing} V$ is quite necessary. It is not even enough to assume that $W$ is a component of $V$ and a nonsingular variety. For instance the algebraic set $V \subset \mathbb{R}^2$ given by

$$0 = (x^2(x - 1) - y^2)(x(x + 1) + y^2)$$

is a smooth manifold (see Figure 1) but if we take away the algebraic subset $W$ given by $x(x + 1) + y^2 = 0$, we no longer have an algebraic set; we must add back the point $(0, 0)$. The problem is that although $W$ by itself is a nonsingular algebraic set, the point $(0, 0) \in W$ is a singular point of $V$. In general, all we know is that $(V - W) \cup \text{Sing} V$ is an algebraic set.
LEMMA 1.7. If \( p: \mathbb{R}^n \to \mathbb{R} \) is a polynomial with \( p^{-1}(0) \) compact and if \( N \) is any neighborhood of \( p^{-1}(0) \), then there are a number \( d \) and a positive number \( c \) so that \( |p(x)| \geq c|x|^d \) for all \( x \in \mathbb{R}^n - N \).

Proof. Assume \( p \) is not constant; otherwise the lemma is trivial. We may also assume \( \text{int}(N) \) is nonempty. Let \( p^*: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) be the homogenization of \( p \). Pick \( z \in \text{int}(N) \). Let \( \theta: \mathbb{R}^n - z \to \mathbb{R}^n - z \) be inversion through the unit sphere around \( z \), \( \theta(x) = x + (x - z)/(x - z)^2 \). Note \( \theta = \theta^{-1} \).

Let \( q: \mathbb{R}^n \to \mathbb{R} \) be the polynomial \( q(x) = p^*(|x - z|^2, z|x - z|^3 + x - z) \). Then \( q^{-1}(0) = \{z\} \cup \theta(p^{-1}(0) - z) \). Since \( p^{-1}(0) \) is compact, \( z \) is an isolated zero of \( q \). Also \( \theta(\mathbb{R}^n - N) \) is bounded so by the Lojasiewicz inequality [9] there are positive numbers \( b \) and \( a \) so that \( |q(x)| \geq a|x - z|^b \) for all \( x \in \theta(\mathbb{R}^n - N) \).

If \( p \) has degree \( e \), then \( q(x) = |x - z|^{2e} p \theta(x) \) so for \( x \in \mathbb{R}^n - N \),

\[
|p(x)| = |\theta(x) - z|^{-2e} |\theta(x)| \geq |\theta(x) - z|^{-2e} a |\theta(x) - z|^{b}
\]

but we can find \( c > 0 \) so \( a|x - z|^{2e-b} \geq c|x|^{2e-b} \) for all \( x \in \mathbb{R}^n - N \); hence we are done.

As a consequence of Lemma 1.7 we see that any compact algebraic set \( V \) is the set of zeros of a proper polynomial (proper means inverse images of compacta are compact). To see this take any polynomial \( p \) with \( V = p^{-1}(0) \) and find \( c \) and \( d \) as in Lemma 1.7. Then \( (1 + |x|^3)^{d+1} p(x) \) is a proper polynomial whose zeros are \( V \).

2. Constructing smooth algebraic sets

We now wish to approximate smooth manifolds by algebraic sets. This section includes the Seifert-Nash-Tognoli results but we must make them much fancier to be useful later. If one strips away all the extra trimmings, one obtains easier proofs of the Nash-Tognoli results, mostly because of Lemma 2.4.

Lemma 2.2 is a generalization of Seifert’s codimension 1 result, and Proposition 2.8 generalizes Nash’s and Tognoli’s results. As many of the lemmas are rather technical, we sometimes precede them with a paraphrase so that the reader can understand the gist of the lemma. Lemma 2.1 allows one to approximate a smooth function by an entire rational function which equals some specified rational function on a set \( L \).

LEMMA 2.1. Suppose \( K \subset \mathbb{R}^n \) is compact and \( L \subset K \) is an algebraic set and \( f: \mathbb{R}^n \to \mathbb{R} \) is a smooth function so that for each \( p \) in \( L \) there is an algebraic set \( M \) containing \( L \) with \( f(x) = 0 \) for all \( x \) in \( M \) near \( p \) and also \( p \) is a nonsingular point of dimension \( d \) in \( M \) for some \( d \). Then there are
polynomials \( h: (R^n, L) \rightarrow (R, 0) \) which approximate \( f \) arbitrarily closely near \( K \).

In addition, if \( f \) has compact support then we may approximate \( f \) arbitrarily closely on all of \( R^n \) by entire rational functions \( u: (R^n, L) \rightarrow (R, 0) \). More precisely, given \( \varepsilon > 0 \) and an integer \( k \) we may find an entire rational function \( u: (R^n, L) \rightarrow (R, 0) \) so that \( |D(f - u)_x| < \varepsilon \) for all \( x \in R^n \) and all partial derivatives \( D \) of order \( \leq k \) and \( \geq 0 \).

**Proof.** We know that \( f = \sum_{i=1}^{k} a_i p_i \) in a neighborhood of \( L \) in \( R^n \) where \( p_i \) are polynomials vanishing on \( L \) and \( a_i \) are smooth functions. (To see this, take any \( p \) in \( L \) and let \( M \) be as above. Then as in [6], Lemma 2 or [13], we get such an expression for \( f \) in a neighborhood of \( p \), where the polynomials vanish on \( M \). Now piece together with a partition of unity.) Approximate the \( a_i \) near \( K \) by polynomials \( b_i \). Pick any polynomial \( v: R^n \rightarrow R \) so that \( v^{-1}(0) = L \) and let \( w: R^n \rightarrow R \) be the smooth function which equals \( (f - \sum a_i p_i)/v \) on \( R^n - L \) and is 0 on \( L \). Approximate \( w \) near \( K \) by a polynomial \( q \). Then \( qv + \sum b_i p_i \) is a polynomial approximation of \( f \) near \( K \) which is 0 on \( L \).

We now prove the second part. Let \( S^n \subset R \times R^n \) be the unit \( n \) sphere and let \( \theta: S^n - (1, 0) \rightarrow R^n \) be stereographic projection \( \theta(t, x) = x/(1 - t) \). Then since \( f \) has compact support, the function \( g: S^n \rightarrow R \) is smooth where \( g(1, 0) = 0 \) and \( g(x) = f \theta(x) \) for \( x \in S^n - (1, 0) \). By Lemmas 1.3 and 1.4, \( (1, 0) \cup \theta^{-1}(L) \) is an algebraic set and if \( p, d \) and \( M \) are as in the lemma statement then \( \theta^{-1}(p) \) is a nonsingular point of dimension \( d \) in \( \theta^{-1}(M) \). Also \( g = 0 \) on a neighborhood of \( (1, 0) \) in \( S^n \). Hence by the first part of this lemma we may closely approximate \( g \) by a polynomial \( r: (S^n, (1, 0) \cup \theta^{-1}(L)) \rightarrow (R, 0) \). A little calculus exercise shows that if this approximation is \( C^k \) close, then \( |D(f - r \circ \theta^{-1})_x| < \varepsilon \) for all \( x \in R^n \) and partial derivatives \( D \) of order \( \geq 0 \) and \( \leq k \). So letting \( u = r \circ \theta^{-1} \), we are done since \( \theta^{-1}(x) = (|x|^2 - 1, 2x)/(|x|^2 + 1) \) is an entire rational function.

**Definition.** Let \( W \) be a topological space and \( M \subset W \). We say that \( M \) compactly separates \( W \) if there are closed sets \( W_0 \) and \( W_1 \) so that \( W = W_0 \cup W_1 \) and \( M = W_0 \cap W_1 \) and \( W_1 \) is compact.

**Lemma 2.2 (Paraphrased).** A codimension 1 manifold \( M \) which compactly separates an algebraic set \( W \) may be isotoped in \( W \) to an algebraic set. This isotopy can fix nice subsets \( L \) of \( M \).

**Lemma 2.2.** Suppose \( L, V \) and \( W \) are algebraic sets and \( M \) is a compact, codimension 1, boundaryless smooth submanifold of Nonsing \( W \) which compactly separates \( W \). Suppose also that \( L \subset \text{Nonsing} \ V \subset M \). Then there
are arbitrarily small isotopies of $M$ in Nonsing $W$ which fix $L$ and carry $M$ to a nonsingular algebraic set. If $L$ is projectively closed, these small isotopies can carry $M$ to a nonsingular projectively closed algebraic set.

Proof. By the remark after Lemma 1.7, we may pick a proper polynomial $q: \mathbb{R}^n \to \mathbb{R}$ so that $q^{-1}(0) = L$. Pick $W_0$ and $W_1$ so $M = W_0 \cap W_1$, $W_0 \cup W_1 = W$ and $W_1$ is compact. We may pick a smooth function $f: W \to \mathbb{R}$ so that $f|_{W_0}$ is nonnegative, $f|_{W_1}$ is nonpositive, $f^{-1}(0) = M$, 0 is a regular value of $f$ and $f(x) = q^{2}(x)$ for all $x$ outside some compact subset of $W$. By Lemma 2.1 we may approximate $f - q^{2}$ by a rational function $u: (\mathbb{R}^n, L) \to (\mathbb{R}, 0)$.

Let $v = u + q^{2}$. Then $v$ approximates $f$ so $v^{-1}(0)$ is near $M$ since $f$ is proper. Hence by transversality, there is a small isotopy of $M$ to $v^{-1}(0)$ which fixes $L$. But $v^{-1}(0)$ is a nonsingular algebraic set by Lemmas 1.3 and 1.4, so we have proved the first half of the lemma. In case $L$ is projectively closed, the polynomial $q$ can be overt. A glance at the proof of Lemma 2.1 shows that
\[ v(x) = q^{2}(x) + r(1 - 2/(|x|^2 + 1), 2x/(|x|^2 + 1)) \]
for some polynomial $r$. If $r$ has degree $d$ then
\[ v(x) = (q^{2}(x)(1 + |x|^2)^d + p(x))/(1 + |x|^2)^d \]
where $p(x)$ is some polynomial with degree $p(x) \leq 2d$. Hence the highest degree terms of $q^{2}(x)(1 + |x|^2)^d + p(x)$ are $|x|^{2d}$ times the highest degree terms of $q^{2}$. So $q^{2}(x)(1 + |x|^2)^d + p(x)$ is overt and $v^{-1}(0)$ is projectively closed. Q.E.D.

Consider the algebraic set
\[ G(n, k) = \{ L \in \mathbb{R}^{n^2} | L^2 = L, \text{ L symmetric, Trace L = k} \} . \]
Here $L$ is thought of as an $n \times n$ matrix. The algebraic set $G(n, k)$ is nonsingular as the reader may verify. Notice that the identification of a $k$-plane in $n$-space with the matrix of orthogonal projection onto the plane gives a diffeomorphism of $G(n, k)$ to the Grassmanian of $k$-planes in $n$-space. We also have the universal $k$-plane bundle
\[ E(n, k) = \{ (L, y) \in \mathbb{R}^{n^2+n} | L \in G(n, k), Ly = y \} . \]

The algebraic set $G(n, 1)$ is diffeomorphic to real projective $n - 1$ space. The diffeomorphism $\theta: RP^{n-1} \to G(n, 1)$ is in fact defined by $\theta(y_1; \ldots; y_n) = L$ where $L_{ij} = y_i y_j/\left(\sum_{k=1}^{n} y_k^2\right)$. This is a very nice (classical) map since a subset $V \subset RP^{n-1}$ is a projective algebraic set if and only if $\theta(V)$ is an algebraic subset of $G(n, 1)$. To see this, let $p: \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let $p^*$ be
its homogenization. Then
\[ \theta^{-1}(p^{-1}(0) \cap G(n, 1)) = \{(y_1, \ldots, y_n) \in RP^{n-1} \mid p^*(\sum_{k=1}^{n} y_k^2, y_1, y_2, \ldots, y_n^2) = 0\} \]
which is a projective algebraic set. Conversely, if \( r: R^* \to R \) is a homogeneous polynomial of degree \( d \) then \( r^2 \) has even degree so there is a homogeneous polynomial \( p: R^{n^2} \to R \) such that
\[ r(y_1, y_2, \ldots, y_n)/(y_1 y_2)^d = p(y_1, \ldots, y_n) . \]

Hence \( \theta(r^{-1}(0)) = \theta(r^2)^{-1}(0) = p^{-1}(0) \) is an algebraic set. For instance if \( r: R^* \to R \) is a homogeneous polynomial then \( r^2 \) has even degree so there is a homogeneous polynomial \( p: R^{n^2} \to R \) such that
\[ r(y_1, y_2, \ldots, y_n)/(y_1 y_2)^d = p(y_1, \ldots, y_n) . \]

**Lemma 2.3.** Suppose \( V \subset R^* \) is an algebraic set. Then there is a rational function \( u: \text{Nonsing } V \to G(n, k) \) so that for each \( x \in \text{Nonsing } V \), \( u(x) \) is the normal \( k \) plane to \( V \) at \( x \) (\( k = n - \dim V \) and we think of \( G(n, k) \) as the Grassmanian).

**Proof.** Pick a set of generators \( q_i \) for \( \mathcal{I}(V) \), \( i = 1, \ldots, m \). For each cardinality \( k \) subset \( I = (i_1, \ldots, i_k) \) of \( \{1, 2, \ldots, m\} \) let
\[ U_I = \{x \in V \mid \text{the gradients } \nabla q_{i_1}, \ldots, \nabla q_{i_k} \text{ at } x \text{ are linearly independent}\} . \]

Then for \( x \in U_I \), the matrix \( K_x \) of orthogonal projection to the \( k \) plane normal to \( V \) at \( x \) is the matrix so that
\[ K_x(y) = \sum_{s=1}^{k} a_s^t(x, y) \nabla q_{i_s}(x) \quad \text{for all } y \in R^* \]
where
\[
\begin{pmatrix}
a_1^t(x, y) \\
\vdots \\
a_k^t(x, y)
\end{pmatrix}
= F_I^{-1}(x)
\begin{pmatrix}
y \cdot \nabla q_{i_1}(x) \\
\vdots \\
y \cdot \nabla q_{i_k}(x)
\end{pmatrix}
\]
and \( F_I(x) \) is the \( k \times k \) matrix with \( s, t^{th} \) component \( \nabla q_{i_s}(x) \cdot \nabla q_{i_t}(x) \). By Cramer's rule the coordinates of \( K_x \) are rational functions so \( K_x = P_I(x)/Q_I(x) \) for \( x \in U_I \) where \( P_I: R^* \to R^{n^2} \) and \( Q_I(x) = \det F_I(x) \). So we may define
\[ P(x) = \sum_I Q_I(x) P_I(x) \]
and
\[ Q(x) = \sum_I Q_I^t(x) . \]

Then \( u(x) = P(x)/Q(x) \) has the required properties. (Recall that \( a/b = c/d \) implies \((a + c)/(b + d) = a/b \). Also if \( Q(x) = 0 \) for \( x \in V \) then \( \det F_I(x) = 0 \) for all \( I \) so \( x \in \text{Sing } V \).

**Lemma 2.4 (Paraphrased).** A smooth function from a compactum \( K \) to a nonsingular algebraic set \( W \) can be approximated by an entire rational function at the expense of wiggling \( K \) a bit. A relative version is also true.
if we desire the approximating function to equal a specified rational function $u$ on some algebraic subset $L$.

**Lemma 2.4.** Let $L$, $V$, $X$, and $W$ be algebraic sets with $L \subset \text{Nonsing } V$, $V \subset X \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ and suppose $K \subset X$ is compact and $N \subset K$ is a neighborhood of $L$ in $V$ and $f: K \to \text{Nonsing } W$ is smooth and there is an entire rational function $u: (\mathbb{R}^n, L) \to (\mathbb{R}^m, W)$ so that $f|_N = u|_N$.

Then there is an algebraic set $U \subset X \times R^m \subset \mathbb{R}^n \times \mathbb{R}^m$ and a subset $J \subset U \cap K \times \mathbb{R}^m$ and an entire rational function $g: U \to W$ and a smooth function $\varphi: (K, L) \to (\mathbb{R}^m, 0)$ so that

i. $J = \{(x, \varphi(x)) | x \in K\}$.

ii. $J$ is a relatively open subset of $U \cap K \times \mathbb{R}^m$.

iii. The function $x \to g(x, \varphi(x))$ can be as close to $f$ as we wish (for some choice of $U$, $J$, $g$, $\varphi$).

iv. $g(x, 0) = f(x)$ for $x \in L$.

v. If $x \in K$ is nonsingular of dimension $s$ in $X$, then $(x, \varphi(x))$ is nonsingular of dimension $s$ in $U$.

**Proof.** By Lemma 2.3 we may pick polynomials $P: \mathbb{R}^m \to \mathbb{R}^m$ and $Q: \mathbb{R}^m \to \mathbb{R}$ so $Q^{-1}(0) = \text{Sing } W$ and $P(x)/Q(x) \in G(m, m - \text{dim } W)$ is orthogonal projection to the normal plane to $W$ at $x \in \text{Nonsing } W$. Using Lemma 2.1, pick a polynomial map $s: (\mathbb{R}^n, L) \to (\mathbb{R}^m, 0)$ which approximates $f - u$ near $K$ and let $v = s + u$. Define

$$U = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | x \in X, v(x) + y \in W, \quad P(v(x) + y)(y) = Q(v(x) + y) \cdot y\},$$

$$g(x, y) = v(x) + y,$$

$$\varphi(x) = \text{the vector from } v(x) \text{ to the closest point to } v(x) \text{ on } W,$$

and

$$J = \{(x, y) \in U | x \in K, y = \varphi(x)\}.$$

We are assuming that the approximation $v$ to $f$ is close enough so that for each $x \in K$, $v(x)$ has a unique closest point on $W$ and this point is in Nonsing $W$. Then $U$ is just the set of $(x, y - v(x))$ where $x \in V$, $y \in W$ and either $y \in \text{Sing } W$ or else $y \in \text{Nonsing } W$ and $v(x) - y$ is perpendicular to $W$ at $y$. With this interpretation it is easy to prove i, ii, iii and iv. To prove v, note $(x, \varphi(x))$ is nonsingular of dimension $s + m$ in $X \times \mathbb{R}^m$ and the function from $X \times \mathbb{R}^m$ to $\mathbb{R}^n \times \mathbb{R}^m$ taking $(x, y)$ to $(v(x) + y, y)$ is an entire rational function transverse to

$$Z = \{(z, y) \in W \times \mathbb{R}^m | P(z)(y) = Q(z) \cdot y\}$$

near $(x, \varphi(x))$. But if $z \in \text{Nonsing } W$ and $(z, y) \in Z$, then $(z, y)$ is nonsingular.
of dimension \( m \) in \( Z \) so by Lemma 1.4, \((x, \varphi(x))\) is nonsingular of dimension \( s + m + m - 2m \) in \( U \).

Notice that if we wished, we could assume \( \dim U = \dim V \) by replacing \( U \) with some \( \text{Sing}(\text{Sing}(\cdots(\text{Sing} U)\cdots)) \) since \( \dim \text{Sing}(Y) < \dim Y \) for any algebraic set \( Y \).

**Definition.** An algebraic set \( X \) has totally algebraic homology if for every \( \alpha \in H_0(X, \mathbb{Z}/2\mathbb{Z}) \) there are a positive integer \( n \) and a nonsingular projectively closed \( k \) dimensional algebraic subset \( V \subset X \times \mathbb{R}^n \) so that \( \pi_*([V]) = \alpha \) where \( \pi: V \to X \) is induced by projection and \([V] \in H_k(V, \mathbb{Z}/2\mathbb{Z})\) is the fundamental class of \( V \).

Actually, the requirements nonsingular and projectively closed could be replaced by the requirement that \( V \) be compact and we would get an equivalent definition of totally algebraic homology. A sketch of why this is true follows. Let \( V' \) be the smallest projective algebraic set containing \( V \). Notice that \( V' - V \) is a projective algebraic set so \( \dim(V' - V) < \dim V = k \). Hence \( V' - V \subset \text{Sing} V' \). By resolution of singularities [4] we have a nonsingular \( k \) dimensional projective algebraic set \( W \subset \mathbb{R}P^m \) and a rational function \( r: W \to V' \) so that \( r^{-1}(\text{Nonsing} V') \) is dense in \( W \) and \( r|_{r^{-1}(\text{Nonsing} V')} \) is a diffeomorphism onto \( \text{Nonsing} V' \). Hence \( r(W) \subset V \) and \( r_*([W]) = [V'] \). Let \( \theta: \mathbb{R}P^m \to G(m + 1, 1) \) be the canonical isomorphism; then \( \{(r(x), \theta(x)) \in V \times G(m + 1, 1) | x \in W\} \) is a nonsingular projectively closed algebraic subset of \( X \times \mathbb{R}^n \times \mathbb{R}^{(m+1)\mathbb{Z}} \), which gives the same homology class as \([V]\).

**Lemma 2.5.** An algebraic set \( X \) has totally algebraic homology if and only if every element of \( \mathfrak{N}_*(X) \) (unoriented bordism of \( X \)) is represented by some \( \pi: V \to X \) where \( V \) is a nonsingular projectively closed algebraic subset of \( X \times \mathbb{R}^n \) for some \( n \) and \( \pi \) is induced by projection.

**Proof.** One way is trivial since \( ev: \mathfrak{N}_*(X) \to H_*(X, \mathbb{Z}/2\mathbb{Z}) \) is onto where \( ev(f: M \to X) = f_*([M]) \). So suppose \( X \) has totally algebraic homology. By [3], generators of the group \( \mathfrak{N}_*(X) \) are compositions of projections \( U_i \times V_j \to V_j \) and maps \( \alpha_j: V_j \to X \) where \( \{U_i\} \) generate \( \mathfrak{N}_*(\text{point}) \) and \( \{ev(\alpha_j)\} \) generate \( H_* (X, \mathbb{Z}/2\mathbb{Z}) \). We may assume for each \( j \) that \( V_j \) is a nonsingular projectively closed algebraic subset of \( X \times \mathbb{R}^{s_j} \) for some \( n_j \), and \( \alpha_j \) is induced by projection. Also by [10] we know that \( U_i \) can be products of real projective spaces \( G(n, 1) \) and nonsingular algebraic sets

\[
H_{st} = \{(K, L) \in G(s, 1) \times G(t, 1) | K' \cdot L = 0\}
\]

where \( s \geq t \) and \( K' \) is the \( t \times t \) submatrix of \( K \) with \( K'_{ij} = K_{ij} \). In particular,
each $U_i$ can be a nonsingular projectively closed algebraic subset of some $R^{m_i}$. Hence the $V_j \times U_i \subset X \times R^{m_j} \times R^{m_i}$ and $\pi_{ij}: V_j \times U_i \rightarrow X$ induced by projection generate $\mathfrak{H}_*(X)$ as a group. But after a translation in the Euclidean factor we may assume all the $V_j \times U_i$ are disjoint. Since the group operation in $\mathfrak{H}_*(X)$ is disjoint union, we are done.

**Lemma 2.6.** Suppose $X_i$, $i = 0, 1$ are algebraic sets with totally algebraic homology. Then $X_0 \times X_1$ has totally algebraic homology.

**Proof.** By the Kunneth formula, $H_*(X_0 \times X_1, Z/2Z)$ is generated by classes $\alpha_0 \times \alpha_i$ with $\alpha_i \in H_*(X_i, Z/2Z)$. Let $V_i$ be nonsingular projectively closed algebraic subsets of $X_i \times R^{m_i}$ so that if $\pi_i: X_i \times R^{m_i} \rightarrow X_i$ is projection then $\pi_i([V_i]) = \alpha_i$. Let $\pi: X_0 \times R^{m_0} \times X_1 \times R^{m_1} \rightarrow X_0 \times X_1$ be projection. Then $\pi_*([V_0 \times V_1]) = \alpha_0 \times \alpha_1$, so we are done.

**Lemma 2.7.** The Grassmanian $G(n, k)$ has totally algebraic homology.

**Proof.** The $Z/2Z$ homology of $G(n, k)$ is generated by the Schubert cycles $C_{n_1, \ldots, n_k}$ where $n_1 < n_2 < \ldots < n_k$ and $C_{n_1, \ldots, n_k}$ is the set of $k$ dimensional subspaces whose intersection with $R^{m_i}$ has dimension $\geq i$ for each $i = 1, \ldots, k$. In terms of the matrix $L$ of orthogonal projection onto the subspaces, this just means that if $L_{n_i}$ is the first $n_i$ columns of the matrix $I - L$ then all $(n_i - i + 1) \times (n_i - i + 1)$ minors of $L_{n_i}$ have $0$ determinant. Hence the Schubert cycles are algebraic subsets of $G(n, k)$. By the remark after the definition of totally algebraic homology, this means that $G(n, k)$ has totally algebraic homology. (The Schubert cycles are sometimes singular but after resolving their singularities they become nonsingular.) This is a bit unsatisfactory since it requires a powerful theorem (resolution of singularities). One could get around this by noting that to prove the theorems in this paper we only need to represent relatively low dimensional homology, say only $H_i(G(n, k), Z/2Z)$ for $i \leq n$. All this homology is generated by nonsingular algebraic subsets

$$\prod_{j=1}^a G(n_j, k_j) \text{ with } n = \sum_{j=1}^a n_j, \quad k = \sum_{j=1}^a k_j .$$

So by a slight modification of the wording in the proofs one could avoid the singular Schubert cycles if one wished.

**Proposition 2.8 (Paraphrased).** Any closed manifold $M$ may be isotoped to a nonsingular algebraic set. A smooth function from $M$ to a nice algebraic set may simultaneously be approximated by a rational function. A relative version is also true.

**Proposition 2.8.** Let $X$ and $W$ be nonsingular algebraic sets with totally algebraic homology and let $M$ be a compact, boundaryless submani-
fold of $X$ and let $f: M \to W$ be a smooth map. Suppose also that $L \subset M$ is an algebraic set and the germ of $M$ at $L$ is the germ of a nonsingular algebraic set $V$ and the germ of $f$ at $L$ is the germ of an entire rational function $v: V \to W$.

Then there is an $m$, an isotopy $h_t$ of $X \times R^n$, a nonsingular algebraic set $S \subset X \times R^n$ and an entire rational function $r: S \to W$ so that

1) $h_t(M \times 0) = S$,

2) $h_t(x, 0) = (x, 0)$ for all $x \in L$, $t \in [0, 1]$,

3) $r(x, 0) = f(x)$ for all $x \in L$,

4) $h_t$ can be as small as we want and $rh_t|_{M \times 0}$ can be as close an approximation to $f$ as we want (for some choice of $S$ and $r$).

5) If $L$ is projectively closed then $S$ is projectively closed.

Proof. We will first prove a weaker version where $S$ might be singular, 1) is changed to 1'): $h_t(M \times 0)$ is a union of nonsingular components of $S$, 2), 3) and 4) remain the same but 5) is omitted. In addition, $W$ need not have totally algebraic homology.

There are a neighborhood $N$ of $L$ in $M$, a tubular neighborhood $T$ of $M$ in $X$, integers $n$ and $k$, a smooth map $g: T \to E(n, k)$ and an entire rational function $u: V \to G(n, k)$ so that

a) $g$ is transverse to $G(n, k)$;

b) $g^{-1}(G(n, k)) = M$;

c) $g|_N = u|_N$;

d) $f|_N = v|_N$.

To construct $g$, pick $n$ so that $X \subset R^n$ and let $k$ be the codimension of $M$ in $X$. Pick any $x \in T$. Let $y$ be the closest point to $x$ in $M$. Let $A \in G(n, k)$ be orthogonal projection to the tangent plane of $X$ in $R^n$ composed with orthogonal projection to the normal plane of $M$ in $R^n$. Then we let $g(x) = (A, A(y - x)) \in E(n, k)$. Notice that $g|_N$ is a rational function by Lemma 2.3. We may extend $f$ to all of $T$ by composing $f$ with projection from $T$ to $M$. Let $m$ be such that $W \times E(n, k)$ is an algebraic subset of $R^m$. Then by Lemma 2.4 there are an algebraic set $U \subset X \times R^m$ and a smooth $\varphi: (T, L) \to (R^m, 0)$ and an entire rational function $w: U \to W \times E(n, k)$ so that

A) $J = \{(x, \varphi(x))| x \in \text{Int } T\}$ is an open subset of Nonsing $U$,

B) The function $x \mapsto w(x, \varphi(x))$ is close to $f \times g$,

C) $w(x, 0) = (u(x), u(x)) = (f(x), g(x))$ for all $x \in L$.

Notice that by transversality and a, b, B and C above, $M \times 0$ is isotopic fixing $L \times 0$ to $\{x \in T| w(x, \varphi(x)) \in W \times G(n, k)\}$ which is isotopic fixing $L \times 0$ to $J \cap w^{-1}(W \times G(n, k))$. Let $Z' = J \cap w^{-1}(W \times G(n, k))$ and $Z'' =
U \cap w^{-1}(W \times G(n, k))$. Notice that by Lemma 1.4 all the points of $Z'$ are nonsingular of dimension $\dim M$ in $Z''$. We then let $Z = \text{Sing}(\text{Sing}(...(\text{Sing} Z''))) \cdot \cdot \cdot )$ where we take \text{Sing} a number of times so that $\dim Z = \dim M$. We have now proved the weakened theorem since we have a small isotopy $h_t$ of $M \times 0$ to $Z'$ which fixes $L \times 0$. For the rational function $r$ we pick $r = \pi w$ where $\pi: W \times G(n, k) \to W$ is projection.

We will now prove the proposition using this weaker version. By Lemmas 2.5 and 2.6 there are a $b$ and a compact smooth submanifold $N \subset X \times W \times R^b$ so that $\partial N$ is the disjoint union of a projectively closed nonsingular algebraic set $Q$ and the graph

$$\{(x, y, z) \in X \times W \times R^b | x \in M, y = f(x) \text{ and } z = 0\}.$$  

After translating, we may also assume $Q \cap X \times W \times 0$ is empty. Let $c$ be such that $W \subset R^c$ and let $N' = \partial(N \times [-1, 1])$. We may find an imbedding $\alpha: N' \to X \times R^c \times R^b \times R$ so that $\alpha(x, t) = (x, t)$ for all $x \in Q \subset X \times R^c \times R^b$ and $t \in [-1, 1]$ and so that $\alpha(x, f(x), 0, t) = (x, 0, 0, t)$ for all $x \in M$ and $t \in [-1, 1]$. (For instance send $(x, y, z, t)$ to $(x, \beta(x, y, z), y, z, t)$ where $\beta: N \to [0, 1]$ is some function which equals 1 on $Q$ and is zero on $\partial N - Q$. Assuming $b$ and $c$ were chosen large enough, we could then perturb this function to an imbedding $\alpha$.) Let $M' = \alpha(N')$. We have a smooth function $f': M' \to W$, namely $f' \alpha(x, y, z, t) = y$.

Let $L' = L \times 0 \times 0 \times 0 \cup Q \times 0$. Notice that the germ of $M'$ at $L'$ is the germ of a nonsingular algebraic set since the germ of $M'$ at $Q \times 0$ is $Q \times R$ and the germ of $M'$ at $L \times 0 \times 0 \times 0$ is the germ of $V \times 0 \times 0 \times R$ at $L \times 0 \times 0 \times 0$. Also the germ of $f'$ at $L'$ is the germ of an entire rational function by Lemma 1.5. Hence by the weak form of this proposition, there are an integer $d$ and an isotopy $h'_t$ of $X \times R^c \times R^b \times R \times R^d$ and an entire rational function $r: X \times R^c \times R^b \times R \times R^d \to W$ so that $h'_t(M')$ is a nonsingular component $Z'$ of an algebraic set $Z$, $h'_t(x, 0) = (x, 0)$ for all $x \in L'$ and $t \in [0, 1]$, $r' h'_t \l_{\cdot \times 0}$ is close to $f'$ and $r'(x, 0) = f'(x, 0)$ for all $x \in L'$. If we make sure this isotopy is small enough so that $h'_t(M' \times 0)$ keeps transverse to $X \times R^c \times R^b \times 0 \times R^d$, then there is a small isotopy of $Z'$ taking $h'_t(M \times 0 \times 0 \times 0)$ to a union of nonsingular components of the algebraic set $Z \cap X \times R^c \times R^b \times 0 \times R^d$. (This is because $M \times 0 \times 0 \times 0$ is a union of components of the set $M' \times 0 \cap X \times R^c \times R^b \times 0 \times R^d$.) By extending this isotopy to all of $X \times R^c \times R^b \times R \times R^d$, we may assume that $h'_t(M \times 0 \times 0 \times 0 \times 0)$ is a union of nonsingular components of $Z \cap X \times R^c \times R^b \times 0 \times R^d$. Thus by Lemma 2.2 there is a nonsingular algebraic set $K \subset Z'$ so that $L' \times 0 \subset K$ and there is a small isotopy of $h'_t(M \times 0 \times 0 \times 0 \times 0 \cup Q \times 0 \times 0)$ to $K$ which fixes
Again by extending this isotopy to all of \( X \times R^c \times R^b \times R \times R^t \), we may assume in fact that \( h'_i(M \times 0 \times 0 \times 0 \times 0 \cup Q \times 0 \times 0) = K \). But
\( h'_i(Q \times 0 \times 0) = Q \times 0 \times 0 \) is a nonsingular algebraic set, so by Lemma 1.6, \( h'_i(M \times 0 \times 0 \times 0 \times 0) \) is a nonsingular algebraic set. Also \( rh'_i|_{M \times 0 \times 0 \times 0 \times 0} \) is close to \( f'|_{M \times 0 \times 0 \times 0 \times 0} = f|_{M \times 0 \times 0 \times 0 \times 0} \), hence the proposition is proved.

In case \( L \) is projectively closed, \( L' \) is projectively closed also, so we could have made \( K \) and hence \( h'_i(M \times 0 \times 0 \times 0 \times 0) \) projectively closed.

**Definition.** A finite collection of closed submanifolds \( M_a \subset W, \alpha \in A \) is in general position if for any \( \alpha \in A \) and \( A' \subset A - \alpha \), \( M_a \) intersects \( \bigcap_{\alpha' \in A'} M_{\alpha'} \) transversely.

**Lemma 2.9.** Suppose \( M_a \subset W, \alpha \in A \) is a finite collection of submanifolds in general position and \( h^\alpha_t: M_a \rightarrow W, t \in [0, 1], \alpha \in A \) are small isotopies of imbeddings with \( h^\alpha_0 = \text{inclusion} \). Then there is an isotopy \( h_t: W \rightarrow W \) so that \( h_t(M_a) = h_t^\alpha(M_a) \) for each \( t \in [0, 1] \) and \( \alpha \in A \) and \( h_0 \) is the identity. Furthermore, \( h_t \) can be arbitrarily small if the \( h^\alpha_t \) are arbitrarily small.

**Proof.** First, notice that by requiring the \( h^\alpha_t \) to be \( C^1 \) small we insure that for each \( t \in [0, 1] \), the submanifolds \( h^\alpha_t(M_a), \alpha \in A \) are in general position. This is because any intersections of the \( h^\alpha_t(M_a) \) must occur near intersections of the \( M_a \) and then the tangent planes to the \( h^\alpha_t(M_a) \) at an intersection point will be very near the tangent planes to the \( M_a \) which were transverse. Also by the isotopy extension theorem, we may extend each \( h^\alpha_t \) to an isotopy \( h^\alpha_t: W \rightarrow W \) with \( h^\alpha_0 = \text{identity} \).

We wish to construct a vector field \( (v, \partial/\partial t) \) on \( W \times [0, 1] \) which is always tangent to each \( \{(h^\alpha_t(x), t)| x \in M_a, \ t \in [0, 1]\} \); then we will obtain \( h_t \) by integrating the vector field. Since we may piece together with a partition of unity, we need only construct \( v \) locally. So take any \((p, s) \in W \times [0, 1] \). There are coordinate neighborhoods \( U \) of \( p \) in \( W \) and a smooth \( f: U \rightarrow \mathbb{R}^m \) and a subset \( A' \subset A \) (which we call \( A' = \{1, 2, \ldots, k\} \) for convenience) so that \( f \) has rank \( m \) on \( U \), and \( U \cap h^\alpha_i(M_a) = (\pi_i f)^{-1}(0), i = 1, \ldots, k \) and \( U \cap h^\alpha_t(M_a) = \emptyset \) for \( \alpha \in A' \). Here \( m_i \) = codimension of \( M_i \) and we identify \( \mathbb{R}^m \) with \( R^{m_1} \times R^{m_2} \times \cdots \times R^{m_k} \) and let \( \pi_i: \mathbb{R}^m \rightarrow \mathbb{R}^{m_i} \) be projection. We define \( F: U' \rightarrow \mathbb{R}^m \) for a neighborhood \( U' \) of \((p, s) \) in \( W \times [0, 1] \) by \( \pi_i F(x, t) = \pi_i f(h^\alpha_t(x))^{-1}(x) \). Then for \( U' \) small, \( F \) will have rank \( m \). Let \( F'_i(x) = F(x, t) \) and pick local coordinates \( x_1, \ldots, x_n \) in \( U \). Then there is a unique function \( a: U' \rightarrow \mathbb{R}^m \) so that \( F'_i \sum_{j=1}^m a^j \nabla F'_j, \partial/\partial t) = 0 \) where \( F_* \) is the map on tangent spaces and \( a^j \) and \( F'_i \) are the \( j \)-th coordinates of \( a \) and \( F'_i \). In particular, \( a = B^{-1}(-\partial F/\partial t) \) where \( B(x, t) \) is the \( m \times m \) matrix with \( i, j \) coordinate \( \nabla F'_i \cdot \nabla F'_j \). Then \( \sum a^j \nabla F'_j, \partial/\partial t) \) is the required local vector
field since  
\[
(p_*F)^{-1}(0) = U' \cap \{(h_i(x), t) | x \in M_i , \ t \in [0, 1]\}
\]
and \(F\) is constant on the flow generated by the vector field:

**Theorem 2.10.** Let \(V\) be a nonsingular algebraic set with totally algebraic homology. Let \(M\) be a boundaryless compact smooth submanifold of \(V\) and let \(M_i, i = 1, \ldots, k\) be boundaryless compact smooth submanifolds of \(M\) in general position. Then for some \(n\), there are arbitrarily small isotopies of \(V \times \mathbb{R}^n\) which simultaneously take \(M \times 0\) and each \(M_i \times 0\) to a nonsingular projectively closed algebraic set.

In particular, there are an algebraic set \(X\) and algebraic subsets \(X_i \subset X, i = 1, \ldots, k\) and a diffeomorphism \(h : M \to X\) so that \(h(M_i) = X_i, i = 1, \ldots, k\).

**Proof.** This is proved in [18]. For this paper we only need to prove Theorem 2.10 for the special case where each \(M_i\) has codimension one in \(M\). Therefore we give a simple proof for this special case.

Pick \(n\) very large. For each \(i = 1, 2, \ldots, k\) there is a smooth map \(f_i : M \to G(n, 1) = \mathbb{R}P^{n-1}\) so that \(f_i\) is transverse to \(G(n - 1, 1) = \mathbb{R}P^{n-2} \subset \mathbb{R}P^{n-1}\) and so that \(f_i^{-1}(G(n - 1, 1)) = M_i\). (This \(f_i\) exists because the canonical line bundle over \(\mathbb{R}P^{n-2}\) classifies line bundles and \(\mathbb{R}P^{n-1}\) is the Thom space of this canonical bundle.) Let \(G = G(n, 1) \times \cdots \times G(n, 1) (k\ times)\) and let \(f : M \to G\) be the map with \(i^{th}\) coordinate \(f_i\). \(G\) has totally algebraic homology by Lemmas 2.6 and 2.7. Hence by Proposition 2.8 (with \(L\) empty) we know there are a nonsingular projectively closed algebraic set \(X \subset V \times \mathbb{R}^n\), an entire rational function \(r : X \to G\) and a diffeomorphism \(h' : M \to X\) so that \(rh'\) is a \(C^1\) close approximation to \(f\). Thus there is a \(C^1\) small isotopy of each \(h'(M_i) = h'f_i^{-1}(G_i)\) to \(r^{-1}(G_i)\) where \(G_i = G(n, 1) \times \cdots \times G(n - 1, 1) \times \cdots \times G(n, 1)\) with the \(G(n - 1, 1)\) in the \(i^{th}\) place. By Lemma 2.9 there is a diffeomorphism \(h'' : X \to X\) so that \(h''h'(M_i) = r^{-1}(G_i)\). Letting \(h = h''h'\) and \(X_i = r^{-1}(G_i)\), we are done. (The \(X_i\) are nonsingular by Lemma 1.4.)

3. **Topological blowing up and algebraic blowing down**

In this section we show a useful form of algebraic blowing down, namely one that can crush algebraic subsets of an algebraic set to points and get an algebraic set. This construction is generalized in later papers where we blow down over algebraic sets rather than just over a finite number of points.

We also state a result from another paper which gives a nice topological resolution of the cone on a smooth bounding manifold.
Proposition 3.1. Let \( X \) be a projectively closed algebraic set and let \( X_i \subset X, \ i = 1, \cdots, k, \) be disjoint algebraic subsets. Let \( Y \) be the quotient space obtained from \( X \) by collapsing each \( X_i \) to a distinct point \( i = 1, \cdots, k. \) Then \( Y \) is homeomorphic to an algebraic set. In fact there are an algebraic set \( Z \) and a homeomorphism \( h: Y \to Z \) so that \( h \) restricted to \( \text{Nonsing} \ X - \bigcup_{i=1}^k X_i \) is a diffeomorphism onto an open subset of Nonsing \( Z. \)

Proof. By Lemma 1.5 we may pick an entire rational function \( r: X \to R \) so that \( r(x) = i \) for all \( x \in X_i, \ i = 1, \cdots, k. \)

Let \( p: R^n \to R \) be an overt polynomial of degree \( d \) so that \( X = p^{-1}(0). \)
Let \( q: R^n \to R \) be a polynomial of degree \( e \) so that \( q^{-1}(0) = \bigcup_{i=1}^k X_i. \) Define \( u: R^2 \to R \) by \( u(s, t) = s^2 + (t-1)^2(t-2)^2 \cdots (t-k)^2. \) We define

\[
Z = \{(x, s, t) \in R^n \times R \times R | \text{there is a } y \in X \text{ so that } s = q(y), t = r(y) \text{ and } x = u(s, t) \cdot y \}
\]

To see that \( Z \) is an algebraic set, note that if \( p^* \) is the homogenization of \( p \) then \( Z \) is the set of common zeros of the polynomial \( p^*(u(s, t), x) \) and the rational functions \( s - q(x/u(s, t)) \) and \( t - r(x/u(s, t)) \). This is clear if \( u(s, t) \neq 0. \) But if \( u(s, t) = 0 \) then \( p^*(u(s, t), x) = p^*(0, x) \) which equals zero only when \( x = 0 \) (since \( p \) is overt). Notice that \( u(s, t) = 0 \) only when \( s = 0 \) and \( t = 1, 2, \cdots \) or \( k. \) Hence \( g: X \to Z \) defined by \( g(y) = (u(q(y), r(y)) \cdot y, q(y), r(y)) \) is a quotient map collapsing each \( X_i \) to the point \( (0, 0, i). \) Hence \( g \) induces a homeomorphism \( h: Y \to Z. \) Since \( g|_{X - \bigcup X_i} \) is a rational function onto \( Z - \bigcup_{i=1}^k (0, 0, i) \) with a rational inverse, it takes nonsingular points to nonsingular points. Hence the conclusion holds.

Fact 3.2. Suppose \( M \) is a smooth compact manifold which bounds. Then there are a compact smooth manifold \( W \) and compact boundaryless submanifolds \( L_i \subset \text{int} \ W, \ i = 1, \cdots, k, \) so that

1) \( M = \partial W. \)

2) The \( L_i \) are in general position.

3) There is a diffeomorphism \( \eta: M \times [0, 1) \to W - \bigcup_{i=1}^k L_i. \)

This fact is proved in [15]. Since this has not yet appeared in print, we give a sketch of the proof. Let \( X \) be a compact smooth manifold so that \( \partial X = M. \) Suppose \( X \) has dimension \( n. \) The first step is to find \( n - 1 \) spheres \( S_i, \ i = 1, \cdots, b, \) imbedded in \( \text{int} X \) so that the \( S_i \) are in general position and \( X - \bigcup_{i=1}^b S_i \) is diffeomorphic to the disjoint union of an open collar on \( \partial X \) and a collection of open \( n \)-discs. For instance, we could take a smooth triangulation of \( X; \) then for each vertex \( v \) not in \( \partial X \) we could shrink the link of \( v \) a little bit to get one of these spheres. Alternatively, it would
probably work to put a Riemannian metric on $X$ and let the $S_i$ be boundaries of small geodesically convex balls. In any case once we get these $S_i$, then for each component of $X - \bigcup_{i=1}^{k} S_i$ which does not intersect $\partial X$, we pick an $n$-disc $B_j$ imbedded in that component, $j = 1, \cdots, c$. Let $X' = X - \bigcup_{j=1}^{c} \text{int } B_j$. Then notice that there is a diffeomorphism of $X' - \bigcup_{i=1}^{k} S_i$ with $\partial X' \times [0, 1)$. Since $\partial X'$ is $M$ union $c$ copies of the $n-1$ sphere we almost have our result; we just have a lot of extra $n-1$ spheres $\partial B_j$. We get rid of them by adding 1-handles. Pick a $j$ so that there is a smooth imbedded arc $\alpha_j: [-1, 1] \to X'$ so that $\alpha_j(-1) \in \partial X'$, $\alpha_j(1) \in \partial B_j$, $\alpha_j$ intersects only one $S_i$ and that intersection is transverse.

We now attach a 1-handle to $X'$ along the boundary at $\alpha(1)$ and $\alpha(-1)$. That is, we make $X'' = [-1, 1] \times D^{n-1} \cup X'$ where we identify $-1 \times D^{n-1}$ with a neighborhood of $\alpha(-1)$ in $\partial X'$ and we identify $1 \times D^{n-1}$ with a neighborhood of $\alpha(1)$ in $\partial X'$. We then let $R_j$ be the circle $[-1, 1] \times 0 \cup \alpha([-1, 1])$. Notice that $(X'' - \bigcup_{i=1}^{k} S_i) - R_j$ is diffeomorphic with $\partial X'' \times$
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[0, 1) and \( \partial X'' \) is diffeomorphic with \( M \) union \( c - 1 \) copies of the \( n - 1 \) sphere, so we have improved matters. We keep on adding 1-handles (we add \( c \) of them in all) and finally get a manifold \( W \) containing \( n - 1 \) spheres \( S_i, i = 1, \ldots, b, \) and circles \( R_j, j = 1, \ldots, c, \) so that the \( S_i \) and \( R_j \) are in general position and there is a diffeomorphism from \( M \times [0, 1) \) to \( (W - \bigcup_{i=1}^{b} S_i) - \bigcup_{j=1}^{c} R_j. \) We illustrate this in Figure 2 for the case where \( X \) is a 2-disc, \( b = 2 \) and \( c = 3. \)

**Lemma 3.3.** We may stipulate that the \( L_i \) in the conclusion of Fact 3.2 all have codimension one in \( W. \)

**Proof.** We just blow up \( W \) along each \( L_i. \) A more concrete explanation follows. The \( L_i \)'s constructed for Fact 3.2 are of two types: codimension 1 spheres \( S_i \) and circles \( R_j. \) These circles \( R_j \) have trivial normal bundle. We now form a smooth manifold \( W' \) by replacing each \( R_j \) in \( W \) by \( R_j \times RP^{n-2} \) which has codimension 1. If \( E \) is the total space of the canonical line bundle over \( RP^{n-2}, \) then the diffeomorphism from \( E - RP^{n-2} \) to \( R^{n-1} - 0 \) gives a smooth structure on \( W'. \) This process changes the \( S_i \) into \( S'_i \) which are connected sums of \( RP^{n-1} \)'s. Since

\[
W' - \bigcup S'_i - \bigcup R_j \times RP^{n-2} = W - \bigcup S_i - \bigcup R_j \approx \partial W \times [0, 1),
\]

we are done after replacing \( W \) by \( W'. \)

4. **Topological characterization of algebraic sets with isolated singularities**

Theorem 4.1 below characterizes only compact algebraic sets with isolated singularities but in combination with Proposition 4.2 it characterizes all real algebraic sets with isolated singularities. Also we get the fact that up to diffeomorphism, nonsingular algebraic sets are just interiors of smooth compact manifolds with (possibly empty) boundary.

For a topological space \( X \) let \( eX \) denote the cone on \( X, \) i.e., \( eX = X \times [0, 1]/X \times 0. \) We use the convention that collapsing the empty set to a point is the same as adding a disjoint point. Hence the cone on the empty space is a point.

![Fig. 3.](image)
We illustrate the implication iii $\implies$ i of Theorem 4.1 by an example. Let $W$ be the cylinder $S^1 \times [0, 1]$ and let $M_i = S^1 \times 0 \cup S^1 \times 1$ and $M_i = \emptyset$ space. Then $X$ is as in Figure 3.

First we observe that it suffices to make each connected component of $X$ homeomorphic to an algebraic set. Since a point is clearly an algebraic set, we just need to make the left hand component $X'$ algebraic. We first find a topological resolution of $X'$. For instance, $X'$ is obtained from the two dimensional sphere by identifying two points with each other. Since the sphere is an algebraic set and those two points are an algebraic subset, Proposition 3.1 tells us that $X'$ is homeomorphic to an algebraic set. Hence $X$ is also. Of course, different resolutions are possible. For instance $X'$ is the torus with a meridian circle collapsed to a point.

For another example, let $W$ be a plumbing of disc bundles over spheres and let $M_i = \partial W$. Then $X = W \cup c\partial W$ has a resolution obtained by taking the double of $W$ and collapsing the spheres in one copy of $W$ to a point. Since these spheres are in general position, we may use Theorem 2.10 to make the double of $W$ a nonsingular algebraic set and make the spheres algebraic subsets. Then by Proposition 3.1 we may collapse the spheres to a point and hence make $X$ into an algebraic set.

Notice that Theorem 4.1 and Proposition 4.2 also classify homotopy types of algebraic sets; in particular they are exactly the homotopy types of finite complexes. In fact, any finite complex is homotopy equivalent to a nonsingular algebraic set.

**Theorem 4.1.** Let $X$ be a compact topological space. Then the following are equivalent:

i. $X$ is homeomorphic to a real algebraic set with isolated singularities.

ii. $X$ is homeomorphic to the quotient space obtained by taking a smooth closed manifold $W$ and collapsing each $K_i$ to a point where $K_i$, $i = 1, \ldots, k$, is a collection of disjoint smooth subpolyhedra of $W$.

iii. $X = W \cup \bigcup_{i=1}^{k} cM_i$ where $W$ and $M_i$ are smooth compact manifolds, $\partial W$ is the disjoint union of the $M_i$'s, each $M_i$ bounds a smooth compact manifold and $M_i \times 1 \subset cM_i$ is identified with $M_i \subset W$.

In ii and iii some of the $K_i$'s and $M_i$'s could be empty; in fact there will be one such $K_i$ or $M_i$ for each isolated point of $X$ (as long as dim $X > 0$).

**Proof.** The implication ii $\implies$ iii follows from [5] by taking a smooth regular neighborhood of each subpolyhedron.

The implication i $\implies$ ii follows from Hironaka's resolution of singularities [4] which implies the existence of a continuous map $\pi: W \to X$ with $W$ a
compact nonsingular algebraic set (hence a smooth closed manifold) with some nonsingular algebraic subsets in general position (hence smooth subpolyhedra) \( K_i \subset W, i = 1, \cdots, m \), so that \( \pi |_{\omega - \cup K_i} \) is a diffeomorphism onto \( X \) minus a finite number of points (the singularities of the algebraic set). So \( X \) is obtained from \( W \) by collapsing each \( K_i \) to a point and then perhaps collapsing the empty set to a point a few times to get any isolated points of \( X \).

So the meat of the theorem is the implication \( iii \Rightarrow i \). We may assume that no \( M_i \) is empty since every empty \( M_i \) corresponds to adding a disjoint point to \( X \). Certainly if \( X \) is homeomorphic to an algebraic set then \( X \cup \) point is also.

By Lemma 3.3 there are for each \( i = 1, \cdots, k \) a smooth compact manifold \( W_i \), smooth compact boundaryless codimension one submanifolds \( L_{ij} \subset \text{int} W_i, j = 1, \cdots, a_i \), and a diffeomorphism \( \alpha_i: M_i \times [0, 1) \rightarrow W_i - \bigcup_{j=1}^{a_i} L_{ij} \) so that the \( L_{ij} \) are in general position. Notice that \( \alpha_i |_{M_i \times 0} \) is a diffeomorphism from \( M_i \times 0 \) to \( \partial W_i \). Hence we may consider the smooth compact boundaryless manifold \( U = W \cup \bigcup_{i=1}^{k} W_i \) with each \( M_i \subset \partial W_i \) identified with \( \partial W_i \). By Theorem 2.10 we know that there are an integer \( n \), a nonsingular projectively closed algebraic set \( V \subset \mathbb{R}^n \), nonsingular algebraic subsets \( N_{ij} \subset V, i = 1, \cdots, k, j = 1, \cdots, a_i \), and a diffeomorphism of \( U \) to \( V \) which takes each \( L_{ij} \) to \( N_{ij} \). By Proposition 3.1 there are an algebraic set \( Z \) and a homeomorphism \( h: Y \rightarrow Z \) where \( Y \) is obtained from \( U \) by crushing each \( \bigcup_{j=1}^{a_i} N_{ij} \) to a point. Also \( Z \) must have isolated singularities since \( V \) is nonsingular. Notice \( Y \) is homeomorphic to \( U \) with each \( \bigcup_{j=1}^{a_i} L_{ij} \) crushed to a point. Also \( W_i/\bigcup_{j=1}^{a_i} L_{ij} \) is homeomorphic to \( cM_i \) since \( W_i - \bigcup_{j=1}^{a_i} L_{ij} \approx M_i \times [0, 1) \). Hence \( Y \) is homeomorphic to \( X \) so \( Z \) is homeomorphic to \( X \) and we are done.

PROPOSITION 4.2. If \( V \) is an algebraic set then there is an algebraic set \( W \) homeomorphic to the one point compactification of \( V \). This homeomorphism takes Nonsing \( V \) into Nonsing \( W \). Also, if \( W \) is an algebraic set and \( z \in W \) there is an algebraic set \( V \) homeomorphic to \( W - z \). This homeomorphism carries Nonsing \( W - z \) into Nonsing \( V \).

In particular, a topological space \( X \) is homeomorphic to an algebraic set if and only if \( X \) is locally compact and the one point compactification of \( X \) is homeomorphic to a real algebraic set.

Proof. Consider an algebraic set \( V \subset \mathbb{R}^n \); we may assume \( V \neq \mathbb{R}^n \). Take \( z \in \mathbb{R}^n - V \). Define an involution \( \varphi: \mathbb{R}^n - z \rightarrow \mathbb{R}^n - z \) by \( \varphi(x) = x + (x - z)/|x - z|^2 \) (inversion through the unit sphere about \( z \)). Then \( \varphi^{-1}(V) \cup z \) is homeomorphic to the one point compactification of \( V \). It is an algebraic set by Lemma 1.3.
and \( \varphi^{-1}(\text{Nonsing } V) \subset \text{Nonsing } (\varphi^{-1}(V) \cup z) \) by Lemma 1.4.

Now take \( W \subset R^n \) and \( z \in W \). Define \( V = \{(x, t) \in W \times R \mid t|z - z|^2 = 1\} \). Then \( V \) is homeomorphic to \( W - z \) and \( \text{Nonsing } V = (\text{Nonsing } W \times R) \cap V \).

**Corollary 4.3.** Up to diffeomorphism, the set of nonsingular real algebraic sets is exactly the set of interiors of compact smooth manifolds with (possibly empty) boundary.

**Proof.** If \( X \) is a nonsingular algebraic set then \( X^* \), the one point compactification of \( X \), is an algebraic set with at most one singularity, namely the compactifying point \( z \). So by Theorem 4.1, \( X^* = W \cup c\partial W \) for some smooth \( W \), but then \( X \) is diffeomorphic to the interior of \( W \).

Conversely, if \( W \) is a compact smooth manifold then \( W \cup c\partial W \) is an algebraic set \( Y \) with at most one singularity \( z \) so \( Y - z \) is a nonsingular algebraic set diffeomorphic to int \( W \).

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**References**


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