4-Manifolds (v.19.2)

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Preface

This book is an attempt to present the topology of smooth 4-manifolds in an intuitive self contained way, as it was developed over the years. I tried to refrain long winded pedagogic explanations, instead favoring concise direct approach to key constructions and theorems, and then supplementing them with exercises to help reader fill in the details which are not covered in the proofs. In short, I tried to present the material direct way I learnt and thought to myself over the years. When there was a chance to explain some idea with pictures I always went with pictures staying true to the saying “A picture is worth a thousand words”. Clearly the material covered in this book is biased towards my interest. I tried to order the material in historical order when possible, while avoiding to be encyclopedic complete referencing when discussing various results. For the sake of simplicity, I made no attempts to cover all the related results around the materials discussed, and later I picked various examples to show how to analyze them with the techniques developed in the earlier chapters. I want to thank many of the friends and the collaborators Robion Kirby, Cliff Taubes, Rostislav Matveyev, Yakov Eliashberg, Ron Fintushel, Burak Ozbagci, Anar Akhmedov, Selahi Durusoy, Cagri Karakurt, Firat Arikan, and Kouichi Yasui who helped me understand this rich filed of 4-manifolds as well as patiently tutoring me in the related fields of symplectic topology and gauge theory over the years. I also want to thank Rafael Torres for reading and helping me to catch many misprints in the manuscript.
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Chapter 1

4-manifold handlebodies

Let $M^m$ and $N^m$ be smooth manifolds. We say $M^m$ is obtained from $N^m$ by attaching a $k$-handle and denote $M = N \circlearrowleft h^k$, if there is an imbedding $\varphi : S^{k-1} \times B^{m-k} \hookrightarrow \partial N$, such that $M$ is obtained from the disjoint union $N$ and $B^k \times B^{m-k}$ by identification

$$M^m = \left[ N \sqcup B^k \times B^{m-k} \right] / \varphi(x) \sim x \quad \text{ (and the corners smoothed)}$$

Here $\varphi(S^{k-1} \times 0)$ is called the attaching sphere, and $0 \times S^{m-k-1}$ is called the belt sphere of this handle. We say $\partial M$ is obtained from $\partial N$ by surgering the $k-1$ sphere $\varphi(S^{k-1} \times 0)$. There is always a Morse function $f : M \to \mathbb{R}$ on $M$ inducing a handle decomposition $M = \bigcup_{k=0}^m M_k$, with $\phi = M_{-1} \subset M_0 \subset \ldots \subset M_m = M$, and $M_k$ is obtained from $M_{k-1}$ by attaching $k$-handles [M1]. This is called a handlebody structure of $M$. If $M^m$ is compact and connected we can assume that it has one 0-handle and one $m$-handle. So in particular in the case of dimension four, $M^4$ is obtained from $B^4$ (the zero handle) by attaching 1, 2, and 3-handles and then capping it off with $B^4$ at the top (the 4-handle).

In [S] Smale, and also independently in [W] Wallace, defined and studied handlebody structures on smooth manifolds. Smale went further by turning the operations on handlebody structures into a great technical machine to solve difficult problems about smooth manifolds, such as the smooth $h$-cobordism theorem which implied the proof of the topological Poincare conjecture in dimensions $\geq 5$. The two basic techniques which he utilized were the handle sliding operation and handle canceling operation [H].

Given a smooth connected 4-manifold $M^4$, by using its handlebody we can basically see the whole manifold as follows: As shown in Figure 1.1 we place ourselves on the boundary $S^3$ of its 0-handle $B^4$ and watch the feet (the attaching regions) of the 1- and 2- handles. The feet of the 1-handles will look like the pair of balls of same color, and the feet of 2-handles will look like imbedded framed circles (framed knots) which might go over the 1-handles. This is because the 2-handles are attached after the 1-handles.
Fortunately we don’t need to visualize the attaching $S^2$’s of the 3-handles, because of an amazing theorem of Laudenbach and Poenaru [LP], which says that connected smooth 4-manifolds are determined by their 1- and 2-handles (i.e. if there are 3-handles they are attached uniquely!)

The feet of a 1-handle in $S^3$ will appear as a pair of balls $B^3$, where on the boundary they are identified by the map $(x, y, z) \mapsto (x, -y, z)$ (with respect to the standard coordinate axis placed at their origins)

Any 2-handle which does not go over the 1-handles is attached by an imbedding $\varphi : S^1 \times B^2 \to \mathbb{R}^3 \subset S^3$, which we call a framing. This imbedding is determined by the knot $K = \varphi(S^1 \times (0, 0))$, together with an orthonormal framing $e = \{u, v\}$ of its normal bundle in $\mathbb{R}^3$. This framing $e$ determines the imbedding $\varphi$ by

$$\varphi(x, \lambda, \nu) = (x, \lambda u + \nu v)$$
Also note that by orienting $K$ and using the right hand rule, one normal vector field $u$ determines the other normal vector $v$. Hence any normal vector field of $K$ determines a framing, which is parametrized by $\pi_1(SO_2) = \mathbb{Z}$. We make the convention that the normal vector field induced from the collar of any Seifert surface of $K$ to be the zero framing $u_0$ (check this is well defined, because linking number is well defined). Once this is done, it is clear that any $k \in \mathbb{Z}$ corresponds to the vector field $u_k$ which deviates from $u_0$ by $k$-full twist. Put another way, $u_k$ is the vector field when we push $K$ along it, we get a copy of $K$ which has linking number $k$ with $K$. So we denote framings by integers. The following exercise gives a useful tool of deciding the $k$-framing.

**Exercise 1.1.** Orient a diagram of the knot $K$, and let $C(K)$ be the set of its crossing points, define writhe $w(K)$ of a knot $K$ to be the integer

$$w(K) = \sum_{p \in C(K)} \epsilon(p)$$

where $\epsilon(p)$ is $+1$ or $-1$ according to right or left handed crossing at $p$. Show that the blackboard framing (parallel on the surface of the paper) of a knot equals to its writhe

![Figure 1.3: $w(K) = 3$ and $u_0$ is the zero framing](image)

When the attaching framed circles of 2-handles go over 1-handles their framing can not be a well defined integer. For example, by the isotopy of Figure 1.4 the framing can be changed by adding or subtracting 2. One way to prevent this framing changing isotopy is to fix an arc, as shown in Figure 1.2, connecting the attaching balls of the 1-handle, and make the rule that no isotopy of framed knots may cross this arc. Only after this unnatural rule, we can talk about well defined framed circles. Most of the work of [AK1] is based on this convention.
1.1 Carving (invariant notation of 1-handle)

Carving is based on the following simple observation: If the attaching sphere $\varphi(S^{k-1} \times 0)$ of the $k$-handle of $M = N^m \ smashed h^k$, bounds a disk in $\partial N$, then $M$ can be obtained from $N$ by excising (drilling) out an open tubular neighborhood of a properly imbedded disk $D^{m-k-1} \subset N$. This is because we can cancel the k-handle by the obvious $(k+1)$-handle and get back $N$ (Section 1.3), then we can undo the $(k+1)$-handle by puncturing (carving) it, as indicated in Figure 1.5. In particular if $M^4$ is connected, attaching a 1-handle to $M^4$ is equivalent to carving out a properly imbedded 2-disk $D^2 \subset B^4$ from the 0-handle $B^4 \subset M$. This simple observation has many useful consequences (e.g. [A1]).

To distinguish the boundary of the carved disk $\partial D \subset S^3$ from the attaching circles of the 2-handles, we will put a “dot” on that circle. In particular this means a path going through this dotted circle is going over the associated 1-handle. So the observer in Figure 1.1 standing on the boundary of $B^4$ will see the 4-manifold as in Figure 1.6. As discussed before, the framing of the attaching circle of each 2-handle is specified by an integer, so the knots come with integers (in this example the framings are 2 and 3).
1.1 Carving

This notation of 1-handle has the advantage of not creating ambiguity on the framings of framed knots going through it, as it happens in Figure 1.4. Also this notation can be helpful in constructing hard to see diffeomorphisms between the boundaries of 4-manifolds (e.g. Exercise 1.3, Section 2.6).

![Figure 1.6](image)

We will denote an \( r \)-framed knot \( K \) by \( K^r \), which will also denote the corresponding 4-manifold \( (B^4 + 2\text{-handle}) \). Since any smooth 4-manifold \( M^4 \) is determined by its 0, 1 and 2-handles, \( M \) can be denoted by a link of framed knots \( K_1^{r_1}, K_2^{r_2}, \ldots \) (2-handles), along with “circle with dots” \( C_1, C_2, \ldots \) (1-handles). Note that we have the option of denoting the 1-handles either by dotted circles, or pair of balls (each notation has its own advantages). Also for convenience, we will orient the components of this link:

\[
\Lambda = \{ K_1^{r_1}, \ldots, K_n^{r_n}, C_1, \ldots, C_s \}
\]

We notate the corresponding 4-manifold by \( M = M_\Lambda \). For example, Figure 1.6 is the 4-manifold \( M_\Lambda \) with \( \Lambda = \{ K_1^2, K_2^3, C \} \), where \( K_1 \) is the trefoil knot, and \( K_2 \) is the circle which links \( K_1 \), and \( C \) is the 1-handle. Notice that a 0-framed unknot denotes \( S^2 \times B^2 \), it also denotes \( S^4 \) if we state that there is a 3-handle is present. Also a \( \pm 1 \)-framed unknot denotes a punctured (that is a 4-ball removed) \( \pm \mathbb{C}P^2 \).

**Exercise 1.2.** Show that \( S^1 \times B^3 \) and \( B^2 \times S^2 \) are related to each other by surgeries of their core spheres (i.e. the zero and dot exchanges in Figure 1.7)

![Figure 1.7](image)
**Exercise 1.3.** Show that the manifold $W$ in Figure 1.8 is a contractible manifold, and surgeries in its interior (corresponding to zero and dot exchanges on the symmetric link) gives an involution on its boundary $f : \partial W \to \partial W$, where the indicated loops $a, b$ are mapped to each other by the involution $f$.

![Figure 1.8](image)

**1.2 Sliding handles**

As explained in [S], [H] we can change a given handlebody of $M$ to another handlebody of $M$, by sliding any $k$-handle over another $r$-handles with $r \leq k$.

![Figure 1.9: Sliding a 2-handle over another 2-handle](image)

In particular in a 4-manifold we can perform the following handle sliding operations: We can (a) slide a 1-handle over another 1-handle, or (b) slide a 2-handle over a 1-handle or (c) slide a 2-handle over another 2-handle. This process alters the framings of the attaching circles of the 2-handles in a well defined way as follows.
Exercise 1.4. Let $\Lambda = \{K_1^{r_1}, ..., K_n^{r_n}, C_1, ..., C_s\}$ be an oriented framed link with circle with dots, let $\mu_{ij}$ be the linking number of $K_i$ and $C_j$, and $\lambda_{ij}$ be the linking number of $K_i$ and $K_j$. Also let $K'_i$ denote a parallel copy of $K_i$ ($K_i$ pushed off by the framing $r_i$), and $C'_i$ be a parallel copy of the circle $C_i$. Show that the handle slides (a), (b), (c) above corresponds to changing one of the elements of $\Lambda$ as indicated below ($j \neq s$):

(a) $C_i \mapsto C_i + C'_j := \text{The circle obtained, by connected summing } C_i \text{ to } C'_j \text{ along any arch which does not go through any of the } C'_k \text{'s.}$

(b) $K_i^{r_i} \mapsto K_i + C'_j := \tilde{K}_i^{\tilde{r}_i} \text{ The framed knot obtained, by connected summing } K_i \text{ to } C'_j \text{ along any arc, with framing } \tilde{r}_i = r_i + 2\mu_{ij}$

(c) $K_i^{r_i} \mapsto K'_i + K_j^{r_j} := \tilde{K}_i^{\tilde{r}_i} \text{ The framed knot obtained, by connected summing } K_i \text{ to } K'_j \text{ along any arc, with framing } \tilde{r}_i = r_i + r_j + 2\lambda_{ij}$

These rules are best explained by the following examples:

![Diagram](image1.png)

Figure 1.10: Examples of handle slides (a), (b), (c)
1.3 Canceling handles

We can cancel a $k$-handle $h^k$ with a $(k + 1)$-handle $h^{k+1}$ provided that the attaching sphere of $h^{k+1}$ meets the belt sphere of $h^k$ transversally at a single point (as explained in [S] and [H]). This means that we have a diffeomorphism:

$$N \sim h^k \sim h^{k+1} \approx N$$

For example, in Figure 1.5 a canceling 1 and 2-handle pairs, of a 3-manifold, was drawn. Any 1-handle, and a 2-handle whose attaching circle (framed knot) goes through the 1-handle geometrically once, forms is a canceling pair. If no other framed knot goes through the 1-handle of the canceling pair, simply erasing the pair from the picture corresponds to canceling operation. Figure 1.11 gives three equivalent descriptions of a canceling handle pair, so if you want to cancel just erase them from the picture.

![Figure 1.11](image-url)

It follows from the handle sliding description that, if there are other framed knots going through the 1-handle of a canceling handle pair, the extra framed knots must be slid over over the 2-handle of the canceling pair, before the canceling operation is performed (erasing the pair from the picture)

![Figure 1.12](image-url)

Notice that since 3-handles are attached uniquely, introducing a canceling 2- and 3-handle pair is much simpler operation. We just draw the 2-handle as 0-framed unknot,
which is \( S^2 \times B^2 \), and then state that there is a canceling 3-handle on top of it. In a handle picture of 4-manifold, no other handles should go through this 0-framed unknot to be able to cancel it with a 3-handle.

**Exercise 1.5.** Verify the diffeomorphisms of Figure 1.13 by handle slides (the integers \( \pm 1 \) across the stands indicate \( \pm 1 \) full twist). Also show that this operation changes the framing of any framed knot, which links the \( \pm 1 \) framed unknot \( r \) times, by \( r^2 \).

![Figure 1.13](image)

**Exercise 1.6.** Show that \( S^2 \times S^2 \# CP^2 \) and \( CP^2 \# CP^2 \# \overline{CP^2} \) are diffeomorphic to each other by sliding operations as shown in Figure 1.14 (the identification of these two 4-manifolds is originally due to Hirzebruch).

![Figure 1.14](image)

The operation \( M \mapsto M \# \pm CP^2 \) is called blowing up operation, and its converse is called blowing down operation. If \( C^{\pm 1} \) is an unknot with \( \pm 1 \) framing in a framed link \( \Lambda \), then by the handle sliding operations of Exercise 1.5, we obtain a new framed link \( \Lambda' \) containing \( C^{\pm 1} \) such that \( C^{\pm 1} \) does not link any other components of \( \Lambda' \). So we can write \( \Lambda' = \Lambda'' \sqcup \{ C^{\pm 1} \} \). Therefore we have the identification \( M_\Lambda = M_{\Lambda''} \# (\pm CP^2) \), and hence a diffeomorphism of 3-manifolds \( \partial M_\Lambda \approx \partial M_{\Lambda''} \). Sometimes the operations \( \Lambda \leftrightarrow \Lambda'' \) on framed links are called blowing up/down operations (of the framed links).
Recall that the set $\Lambda$ of framed links encodes handle information on the corresponding 4-manifold $M = M_\Lambda$, which comes from a Morse function $f : M \to \mathbb{R}$. Cerf theory studies how two different Morse functions on a manifold are related to each other [C]. In [K1], by using Cerf theory, Kirby studied the map $\Phi$ defined by $\Lambda \mapsto \partial M_\Lambda$, from framed links to 3-manifolds. It is known that $\Phi$ is onto [Li] (see Chapter 5).

$$\Phi : \{\text{Framed links}\} \mapsto \{\text{closed oriented 3-manifolds}\}$$

In [K1] it was shown that under $\Phi$ any two framed links are mapped to the same 3-manifold if and only if they are related to each other by handle sliding operation of Exercise 1.4 (c), and blowing up or down operations. For this reason, in literature manipulating framed links by these two operations is usually called Kirby calculus.

### 1.4 Carving ribbons

In some cases carving operation allows us to slide a 1-handles over a 2-handles, which is in general prohibited (the attaching circle of the 2-handle might go through the 1-handle). But as in the configuration of Figure 1.16, we can imagine the carved disk (painted yellow inside) as a trough (or a groove) just below the surface; then sliding it over the 2-handle (imagine as a surface of a helmet) by indicated isotopy makes sense.
Exercise 1.7. Justify the isotopy of Figure 1.16, by first breaking the 1-handle into a 1-handle and a canceling 1 and 2-handle pair, as in the first picture of Figure 1.17, then by sliding the new 2-handle over the 2-handle of Figure 1.16, then at the end by canceling back the 1 and 2-handle pair. Also verify the second picture of Figure 1.17.

Figure 1.17

This move can turn a 1-handle, which is a carved disk in $B^4$ bounding an unknot (in a nonstandard handlebody of $B^4$) into a carved ribbon disk bounding a ribbon knot (in the standard handlebody of $B^4$), as in the example of Figure 1.18.

Figure 1.18

We can also consider this operation in the reverse, namely given a ribbon knot, how can we describe complement of the slice disk which ribbon knot bounds in $B^4$? i.e. what is the ribbon complement obtained by carving this ribbon disk from $B^4$? Clearly by ribbon moves (i.e. cutting and regluing along bands) we can turn the ribbon knot into disjoint union of unknotted circles, which we can use to carve $B^4$ along the disks they bound, getting some number of connected sum $k(S^1 \times B^3)$. Now we can do the reverse of the ribbon moves, which describes a cobordism from the boundary $k(S^1 \times S^2)$ of the carved $B^4$ to the ribbon complement in $S^3$. During this cobordism every time two circles coalesce the complement gains a 2-handle as in Figure 1.19. To sum up, to construct the complement of the ribbon which $K$ bounds in $B^4$, we perform $k$ ribbon moves (for some $k$) turning $K$ into $k$ disjoint unknots $\{C_j\}_{j=1}^k$, and after every ribbon move we add a 2 handle as indicated in Figure 1.19, and put dots on the circles $\{C_j\}_{j=1}^k$. 

15
1 4-manifold handlebodies

For example, applying this process to the ribbon knot $K$ of Figure 1.20 gives the handlebody at the bottom right picture of Figure 1.20 (this knot appears in [A2] playing important role in exotic smooth structures of 4-manifolds). Note that this process turns carving a ribbon from the standard handlebody picture of $B^4$ into carving the trivial disks from a non standard handlebody picture of $B^4$. Clearly the top left picture is the handlebody of the complement of an imbedding $S^2 \hookrightarrow S^4$ obtained by doubling this ribbon complement in $B^4$ as in Figure 1.21 (we build this handlebody from the bottom $\#_2(S^1 \times B^3)$ to top by attaching 2-handles corresponding to the reverse ribbon moves).
Exercise 1.8. Show that the indicated ribbon move to the ribbon knot of Figure 1.22 gives the ribbon complement in $B^4$ (the second picture), which consisting of two 1-handles and one 2-handle, and the third picture (with an extra 2-handle and an invisible 3-handle) is the complement of the double of this ribbon disk (i.e. a copy of $S^2$) in $S^4$. Also by handle slides verify the last diffeomorphism.

We can define an imbedding $S^2 \subset S^4$ by doubling a ribbon disk $D \subset B^4$ (consistent with the doubling process of Section 3.1). Figure 1.23 $\Rightarrow$ Figure 1.24 is the process of carving the ribbon disk $D$ from $B^4$, and then doubling what is left; ending with $S^4 - S^2$. 

Figure 1.21

Figure 1.22

Figure 1.23: A ribbon disk $D \subset B^4$  
Figure 1.24: Doubling $B^4 - D$
1.5 Non-orientable handles

If we attach a 1-handle $B^1 \times B^3$ to $B^4$ along a pair of balls $\{B^3_-, B^3_+\}$ with the same orientation of $\partial B^4$ we get a orientation reversing 1-handle, which is the nonoriented $B^3$-bundle over $S^1$. So to an observer, located at the boundary of $B^4$, this will be seen as a pair of balls with their interiors identified by the map $(x, y, z) \mapsto (x, -y, -z)$. To denote this in pictures, we use the notation adapted in [A3], by drawing a pair of balls with little arcs passing through their centers, which means that the usual oriented 1-handle identification is augmented by reflection across the plane which is orthogonal to these arcs (Figure 1.25). When drawing the frame knots going through these handles, one should not forget which points of the spheres $\partial B^3_-$ and $\partial B^3_+$ correspond to each other via the 1-handle. Also framings of the framed knots going through these handles are not well defined, not just because of the isotopies of Figure 1.4, but also going through an orientation reversing handle the framing changes sign. For that reason when these handles are present, we will conservatively mark the framings by circled integers (with the knowledge that pushing them through the handle that integer changes its sign). For the same reason, any small knot tied to a strand going through this handle will appear as its mirror image when its is pushed though this handle, as indicated in Figure 1.26. 

Sometimes to simplify the handlebody picture, we place one foot of a orientation reversing 1-handle at the point of infinity $\infty$ (continue traveling west to $\infty$ you will appear to be coming back from the east with your orientation reversed), as in the example of Figure 1.27. Notice that when we attach $B^4$ an orientation reversing 1-handle, we get the orientation reversing $D^3$ bundle over $S^1$, denoted by $D^3 \times S^1$. 

![Figure 1.25]

![Figure 1.26]
Exercise 1.9. Show that the diffeomorphism type of the manifold $M(p,q)$ in Figure 1.27 depends only on parity of $p+q$, it is $D^2 \times \mathbb{RP}^2$ if the parity is even, or else it is $D^2 \tilde{\times} \mathbb{RP}^2$ (a twisted $D^2$-bundle over $\mathbb{RP}^2$). Also prove that there is the identification $\partial(D^2 \tilde{\times} \mathbb{RP}^2) \approx \partial(D^3 \tilde{\times} S^1)$.

![Figure 1.27: M(p,q)](image)

Now if $D^2 \tilde{\times} \mathbb{RP}^2 \subset M^4$, we define Blowing down $\mathbb{RP}^2$ operation as:

$$M \mapsto (M - D^2 \tilde{\times} \mathbb{RP}^2) \cup_\partial (D^3 \tilde{\times} S^1)$$

Figure 1.28 shows how this operation is performed in a handlebody.

![Figure 1.28: Blowing down $\mathbb{RP}^2$ operation](image)

By using this, in [A3], an exotic copy of $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ was constructed (the second picture of Figure 1.29). A discussion of this manifold is given in Section 9.4.

![Figure 1.29: Standard and exotic $S^3 \tilde{\times} S^1 \# S^2 \times S^2$](image)
1 4-manifold handlebodies
Chapter 2

Building low dimensional manifolds

Just as we visualized 4-manifolds by placing ourselves on the boundary of its 0-handle (Figure 1.1) and observing the feet of 1 and 2-handles, we can visualize 2 and 3-manifolds by placing ourselves on the boundary of their 0 handles. By this way we get analogous handlebody pictures as in Figures 2.1 and 2.2, except in this case we don’t need to specify the framings of the attaching circles of the 2-handles. The 3-manifold handlebody pictures obtained this way are called Heegaard diagrams. Clearly we can thicken these handlebodie pictures by crossing them with balls to get higher dimensional handlebodies as indicated in these figures. For example, the middle picture of Figure 2.1 is a Heegard diagram of $T^2 \times I$ and the last picture is $T^2 \times B^2$, and the Figure 2.2 is a Heegaard diagram of $S^3$, and a handlebody of $S^3 \times B^1$.

Figure 2.1
After this, by changing the 1-handle notations to the circle with dot notation, in Figure 2.3 we get a handlebody picture of $T^2 \times B^2$, and proceeding in the similarly way we get a handlebody of $F_g \times B^2$, where $F_g$ denotes the surface of genus $g$.

**Exercise 2.1.** By using this method verify Exercise 1.9, and also show that the handlebody pictures in Figure 2.4 are twisted $D^2$ bundles over $S^2$, $\mathbb{R}P^2$, and $F_g$, respectively (more specifically they are all Euler class $k$ $D^2$-bundles).
2.1 Plumbing

For $i = 1, 2$ let $D_i \hookrightarrow E_i \to S_i$ be two Euler class $k_i$ disk bundles over the 2-spheres $S_i$, and let $B_i \subset S_i$ be the disks giving the trivializations $E_i|_{B_i} \cong B_i \times D_i$. Plumbing $E_1$ and $E_2$ is the process of gluing them together by identifying $D_1 \times B_1$ with $B_2 \times D_2$ (by switching base and fiber directions). Hence, in this manifold if we locate ourselves on the boundary of $D_1 \times B_1 = B^4$ we will see a two 2-handles attached to $B^4$ along the link of Hopf circles $\{\partial D_1, \partial B_1\}$ with framings $k_1, k_2$. This plumbed manifold is symbolically denoted by a graph with one edge and two vertices weighted by integers $k_1$ and $k_2$.

![Figure 2.5](image)

By iterating this process we can similarly construct 4-manifolds corresponding to any graph whose vertices weighted with integers. Also in this construction $S_i$’s don’t have to be spheres, they can be any surfaces (orientable or not), but in this case when we abbreviate these plumbings by graphs, to each of its vertex we need to specify a specific surface along side an integer weight. Reader can check that the manifolds corresponding to the graphs in Figure 2.6 are given by the handlebodies on the right.

![Figure 2.6](image)
The following plumbed manifolds have special names: \( E_8 \) and \( E_{10} \).

\[
E_8 = \begin{array}{cccccccc}
\end{array} = \begin{array}{cccccccc}
\text{Figure 2.7}
\end{array}
\]

An alternative way to denote a plumbing diagram of 2-spheres is to draw the dual of the plumbing graph, where the weighted vertices are replace by weighted intersecting arc segments, intersecting according to the graph (e.g. Figure 2.8).

\[
E_{10} = \begin{array}{cccccccc}
\end{array} = \begin{array}{cccccccc}
\end{array}
\]

\[
\text{Figure 2.8}
\]

### 2.2 Self plumbing

By a method similar to the one above, we can plumb a disk bundle \( D \to E \to S \) to itself. So the only difference is, when we locate ourselves on the boundary of \( D_1 \times B_1 = B^4 \), we will see a link of thickened Hopf circles being identified with each other by a cylinder \((S^1 \times [0, 1]) \times D^2\). This is equivalent to first attaching a 1-handle to \( B^4 \) (thickened point \( \times [0, 1] \)), followed by attaching a 2-handle to the connected sum of the two circles over the 1-handle (a tunnel). Figure 2.9 gives self-plumbings of Euler class \( k \) bundles over \( S^2 \) and over \( T^2 \) (see the “round handle” discussions in Definition 7.7 and Figure 3.10).
2.3 Some useful diffeomorphisms

As shown in Figure 2.10, by introducing a canceling pair of 1 and 2-handles, and sliding new 1-handle over the existing 1-handle, and then canceling the resulting 1 and 2-handle pair, we get an interesting diffeomorphism between the first and last handlebodies of this figure. These diffeomorphisms are used in [AK2] identifying various 3-manifolds.

Exercise 2.2. Let $W^\pm(l, k)$ be the contractible manifolds given in Figure 2.11 (where the integer $l$ denotes $l$-full twist between the stands). By using the above diffeomorphisms construct the following diffeomorphisms:

- $W(l, k) \approx W(l + 1, k - 1)$
- $W^-(l, k) \approx -W^+(l, -k + 3)$
2.4 Algebraic topology

Algebraic topology of a 4-manifold $M^4$ can easily be read off from its handlebody description. For example if $M = M_4$, where $\Lambda = \{K_1^{r_1}, ..., K_n^{r_n}, C_1, ..., C_s\}$ and no 3-handle present, to compute $\pi_1(M)$ for each circle-with-dot $C_i$ we introduce a generator $x_i$, then each loop $K_j$ gives a relation $r_j(x_1, ..., x_s)$ recording how it goes through the 1-handles. Also homology groups of $M$ can be computed from the cell complex given by its 1 and 2-handles. For example, to any $\alpha = \sum c_j K_j$ with $c_j \in \mathbb{Z}$ we can associate $[\alpha] \in H_1(M)$ (by thinking $K_j$ as loops). Alternatively, when $[\alpha] = 0$, we can view $\alpha$ as a CW 2-cycle and think of $\alpha \in H_2(M)$. The intersection form $(\alpha, \beta) \mapsto \alpha.\beta$ on $H_2(M)$

$$q_M : H_2(M; \mathbb{Z}) \otimes H_2(M; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

can be computed from these classes: For example if $\alpha = \sum c_i K_i^{r_i}$ and $\beta = \sum c_j K_j^{r_j}$, then $\alpha.\beta = \sum c_i c_j L(K_i, K_j)$, where $L(K_i, K_j)$ is the linking number of $K_i$ and $K_j$. In particular $L(K_j, K_j)$ is given by the framing $r_j$ of $K_j$. The signature of the bilinear form $q_M$ is defined to be the signature $\sigma(M)$ of $M$. When $M$ is closed and simply connected the intersection form $q_M$ determines the homotopy type of $M$ ([Wh], [M2], [MS]). Note that changing the orientation of the manifold $M_{\Lambda}$ corresponds changing $\Lambda$ to $-\Lambda = \{-K_1^{r_1}, ..., -K_n^{r_n}, C_1, ..., C_s\}$, where $-K$ denotes the mirror image of the knot $K$. So in short $-M_{\Lambda} = M_{-\Lambda}$. Also notice that in $\Lambda$ replacing the dotted circles $C_1, ..., C_s$ with 0-framed unknots does not change the boundary $\partial M_{\Lambda}$ (Exercise 1.2).

Every closed 3-manifold $Y^3$ bounds a compact smooth 4-manifold $M$ consisting of 0- and 2-handles (Chapter 5). When $\Lambda = \{K^r\}$ the following alternative notations will be used interchangeably:

$$M_\Lambda = B^4(K, r) = K^r$$

So $\partial(K^r)$ will be the 3-manifold obtained by surgering $S^3$ along $K$ with framing $r$. If $K \subset Y$, then we will denote the 3-manifold obtained by surgering $Y$ along $K$ with framing $r$ by the notations $Y(K, r)$ or $Y_r(K)$ (provided the framing $r$ in $Y$ is well defined).
2.5 Examples

Links of hypersurface singularities provide rich class of 3-manifolds (Section 12.3), they are obtained by intersecting hypersurfaces in \( \mathbb{C}^3 \) with isolated singularities at the origin with a sphere \( S^5_\epsilon \) of small radius \( \epsilon \), centered at the origin. In particular the Brieskorn 3-manifolds are the boundaries of the following interesting 4-manifolds (Exercise 12.4)

\[
\Sigma(a, b, c) := \left\{ (x, y, z) \in \mathbb{C}^3 \mid x^a + y^b + z^c = 0 \right\} \cap S^5_\epsilon
\]

\[
\begin{align*}
\Sigma(2, 3, 5) &\cong \quad \Sigma(2, 7, 13) \\
\Sigma(2, 3, 7) &\cong \quad \Sigma(2, 3, 9) \\
\Sigma(2, 3, 11) &\cong \quad \Sigma(2, 3, 13)
\end{align*}
\]

Figure 2.12

\( \Sigma(2, 3, 5) \) is called the Poincare homology sphere. There is a useful boundary identification \( f : \Sigma(2, 3, 5) \rightarrow \partial E_8 \) (see Fig 2.7) given by the steps of Figure 2.13

Figure 2.13
Exercise 2.3. By imitating the steps of Figure 2.13 show that $\Sigma(2,3,7) \approx \partial E_{10}$. Also by justifying the diffeomorphisms in Figure 2.14 show that $\partial E_{10} \approx M^4$, where $M^4$ is a manifold obtained from $E_8$ by attaching pair of 2-handles, and $M$ has the intersection form of $E_8 \# (S^2 \times S^3)$.

Exercise 2.4. By justifying steps of Figure 2.15 construct a closed simply connected smooth manifold with signature $-16$ and the second Betti number $22$ (In the figure $M = A + B$ means $M$ contains each of the handlebodies $A$ and $B$, and the handles of $A$ and $B$ has zero algebraic linking number, in which case we can write $M = A + \text{handles} = B + \text{handles}$, and $M$ is homology equivalent to $A \# B$).
\[ \Sigma(2, 3, 13) \text{ bounds a contractible manifold, which can be seen by the identification } \partial\Sigma(2, 3, 13) \approx \partial W^+(1, 0) \text{ as indicated in Figure 2.16.} \]

Exercise 2.5. By blowing up and down operations verify the diffeomorphism of Figure 2.17 (Hint: imitate the initial steps of Figure 2.16).

Exercise 2.6. By verifying the identifications of Figure 2.18, show that \( \Sigma(2, 3, 11) \) bounds a definite manifold with intersection form \( E_8 \oplus (-1) \), and also bounds a smooth simply connected manifold with signature \(-16\) and the second Betti number 20.
Exercise 2.7. By thickening the handlebody of $\mathbb{R}P^2$ in two different ways get two different $D^2$ bundles of $\mathbb{R}P^2$, one is the trivial bundle with nonorientable total space, the other is the twisted bundle with oriented total space (Figure 2.19). The second one is the normal bundle of an imbedding $\mathbb{R}P^2 \subset \mathbb{R}^4$. 

Figure 2.19
2.6 Constructing diffeomorphisms by carving

Given a diffeomorphism \( f : \partial M \approx \partial N \), when does \( f \) extend to a diffeomorphism inside \( F : M \rightarrow N \)? Some instances carving can provide a solution. One necessary condition is that \( f \) must extend to a homotopy equivalence inside, so let us assume this as a hypothesis. Now let us start with \( M = M_\Lambda \), where \( \Lambda = \{ K_1^{r_1}, ..., K_\mathbf{n}^{r_\mathbf{n}}, C_1, ..., C_\mathbf{s} \} \), and let \( \{ \gamma_1, ..., \gamma_k \} \) be the dual circles of the 2-handles. i.e. \( \gamma_j = \partial B_j \), where \( B_j \) is the co-core of the dual 2-handle of \( K_j^{r_j} \). Then if \( \{ f(\gamma_1), ..., f(\gamma_k) \} \) is a slice link in \( N \), that is if each \( f(\gamma_j) = \partial D_j \) where \( D_j \subset N \) are properly imbedded disjoint disks. Then we can extend \( f \) to a diffeomorphism:

\[
 f' : M' := \partial M \cup \cup_j \nu(B_j) \rightarrow \partial N \cup \cup_j \nu(D_j) := N'
\]

where \( \partial M \) and \( \partial N \) are the collar neighborhoods of the boundaries, and \( \nu(B_j) \), \( \nu(D_j) \) are the tubular neighborhoods of the disks \( B_j, D_j \).

This reduces the extension problem to the problem of extending \( f' \) to complements \( M - M' \rightarrow N - N' \). Notice that \( M - M' = \#_s(S^1 \times B^3) \) and \( N - N' \) is a homotopy equivalent to \( \#_s(S^1 \times S^2) \). Since every self diffeomorphism \( \#_s(S^1 \times S^2) \) extends to a unique self diffeomorphism of \( \#_s(S^1 \times B^3) \), the only way \( f' \) doesn’t extend to \( M - M' \rightarrow N - N' \) is \( N - N' \) is an exotic copy of \( \#_s(S^1 \times B^3) \). The case of \( s = 0 \) is particularly interesting, since the only way the last extension problem \( M - M' \rightarrow N - N' \) fails is when the 4-dimensional smooth Poincare conjecture fails (in examples usually this step goes through). Even in the cases of single 2-handle \( M = K^r \) and \( N = L^r \) extending a diffeomorphism \( f : \partial(K^r) \rightarrow \partial(L^r) \) can be difficult task without carving. Because there are no other handles to slide over to construct a diffeomorphisms in a conventional way! Cerf theory says that, if they are diffeomorphic there must be canceling handle pairs, but we don’t know where they are? Carving at least gives us a way to start. For example, Figure 2.21 describes a diffeomorphism between the boundaries of two 4-manifolds ([A1]).

\[
 f : \partial(K^1) \rightarrow \partial(L^1)
\]
Figure 2.21

Figure 2.22 shows the dual circle $\gamma$ and its image $f(\gamma)$ under this diffeomorphism. By carving along $\gamma$ and $f(\gamma)$ we can extend $f$ to a diffeomorphism. To do this we put dots on these circles (making 1-handles). Then we only have to check that the picture on the right is $B^4$. For this we slide two stands of the 2-handle over the 1-handle as indicated in the figure, and then check that we get $B^4$. 

Figure 2.22
The following is a generalization of the previous example from [A4]. We claim that for each \( r \in \mathbb{Z} - 0 \), there are of distinct knot pairs \( K \) and \( L_r \) (one is slice the others are non-slice knots) such that for all \( r \neq 0 \)

\[
K^r \approx L^r_r
\]  

(2.1)

The previous example gives a hint of how to see this by Cerf theory, for this we introduce and then cancel 1- and 2-handle pairs as shown in Figure 2.23.

Exercise 2.8 ([O], [T]). Show that in the boundaries \( \partial M \) and \( \partial N \) of the manifolds of Figure 2.24 the loops \( a \) and \( b \) are isotopic to each other (Hint: slide them over the 2-handle). From this produce distinct knots \( K_r \) and \( L_r \) with \( K^0_r \approx K^0_s \) and \( L^4_r \approx L^4_s \) for all \( r \neq s \) (Hint: consider repeated \( \pm 1 \) surgeries to \( a \) and \( b \)).
2.7 Shake slice knots

Definition 2.1. We call a knot $K$ $r$-shake slice if the link $L = \{K, K+, \ldots, K+, K-, \ldots, K-\}$ consisting of $K$ and an even number of oppositely oriented parallel copies of $K$ (pushed off by the framing $r$) bounds disk with holes. We call $L$ an $r$-shaking of the knot $K$.

As in the (2.1) examples, if $K$ is a knot and $L$ is a slice knot, such that the 4-manifolds $L^r$ and $K^r$ are diffeomorphic, then $K$ is an $r$-shake slice. This is because $L$ being slice there is a smooth 2-sphere $S \subset L^r$ representing the generator of the homology group $H_2(L^r) \cong \mathbb{Z}$ (namely the slice disk which $L$ bounds in $B^4$ together with the core of the handle), then if $F : L^r \to K^r$ is a diffeomorphism, by making $F$ transversal to the cocore of the 2-handle of $K^r$ and by an isotopy, we can make the link $F(S) \cap S^3$ an $r$-shaking of the knot $K$, clearly this link bounds disk with holes (Figure 2.25). Theorem 9.4 gives an example of pair of knots $K_1$ and $K_2$, such that one is slice the other is not, and $K_1^{-1}$ is homeomorphic to $K_2^{-1}$, even though they are not diffeomorphic to each other. So not all knots are shake slice.

![Diagram](Figure 2.25)

Exercise 2.9. Tristram inequality ([Tr]) states that if an oriented link $L$ bounds a disk with holes in $B^4$, then for any prime $p$:

$$|\sigma_p(L)| + n_p(L) \leq \mu(L)$$

where $\sigma_p(L)$, $n_p(L)$ and $\mu(L)$ denote $p$-signature, $p$-nullity, and the number of components of the link $L$. Prove that if $|\sigma_p(K)| \geq 2$ and $p$ divides $r$, then $K$ is not $r$-shake slice (Hint: show that $r$-shaking doesn’t change the $p$-signature, whereas it increases the nullity by $2m$ where $m$ is the number of pairs of oppositely oriented copies to $K$, [A1]).
Chapter 3

Gluing 4 manifolds along their boundaries

Given two connected smooth 4-manifolds with boundary $M$, $N$, and an orientation preserving diffeomorphism $f : \partial M \to \partial N$, we can ask how to draw a handlebody of the oriented manifold obtained by gluing with this map:

$-M \bowtie f N$

Also we can ask when given two disjoint copies of codimension zero submanifolds $L \cup -L \subset \partial M$, how can we draw the manifold $M(f)$ obtained from $M$ by identifying these two copies by a diffeomorphism $f : L \to L$?

$$M(f) = M \cup (L \times [0,1]) / (x,0) \sim (f(x),1), \quad x \in L \cup -L$$

There are two ways of doing this: The upsidedown method, and the cylinder method.

3.1 Constructing $-M \bowtie f N$ by upside down method

Let $M = M_\Lambda$ with $\Lambda = \{K_1^+, ..., K_n^+, C_1, ..., C_s\}$ and $N = N_\Lambda$. Let $\{\gamma_1, ..., \gamma_n\}$ be the dual zero-framed circles of the 2-handles $K_1, ..., K_n$. Then $-M \bowtie f N$ is represented by $Q_{\Lambda''}$ where $\Lambda'' = \Lambda' \sim \{f(\gamma_1), ..., f(\gamma_n)\}$. In particular, if we apply this process to $M^0 := M - \{3 \text{ and } 4\text{-handles}\}$ and any diffeomorphism $f : \partial M^0 \to \partial \# p(S^1 \times B^3)$ ($p$ is the number of 3-handles), we get the upsidedown handlebody of $M$. In the special case of when $M$ has no 3-handles, then clearly the framed link $\{f(\gamma_1), ..., f(\gamma_n)\}$ in $\partial B^4$ gives its upside down handlebody of $M$. 

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The manifold $-M \sim_{id} M$ is called the double of $M$ and denoted by $D(M)$. For example, from the above description we can construct handlebody for $D(T^2 \times B^2) = T^2 \times S^2$. The Cusp is defined to be the 4-manifold $K^0$, where $K$ is the right handed trefoil knot. By using this method, we can easily check that the double $D(C)$ of the cusp $C$ is diffeomorphic to $S^2 \times S^2$ (see Figure 3.2).

**Exercise 3.1.** The Fishtail is defined to be the manifold of the first picture of Figure 3.3. Let $f : \partial F \to \partial F$ be the diffeomorphism on the boundary of the Fishtail described in Figure 3.3 (obtained by the dot and 0 exchange). Show that $-F \sim_f F = S^4$ (given by a canceling pair of 2 and 3-handles).
3.2 Constructing $-M \sim_f N$ and $M(f)$ by cylinder method (Roping)

For this we attach a cylinder $\partial M \times [0,1]$ to disjoint union $-M \sqcup N$, where one end of the cylinder is attached by the identity the other end is attached by the diffeomorphism $f : \partial M \to \partial N$.

$$-M \sim_f N = -M \sim_{id} (\partial M \times [0,1]) \sim_f N$$

Similary, for $M(f)$ we glue a cylinder to $M$ running from $-L$ to $L$

$$M(f) = M \sim_{id \circ f} (\partial L \times [0,1])$$
Think of $f$ as a force field hovering over $-M \sqcup N$ carrying points of $\partial M$ to $\partial N$. To observe the effects of $f$, we lower ropes (from a central point) with hooks tied at their ends, and the hooks go through the cores $\gamma_j$ of the 1-handles of the 3-manifold $\partial M$, then we watch where $f$ takes them, as indicated in Figure 3.5.

![Figure 3.5: The affect of $f$](image)

To describe the process of gluing the two boundary components by $f$, we attach 2-handles to $\gamma_j \neq f(\gamma_j)$ by using the ropes as guide, as shown in Figure 3.6, which is $-M \bowtie_f N$. Put another way, $f$ connects the 0-handle of $M$ with 0-handle of $N$ by a 1-handle (so this 1-handle cancels one of the two 0-handles), then by going over this 1-handle it identifies the neighborhoods of $\gamma_j$ with $f(\gamma_j)$, which amounts to attaching 2-handles to $\gamma_j \neq f(\gamma_j)$ (i.e. creating tunnels). We don’t need to describe how $f$ identifies the 2- and 3-handles of $M$ with that of $N$, since by the similar description, they amount to adding 3- and 4-handles to $-M \sqcup N$. Recall that describing 4-manifold handlebodies we don’t need to specify 3- and 4-handles, because they are always attached canonically.
3.2 Constructing \( -M \sim_f N \) and \( M(f) \) by cylinder method (Roping)

The handlebody of \( M(f) \) can be constructed almost the same way, except we must have an extra 1-handle as shown in Figure 3.7, because in this case the identifying cylinder \( L \times [0,1] \) is attached to a single connected manifold \( M \), as opposed between two disjoint manifolds \( -M \sqcup N \). In the former case, this 1-handle was cancelled by one of the 0-handles of \( -M \sqcup N \).

![Figure 3.8: \( -M \sim_f N \)](image)

A simple but useful example of this process is so called *round 1-handle* attachment (Definition 7.7), which is obtained by identifying the tubular neighborhoods of two disjoint framed circles \( \{C_1, C_2\} \) lying in \( \partial M^4 \), hence it can be described by attaching a 1 and 2 handle pair, where the 2-handle is attached to the curve obtained by connected summing \( C_1 \) and \( C_2 \) along the 1-handle as shown in Figure 3.10 by the framing induced from the framings of \( C_1 \) and \( C_2 \).

![Figure 3.9: \( M(f) \)](image)

![Figure 3.10: Round 1-handle](image)
3.3 Codimension zero surgery \( M \mapsto M' \)

Let \( N \subset M \) be a codimension zero submanifold giving the decomposition \( M = N \cup C \) where \( N \) is the complement of \( C \), and let \( f : \partial N \to \partial N' \) be a diffeomorphism. We call the process of cutting \( N \) out and gluing \( N' \) by \( f \) a codimension zero surgery.

\[
M \mapsto M' := N' \cup_C f \]

To visualize this process think of all the handles of \( C \) as placed at the top of \( N \). Notice that only the 2-handles of \( C \) interacts with the handles of \( N \). So think of \( N \) as hanging from the 2-handles of \( C \) (Figure 3.11) like hangers in a dress closet, where \( f \) shuffles the hangers below. So while the handles of \( N \) are hanging from the top, they look differently according to how \( f \) rearranged them below.

![Figure 3.11: Codimension zero surgery \( M \mapsto M' \)](image)

More precisely, we apply diffeomorphism \( f : \partial N \to \partial N' \) keeping track of where \( f \) throws the 2-handles of \( C \) in \( N' \). That is, if \( N = N_\Lambda \) and \( N' = N'_\Lambda' \), such that \( M = M_\Lambda \) with \( \Lambda = \Lambda \sim \{K_1^{r_1}, \ldots, K_p^{r_p}\} \), then \( M' = M'_{\Lambda'} \) where \( \Lambda' = N' \sim \{f(K_1^{r_1}), \ldots, f(K_p^{r_p})\} \).

In this case we say that \( M \) is obtained from \( M' \) by cutting out \( N' \), and gluing \( N \) via \( f : \partial N \to \partial N' \). So if \( M' = M'_\Lambda \), with \( \Lambda' = \Lambda' \sim \{L_1^{s_1}, \ldots, L_p^{s_p}\} \), then \( M = M_\Lambda \) where \( \Lambda = \Lambda \sim \{f^{-1}(L_1^{s_1}), \ldots, f^{-1}(L_p^{s_p})\} \).
Chapter 4

Bundles

Here we describe handlebodies of 4-manifolds which are bundles. First a simple example:

4.1 \( T^4 = T^2 \times T^2 \)

We start with \( T^2 \) (Figure 2.1) then thicken it to \( T^2 \times [0, 1] \), then by identifying the front and the back faces of \( T^2 \times [0, 1] \) we form \( T^3 \) Figure 4.1 (recall the recipe of Section 3.2), we then construct \( T^4 \) by identifying the front and back faces of \( T^3 \times [0, 1] \) (Figure 4.2).

\[ \begin{align*}
\text{Figure 4.1: } T^3
\end{align*} \]
By converting the 1-handle notation to the circle-with-dot notation of Section 1.1, in Figure 4.5 we get another handlebody picture of $T^4$. For the benefit of the reader we do this conversion gradually: For example we first flatten the middle 1-handle balls as in Figure 4.3 obtaining Figure 4.4, then convert them to the circle with dot notation with ease obtaining the final Figure 4.5.

**Exercise 4.1.** Show that without the 2-handle $v$, Figure 4.2 describes $T^2_0 \times T^2$, and deleting two 3-handles from $T^2_0 \times T^2$ gives just $T^2_0 \times T^2_0$, where $T^2_0 = T^2 - D^2$ is the punctured $T^2$. 

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4.2 Cacime surface

Cacime is a particular surface bundle over a surface, which appears naturally in complex surface theory [CCM]. Understanding this manifold is instructive, because it is a good test case for understanding many of the difficulties one encounters constructing handlebodies of surface bundles over surfaces. We will first draw a handlebody of this manifold, then from this drive a recipe for drawing surface bundles over surfaces in general.

Let \( F_g \) denote the surface of genus \( g \). Let \( \tau_2 : F_2 \to F_2 \) be the elliptic involution with two fix points, and \( \tau_3 : F_3 \to F_3 \) be the free involution induced by 180° rotation (Figure 4.6). The Cacime surface \( M \) is the complex surface obtained by taking the quotient of \( F_2 \times F_3 \) by the product involution:

\[
M = (F_2 \times F_3)/\tau_2 \times \tau_3
\]

By projecting to the second factor we can describe \( M \) as a \( F_2 \)-bundle over \( F_2 = F_3/\tau_3 \).
Let $A$ denote the twice punctured 2-torus $A = T^2 - D_2^2 \cup D_2^2$. Clearly $M$ is obtained by identifying the two boundary components of $F_2 \times A$ by the involution induced by $\tau_2$ (notice that $A$ is a fundamental domain of the action $\tau_3$).

\[
\begin{array}{c}
F_2 \times \begin{array}{c}
\tau_2
\end{array} \\
\hline
\end{array}
\begin{array}{c}
F_2 \times T^2 \\
\hline
\end{array}
\begin{array}{c}
\tau_2 \\
\hline
\end{array}
\end{array}
\]

Figure 4.7: $T^4$

By deforming $A$ as in Figure 4.7, we see that $M = E \natural E'$ is the fiber sum of two $F_2$ bundles over $T^2$, where $E = F_2 \times T^2 \to T^2$ is the trivial bundle and $E'$ is the bundle:

\[
E' = F_2 \times S^1 \times [0,1]/(x,y,0) \sim (\tau_2(x),y,1) \to T^2
\]

Now by using the techniques developed in Chapter 3, step by step we will construct the following manifolds and diffeomorphisms:

(a) $E_0 := E - F_2 \times D^2 = F_2 \times T_0^2$

(b) $f_1 : \partial E_0 \to F_2 \times S^1$

(c) $E'_0 = E' - F_2 \times D^2$

(d) $f_2 : \partial E'_0 \to F_2 \times S^1$

(e) $M = -E_0 \sim f_2^{-1} \circ f_1 E'_0$
By imitating the construction of $T^4$ in the Figures 4.1 through 4.5, in Figures 4.8 through 4.9 we build $F \times T^2$.

**Exercise 4.2.** Show that if we remove the 2-handle $c$ from the Figure 4.9, we get a handlebody picture for $E_0 = F_2 \times T^2$.

(b) We claim that there is a diffeomorphism $f_1 : \partial E_0 \longrightarrow F_2 \times S^1$ which takes the ropes with hooks of Figure 4.9 to the corresponding ropes as indicated in Figure 4.10 (see Section 3.2 for discussion of ropes).

**Exercise 4.3.** Show that this diffeomorphism $f_1$ can be obtained by the handle slides on the boundary as indicated in Figure 4.11 (Hint just perform the indicated handle slides while keeping track of the ropes)
(c) $E'_0 = E' - F_2 \times D^2$ is a twisted version of $E_0$, we proceed as in Figure 4.8 except that we identify the front and back faces of $F_2 \times [0, 1]$ by $\tau : F_2 \to F_2$ (Figure 4.12 depicts the induced map on $F_2$ by $\tau$) and get Figure 4.13, which is a twisted version of Figure 4.8. To construct $E'$ we simply cross this twisted $F_2$ bundle over the circle $F_2 \times \tau S^1$ with $S^1$. This gives Figure 4.14 (drawn in two different 1-handle notations), which is the analogous version of Figure 4.9.
Exercise 4.4. Show that without the 2-handle \( c \) Figure 4.14 describes \( E'_0 = E' - F_2 \times D^2 \).
(d) Similar to (b) there is a diffeomorphism $f_2 : \partial E'_0 \rightarrow F_2 \times S^1$ obtained by the handle slides described in Figure 4.15. As before, to describe this diffeomorphism geometrically in pictures we lower ropes (with hooks) and trace out what $f$ does to these ropes during the diffeomorphism of Figure 4.15. This gives Figure 4.16.
Finally by applying the recipe of Section 3.2 we construct the manifold $M = -E_0 \sim_{f_2^{-1} \circ f_1} E'_0$, which is Figure 4.17. Notice that the ropes in $\partial E_0$ at the left picture of Figure 4.10 are mapped to the ropes at the left picture of Figure 4.16 by $f_2^{-1} \circ f_1$.

Figure 4.17: Cacime surface
4.3 General surface bundles over surfaces

Now it is clear how to proceed drawing a handlebody picture of a general $F_g$ bundle over $F_p$ denoted by $M = F_g \times F_p$: We first decompose $M$ as a fiber sum of $F_g$ bundles over $T^2$, with monodromies $\tau_j : F_g \to F_g$ ($j = 1, \ldots, p$), as shown in Figure 4.18. By removing $F_g \times D^2$ from each, we write $M = \sqcup \partial E_j$ where each $E_j$ is a $F_g$ bundle over $T_0^2$, then we perform gluing operations along the boundaries described in Chapter 3.

![Figure 4.18](image)

4.4 Circle bundles over 3-manifolds

Let $Y^3 = H_g \sim_f H_g$ be a 3-manifold given by its Heegard decomposition, i.e. $H_g$ is the solid handlebody bounding $F_g$, and $f : F_g \to F_g$ is a diffeomorphism (see Chapter 5).

**Exercise 4.5.** Show that any oriented circle bundle $M^4$ over $Y^3$ is a union of two copies of $S^1 \times H_g$ glued along their boundaries by a diffeomorphism. A handlebody for $M$ can be constructed by applying the gluing techniques of Chapter 3 to two copies of the handlebody of $S^1 \times H_g$ (Figure 4.19).

![Figure 4.19](image)
4.5 3-manifold bundles over the circle

Section 3.2 essentially gave a recipe for drawing the handlebody picture of a 3-manifold bundle over the circle with a monodromy $f: Y^3 \to Y^3$:

$$M^4 = M(f) = Y \times [0,1]/(x,0) \sim (f(x),1)$$

To put this technique in practice, we will apply it to a very interesting example $Q^4$, which is an exotic copy of $\mathbb{RP}^4$. Recall that $\mathbb{RP}^4 = N \sim_\partial C$ is a union of codimension zero submanifolds $N$ and $C$, glued along their common boundaries, where $N$ is a twisted $B^2$-bundle over $\mathbb{RP}^2$, and $C$ is the nonorientable $B^3$-bundle over $\mathbb{RP}^1$ (see Section 1.5). Similarly $Q^4 = N \sim_\partial C_A$ is the union of codimension zero submanifolds $N$ and $C_A$, glued along their common boundaries, where $C_A$ is the mapping torus of the diffeomorphism $f_A: T^3_0 \to T^3_0$ induced by the following integral matrix $A: \mathbb{R}^3/\mathbb{Z}^3 \to \mathbb{R}^3/\mathbb{Z}^3$ (4.1). It is easy to see that $C_A$ is homology equivalent to $C$, and $\pi_1(Q) = \mathbb{Z}_2$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$ (4.1)

The exotic manifold of Figure 1.29 [A3] was derived from $Q^4$. Also, the double covering of $Q$ is the homotopy sphere $\Sigma = D^2 \times S^2 \sim C_B$, where $C_B$ is the mapping torus of the diffeomorphism $f_B: T^3_0 \to T^3_0$ induced from $B = A^2$. $\Sigma$ is the first member $\Sigma_0$ of a similarly defined infinite family of homotopy spheres $\Sigma_m$, which turned out to be diffeomorphic to the standard 4-sphere ([AK1], [G3], [A5]).

Let us draw a handlebody picture of $\Sigma$ (following [AK1]). Notice that $C_B = M(f)$, where $M = T^3 \times [0,1]$ and $f$ is the diffeomorphism $f_B$. By applying the technique of Section 3.2 we draw $M(f)$. For this we start with a Heegaard picture of $T^3$ as drawn 1- and 2-handles on $S^2$ (Figure 4.1).

![Figure 4.20: $T^3$](image)

We then study $f_B: T^3 \to T^3$ by first checking what it does the coordinate axis, i.e. the cores of the 1-handles corresponding to the hooks of the “ropes with hooks”}

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technique of Section 3.2. Then Figure 4.22 gives $M(f)$. For simplicity we put one of the attaching balls of the 1-handle (the companion of the center ball) at $\infty$.

$$B = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$ (4.2)

Figure 4.21: Isotoping $f_B$ so that it takes the coordinate axis to the coordinate axis

Exercise 4.6. Convert the 1-handles of Figure 4.22 to the circle-with dot notation (this is done in Figure 14.2), and locate the framed circle of the 2-handle corresponding to $D^2 \times S^2$ of $\Sigma$, and decide the parity of its framing (in [AK1] this was mistakenly assumed to be even, but in [ARu] it was shown to be odd).
Chapter 5

3-manifolds

Every closed orientable 3-manifold $Y^3$ bounds an closed orientable 4-manifold (e.g. [Li], [W]), in fact it bounds a 4-manifold $M_\Lambda$ consisting of 0- and 2-handles. One can see this from the Heegard decomposition of $Y^3 = H_g \cup_{\tau} -H_g$, which is basically union of two copies of a solid handlebody (i.e. the handles of index $\leq 1$, and the handles of index $\geq 2$ respectively) glued along their boundary $\partial H_g := F_g$ by an orientation preserving diffeomorphism $\tau : F_g \to F_g$. It is known that any surface diffeomorphism can be written as a composition of Dehn twists $\tau = \tau_{C_1}...\tau_{C_n}$ along imbedded curves $C_j \subset F_g$.

Definition 5.1. Dehn twist $\tau_{C} : F \to F$ along $C$ is a diffeomorphism, which is identity outside of a cylinder $C \times [-1,1] \subset F$ centered at $C$, and a (right or left) full rotation inside the cylinder as in Figure 5.1 so that $\tau_{C} : C \to C$ is the antipodal map (c.f. [Iv]).

![Figure 5.1: \(\tau_C\)]

Repeated applications of the following implies that any closed oriented $Y^3$ is obtained by an integral surgery to a framed link, i.e. $Y^3 = \partial M_\Lambda$ for some $\Lambda$.

Exercise 5.1. For $0 \leq k \leq n$ let $Y_k = H_g \cup_{\tau_k} H_g$, where $\tau_k = \tau_{C_1}...\tau_{C_k}$ with $\tau_0 = id$. Show that $Y_0 = \#_g S^1 \times S^2$, and $Y_k$ is obtained from $Y_{k-1}$ by an integral surgery to a knot in $Y_{k-1}$ isotopic to $C_k$ (Hint: Push the last curve $C_k$ to the interior of the solid handlebody $H_g$, then compare the manifolds $Y_{k-1}$ and $Y_k$, cf. [Sa]).

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5.1 Dehn surgery

Dehn surgery of a 3-manifold $Y^3$ is an operation of taking out an imbedded copy of $S^1 \times B^2$ from $Y^3$ and regluing it back by some diffeomorphism of its boundary. Let $K \subset Y^3$ be a null homologous knot, and $N(K)$ be its tubular neighborhood. Let $\mu$ and $\lambda$ be the meridian and longitude of $\partial N(K)$. This means $\lambda$ is the parallel copy of $K$ which is null homologous in $Y - K$. Let $p, q$ be coprime integers. Fix a trivialization:

$$\phi : B^2 \times S^1 \to N(K)$$

such that $\phi(1, 0) = \mu$ and $\phi(0, 1) = \lambda$, where $(1, 0)$ and $(0, 1)$ denotes the two generators of $S^1 \times S^1$. For simplicity we will view $\phi$ as an identification, and parametrize the curves on $\partial N(K)$ by $(p, q) \leftrightarrow p\mu + q\lambda$. Then $r = p/q$ surgery to $Y^3$ is the manifold.

$$Y(K, r) = [Y - N(K)] \varphi_r (B^2 \times S^1)$$

where $\phi_r : \partial B^2 \times S^1 \to \partial N(K)$ is the unique diffeomorphism with $\phi_r (1, 0) = (p, q)$. With above identifications we can express the gluing map by the matrix (cf. [Ro])

$$\phi_r = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

where $ps - qr = 1$. In particular $Y(K, \infty) = Y$. For brevity, when there is no danger of confusion we will abbreviate $K^r = Y(K, r)$, even though previously when $r$ is an integer we used this notation for the 4-manifold obtained $B^4$ by attaching 2-handle to $K$.

**Exercise 5.2.** Let $K \subset S^3$ be a knot. By using the identity (5.1), justify the 3-manifold diffeomorphism in Figure 5.2 (Hint: By the diffeomorphism of Figure 5.3 reduce this to the case $\{K, \mu\}$ is the Hopf link, and also note that in $S^3(K, n)$, the circle $\mu$ isotopes to the center circle in the surgery solid torus of $B^2 \times S^1$)

$$\begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} np - q & nr - s \\ p & r \end{pmatrix}$$

(5.1)

**Exercise 5.3.** By iterating the diffeomorphism of Exercise 5.2 justify the 3-manifold diffeomorphisms of Figure 5.4, where $p/q = [a_1, a_2, ..., a_k]$ is the continued fraction expansion (when $a_i \geq 2$ the expression is unique).

$$p/q = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_k}}}$$
5.2 From framed links to Heegard diagrams

Given a 4-manifold $M^4 = M_\Lambda$, with $\Lambda = \{K_1^{r_1}, \ldots, K_n^{r_n}\}$, how can we construct the Heegard diagram of its boundary 3-manifold $Y_\Lambda^3 := \partial M_\Lambda$? First note that if the complement $C(\Lambda) := S^3 - \cup_j N(K_j)$ of the tubular neighborhood of the framed link $\Lambda$ is a 1-handlebody (i.e. $B^3$ with 1-handles), then $(C(\Lambda), K_1, \ldots, K_n)$ gives a Heegard decomposition of $Y_\Lambda$. To use this fact, we first remove solid pipes (i.e. properly imbedded thickened arcs) from $C(\Lambda)$ to turn it to a 1-handlebody $C'(\Lambda) = C(\Lambda) - \cup N(I_j)$. To undo the damage we caused by removing pipes $I_j$ from the complement, we attach 2-handles $\gamma_j$ to the meridians of $I_j$'s. Then the end result is a Heegaard decomposition, given by the 3-dimensional 1-handlebody $C'(\Lambda)$ along with simple disjoint simple closed curves $\{K_1, \ldots, K_n, \gamma_1, \ldots \gamma_s\}$ lying on its boundary, signifying the 2-handles. Any Heegard diagram can be described by a genus $g$ surface $F$ with two sets of disjoint simple closed curves $\alpha$ and $\beta$ (compressing curves), each describing a solid handlebody. Here in our case $F = \partial C'(\Lambda)$ and $\beta = \{K_1, \ldots, K_n, \gamma_1, \ldots \gamma_s\}$, and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$, $(r = n + s)$ are the obvious collection of curves on $F$ compressing in $C'(\Lambda)$. For example, by applying this process to $\Lambda = \{K^2\}$, where $K$ is the trefoil knot, we get Figure 5.6.
Exercise 5.4. By considering moving pictures (and the above process), identify the complement of a properly imbedded 2-disk in $B^4$, with a single transverse self intersection, with the Fishtail (Hint: Consider the isotopies of Figure 5.7). Compare this with the self plumbing operation of Figure 2.9.
5.3 Gluing knot complements

Let \( K, L \subset S^3 \) be knots, and let \( \phi : \partial N(K) \to \partial N(L) \) be a diffeomorphism, between the boundaries of the tubular neighborhoods of these knots. We want to find a 4-manifold \( M_\phi(K,L) \) whose boundary is the 3-manifold formed by gluing the knot complements:

\[
\partial M_\phi(K,L) \approx (S^3 - N(K)) \cup_{\phi} (S^3 - N(L))
\]

We do this by the method of Section 3.2. First by using \( \{\mu, \lambda\} \) coordinates express

\[
\phi = \left( \begin{array}{c} p \\ q \\ r \\ s \end{array} \right)
\]

When \( K \) and \( L \) are the unknots \( U \), we get \( \partial M_\phi(U,U) = L(p,q) \) (Lens space). Therefore we can take the plumbing \( M_\phi(U,U) = C_{p,q} \) described in Figure 5.8, where \( p/q = [a_1, a_2, \ldots, a_k] \) (e.g. [Ro]). Write \( C = S^3 - N(U) \approx S^1 \times B^2 \) so that \( L(p,q) = C \cup_{\phi} C \).

Figure 5.8: \( C_{p,q} \)

Next we will extend this process to any knots. Let \( \gamma \) and \( \gamma' \) be the dual circles of \( K \) and \( L \) in \( S^3 \), respectively. Let \( N(\gamma) \) denote the tubular neighborhood of \( \gamma \) in \( \partial(K^0) \).

Notice that we have \( \partial(K^0) - N(\gamma) \approx S^3 - N(K) \). We need to identify boundaries two such knot complements with a cylinder \( (S^1 \times S^1) \times [0,1] \) via \( \phi \), which is the boundary of the manifold obtained from the disjoint union \( K^0 \cup C_{p,q} \cup L^0 \), by identifying \( N(\gamma) \) with one copy of \( C \) in \( L(p,q) \), and identifying the other copy of \( C \) with \( N(\gamma') \):

\[
M_\phi(K,L) = K^0 \sim U \sim C_{p,q} \sim U' \sim L^0
\]
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Exercise 5.5. Show that Figure 5.10 gives a handlebody for $M_\phi(K, L)$.

Exercise 5.6. Show that Figure 5.11 gives $\partial M_\phi(K, L)$ when $\phi = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}$.

Exercise 5.7. By modifying above construction verify that Figure 5.12 also describes $\partial M_\phi(K, L)$ for some $\phi$, where $K$ is the trefoil knot, and $L$ is the figure eight knot.
5.4 Carving 3-manifolds

By applying the carving principle of Section 1.1 to 3-manifolds, we can view any 1-handle of a 3-manifold as a carved out (complement of) properly imbedded unknotted thickened arc in $B^3$; for this we can introduce a 3-manifold analogue of “circle with dot” notation: We denote this by a pair of thick dots and a dotted line connecting them signifying the 1-handle membrane (any arc intersecting this is going over the 1-handle) (Figure 5.13).

![Figure 5.13](image)

By this we can give a 3-dimensional analogue of the “Carving ribbons from $B^4$” operation discussed in Section 1.4. Here we are carving knotted arcs in $B^3$. The analogy is completely similar. For example in Figure 5.14 we start with $S^0$ at top, then by moving pictures introduce 1 and 2- canceling handle pair, and end up with a pair of 1-handles (represented by $S^0 \sqcup S^0$) and a 2 handle (the view from the bottom). In this case we get the manifold of Figure 5.13.

![Figure 5.14](image)
5.5 Rohlin invariant

Let $Y^3$ be an orientable closed 3-manifold with $H^1(Y;\mathbb{Z}_2) = 0$. We know that $Y^3 = \partial M_\Lambda$, for some framed link $\Lambda = \{ K_1^{r_1}, ..., K_n^{r_n} \}$. In fact we can assume that $\Lambda$ is an even framed link, i.e. all $r_1, ..., r_n$ are even integers. This can be seen by first choosing a framed knot $K = \lambda_1 K_1^{r_1} + \ldots + \lambda_n K_n^{r_n}$ with $\lambda_j \in \mathbb{Z}_2$, representing a homology class $\alpha \in H_2(M_\Lambda)$ (Section 2.4) satisfying

$$\lambda_1 a_{1j} + \ldots + \lambda_n a_{nj} = r_j \quad j = 1, \ldots, n$$

(5.2)

where $a_{ij} = L(K_i, K_j)$ (so $a_{jj} = r_j$). These equation can easily be solved over $\mathbb{Z}_2$ (why), but solutions are not unique. For example, when $\Lambda$ is an even link $K = 0$ is a solution.

**Exercise 5.8.** Show that and the homology class $\alpha$ obtained from the equations (5.2) satisfies the so called Wu relation $\alpha.x = x.x$ (mod 2), $\forall x \in H_2(M_\Lambda)$. In other words $\alpha$ intersects odd classes odd times, even classes even times

**Remark 5.2.** Let $M := M_\Lambda$ be a closed 4-manifold, then the mod 2 Poincare dual of any integral homology class satisfying the Wu relation represents the second Steifel-Whitney class $w_2(M_\Lambda) \in H^2(M,\mathbb{Z}_2)$. Any such class is called characteristic. Characteristic classes satisfy the following property (e.g. [Bl], [MS]).

$$\sigma(M) = \alpha.\alpha \pmod{8}$$

**Exercise 5.9.** Show that we can choose $\Lambda$ with $\partial M_\Lambda = Y$, so that $\alpha = [K]$, where $K$ is the connected sums of trefoil knots (Hint: Write a Seifert surface $F$ of $K$ as a disk with handles, then by blowing up $M$ respecting the Wu property ($\pm 1$ blowup circles should always link $K$ odd times) make the handles unknotted and unlinked from each other, and then by blowings up across three stands as in Figure 2.16 change the twisting on each handle by 2, so to end up with 1 twisting). Then show that by blowups we can transform $K$ to a 1-framed unknot, and by blowing it down we transform $\Lambda$ to an even framed link.

So by this exercises we can make $K$ the empty knot, i.e. can choose the bounding 4-manifold $M_\Lambda$ to be even, this means that the corresponding $M^4$ has a spin structure (i.e. $w_2(M) = 0$) inducing a spin structure on the boundary $(Y,s)$, which means a trivialization of the tangent bundle $TY$. A theorem of Rohlin states that the signature of a closed spin 4-manifold $(M,s)$ satisfies the following congruence ([Roh], [K3]):

$$\sigma(M) = 0 \pmod{16}$$

So the following is well defined:
**Definition 5.3. (Rohlin invariant)** Let \((Y^3, s)\) be a spin 3-manifold, let \((M^4, s)\) be a spin 4-manifold bunding \((Y, s)\), then

\[
\mu(Y, s) = \sigma(M) \pmod{16}
\]

Note that when \(Y\) is a homology sphere, it has a unique spin structure, so from the definition we can drop the dependency to spin structure \(s\) and write \(\mu(Y, s) = \mu(Y)\). For example, by using Section 2.5 we compute \(\mu(\Sigma(2, 3, 5)) = \mu(\Sigma(2, 3, 7)) = -8\).
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Chapter 6

Operations

6.1 Gluck twisting

It is known that any diffeomorphism $S^2 \times S^1 \to S^2 \times S^1$ is either isotopic to the identity, or isotopic to the map $\varphi : S^2 \times S^1 \to S^2 \times S^1$ defined by (e.g. [Wa3])

$$\varphi(x, t) = (\varphi_t(x), t)$$

where $S^1 \ni t \mapsto \varphi_t \in SO(3)$ is the nontrivial element of $\pi_1 SO(3) = \mathbb{Z}_2$. Let $X$ be a smooth 4-manifold, and $S \subset X$ be a copy of $S^2$ imbedded with trivial normal bundle $S^2 \times B^2 \subset X$ (i.e. homological self intersection is zero: $S.S = 0$). We call the operation

$$X \mapsto X_S := (X - S^2 \times B^2) \sqcup (S^2 \times B^2)$$

Gluck twisting. Best way to understand how a Gluck twisting operation alters the handles of $X$ is by drawing a handlebody of $X$ as handles attached to the top of this particular $S^2 \times B^2$ (an unknot with 0-framing). Then the Gluck twisting corresponds to the operation of putting one twist to all the stands going through the 0-framed circle:

![Figure 6.1](image-url)
Exercise 6.1. Show that the Gluck twisting operation is equivalent to either one of the following operations:

(i) The operation described in Figure 6.2 (where \( n, m \) are arbitrary).

(ii) The zero and dot exchange operation of Figure 6.3.

If \( X \) is simply connected and \( S \subset X \) is null homologous, then it is clear that the operation \( X \mapsto X_S \) does not change the homology groups and the intersection form of \( X \), and also keeps \( X \) simply connected, hence \( X_S \) is h-cobordant to \( X \) and hence it is homomorphic to \( X \) by [F]. Also, when the intersection form of \( X \) is odd then the Gluck twisting does not change the smooth structure of \( X \) under some conditions (e.g. \([AY1]\)).

Theorem 6.1. Let \( X \) be simply connected smooth 4-manifold, and \( S \subset X \) be a 2-sphere with trivial normal bundle, then \( X_S \approx X \), provided in \( X - S \) there is a 2-dimensional spherical homology class with odd self intersection.

Proof. Let \( X^{(i)} \) denote the union of handles index \( \leq i \). Let \( \alpha \in H_2(X, \mathbb{Z}) \) be a spherical homology class represented by \( f : S^2 \to X - S \) with \( \alpha \alpha \) odd. First we claim \( \alpha \) can be represented by a 2-handle in \( X - S \) whose attaching circle is null homotpic in \( X^{(1)} \). This
is because after a homotopy we may assume that \( f \) is an immersion, and its image is contained in \( X^{(2)} \) (since the cocores of a 3-handles are codimension 3). Let \( p_1, p_2, \ldots, p_n \) be the transverse intersection points of \( f(S^2) \) and the cocores of 2-handles of \( X \). Let \( D_i \) (\( 1 \leq i \leq n \)) be a small 2-disk neighborhoods of \( f^{-1}(p_i) \) in \( S^2 \). We may assume that each \( f(D_i) \) is a core of a 2-handle of \( X \) and that \( f(S^2 - \bigsqcup_{i=1}^{n} D_i) \) is contained in \( X^{(1)} \). For each \( i \), take an arc \( \gamma_i \subset S^2 \) connecting \( D_i \) and \( D_{i+1} \), with \( \nu(\gamma_i) \) its tubular neighborhood, so that \( \Sigma := \left( \bigsqcup_{i=1}^{n} D_i \right) \cup \left( \bigsqcup_{i=1}^{n-1} \nu(\gamma_i) \right) \) is a 2-disk in \( S^2 \). We can assume \( f|\nu(\gamma_i) \) are embeddings into \( \partial X^{(1)} \). This gives a 2-handle \( h^2 \) of \( X \) representing \( \alpha \), whose attaching circle \( f(\partial \Sigma) \) is null homotopic in \( X^{(1)} \) (introduce a canceling 2/3-handle pair and slide its 2-handle over the 2-handles of \( X \) until it becomes \( f(\partial \Sigma) \)).

The Gluck twisting operation corresponds to zero and dot exchanges between the zero framed 2-handle \( S \) and the dotted circle \( T \) as in Figure 6.4. Let \( K \) be the attaching circle of \( h \). By first sliding the middle 1-framed handle over \( K \) we can make its framing even, and then by sliding over \( T \) make it zero, and finally by using the “null-homotopy” assumption we can slide it over \( T \) as shown in Figure 6.5 to make it unknotted and end get the third picture of Figure 6.4. Now it is clear that the zero and dot exchange between \( S \) and \( T \) does not change the diffeomorphism type, i.e. \( X_S \approx X \).

\[ \text{Figure 6.4} \]

\[ \text{Figure 6.5} \]
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6.2 Blowing down ribbons

Let us decompose the 2-sphere as a union of upper and lower hemispheres $S^2 = D^2 \cup_\varnothing D^2_+$, then since the diffeomorphism $t \mapsto \varphi_t(x)$ is an isotopy of (doner kebab) rotation of $S^2$ by the north-south-pole axis, along the circle; we can restrict it to pieces $S^1 \times D_+$

\[ \text{Diff}(S^1 \times S^2) \to \text{Diff}(S^1 \times D^k) \]

Also we can arrange any imbedding $S^2 \subset X^4$ so that $D_+ \text{ lies in the top 4-handle } B^4$ as the trivial disk, and $D_-$ as a slice disk in $X_0 := X - B^4$ bounding the unknot $\partial D_+$. Then we can perform the Gluck twisting operation on two parts separately then take their union: (1) Twist $X_0$ along $D_-$, and (2) Twist $B^4$ along $D_+$ which gives back $B^4$.

![Figure 6.6](image)

It is particularly interesting when $D_-$ is a ribbon in $X_0$, since we know how to draw pictures of ribbons (Section 1.4). For example a typical ribbon in $X_0$ will look like Figure 6.7 (plus the handles of $X_0$ which is not drawn in the figure), then the Gluck twisting operation to $X_0$ along $D_-$ corresponds to attaching a $\pm 1$ framed 2 handle to the trivially linking circle $C$ of $K$ (why?). When $X_0 = B^4$ this gives a contractible manifold.

![Figure 6.7: Ribbon complement $B^4 - D_-$](image)

![Figure 6.8: $B^4$ twisted along $D_-$](image)

**Exercise 6.2.** Show that Figure 6.8 is the picture of $S^4$ Gluck twisted by the 2-sphere, which is the double of the ribbon surface described in Figure 6.7. Conclude that Gluck twisting $S^4$ along a 2-knot $S^2 \subset S^4$ which is the double of a ribbon, gives $S^4$ (Hint: By sliding 2-handles over the small 0-framed linking circles move them to trivial position).
6.3 Logarithmic transform

Given a smooth 4-manifold $X^4$ and an imbedding $T^2 \times B^2 \subset X^4$, the operation of removing this $T^2 \times B^2$ and then gluing it back by a nontrivial diffeomorphism of its boundary $\varphi : T^3 \to T^3$ is called $T^2$-surgery operation. It is known that any closed smooth 4-manifold can be obtained from a connected sum of number of copies of $S^1 \times S^3$ and $\pm \mathbb{C}P^2$ by a sequence of these operations. [I]. A special case of this operation is called $p$-log transform operation, when the map $\phi_p : T^2 \times S^1 \to T^2 \times S^1$ ($p \in \mathbb{Z}$) corresponds to:

$$\phi_p = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & p
\end{pmatrix}$$

This operation $X \mapsto X_p$ is called a logarithmic transform of order $p$, because the degree of the composition of the maps below has order $p$.

$$S^1 \xrightarrow{\text{inc}} T^2 \times S^1 \xrightarrow{\phi_p} T^2 \times S^1 \xrightarrow{\text{proj}} S^1$$

By using technique of Section 3.3 we can draw a handlebody picture of this operation (compare to [AY2], [GS]). The recipe of Section 3.3 says that given $T^2 \times D^2 \subset X$, write $X = T^2 \times D^2 \scissor (\text{other handles})$, then carry the other handles by the inverse diffeomorphism $\phi_p^{-1}$ to top of $T^2 \times D^2$.

$$\phi_p^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & p & -1 \\
0 & 1 & 0
\end{pmatrix}$$

Figure 6.9 describes this handlebody operation with pictures (check the three circles of $T^3$ is mapped by $\phi_p^{-1}$). This figure gives a picture recipe of how to modify a 4-manifold handlebody, containing a framed torus $T^2 \times D^2$, in order to get the handlebody of the $p$-log transformed 4-manifold along this torus. For example Figure 6.9 describes how the linking loop B is changed by this operation. To emphasize the core torus, this operation is referred as the $p$-log transformation operation done along $(S^1 \times S^1, S^1)$. In short when $(a, b, c)$ are the homology generators of $T^2 \times S^1$ we call it $(a \times b, c)$ operation.
6.3.1 Luttinger surgery

In the case when \((X^4, \omega)\) is a symplectic manifold (Chapter 8) and \(T^2 \subset X\) a Lagrangian torus (i.e. \(\omega|_{T^2} = 0\)), there is a special kind of \(T^2\)-surgery called a Luttinger surgery ([Lu],[EP],[ADK1]), which preserves the symplectic structure of \(X\). Luttinger surgery operation \(X \mapsto X_{m,n}\) corresponds to the diffeomorphism \(\phi(x, y, \theta) = (x + m\theta, y + n\theta, \theta)\).

\[
\varphi_{m,n} = \begin{pmatrix}
1 & 0 & m \\
0 & 1 & n \\
0 & 0 & 1 \\
\end{pmatrix}
\]

Since \(T^2\) is Lagrangian, in an open neighborhood \(T^2\) the symplectic structure of \((X, \omega)\) can be identified by the symplectic structure \(T^*((T^2)) \cong T^2 \times \mathbb{R}^2\) (e.g. [MSa]) and

\[
\omega = d (r [(cos2\pi\theta) dx + (sin2\pi\theta) dy])
\]

where \([x, y], (r, \theta)\] are the coordinates of \(T^2 \times \mathbb{R}^2\), and the boundary of a closed tubular neighborhood \(T^2 \subset N\) is represented by the hypersurface \(r = 1\). Clearly on \(\partial N\) we have \(\phi^{*}_{m,n} \omega = \omega\), so the symplectic structure of \(N\) extends to a symplectic structure on \(X_{m,n}\). As before, to perform this operation we carry other handles to top of \(T^2 \times D^2\) by the diffeomorphism \(\varphi^{-1}_{m,n} = \varphi^{-m,-n}\) as in the example of Figure 6.10. In the literature sometimes Luttinger surgery just refers to the operation \(X \mapsto X_{0,\pm 1}\) and its related to the \((\pm 1)\)-log transformation by the following Exercise.

Figure 6.9: \(p\) log-transform operation \(\phi^{-1}_p : X \mapsto X_p\)
Exercise 6.3. For simplicity abbreviate \( \varphi_{\pm} := \varphi_{0,\pm 1} \) and \( \phi_{\pm} := \phi_{\pm 1} \). Then show that (a) \( \varphi_{\pm} = \phi_{\pm} \circ \gamma_{\pm} \), where \( \gamma_{\pm} \) is the restriction of the self diffeomorphism of \( T^2 \times D^2 \) e.g.

\[
\phi_{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \varphi_{\pm} \circ \gamma_{\pm}
\]

More specifically \( \gamma_{\pm} \) is the map \((x, y, \theta) \mapsto (x, y, \pm y + \theta)\).

(b) \( \varphi_{\pm}^{-1} = \varphi_{-} \) and \( \gamma_{\pm}^{-1} = \gamma_{-} \), (c) There is a diffeomorphism \( X_{\pm 1} \approx X_{0,\pm 1} \), in particular if \( X \) is symplectic and \( T^2 \subset X \) Lagrangian, then log transformation \( X \mapsto X_{\pm 1} \) produces a symplectic manifold.

![Luttinger operation](image1)

**Figure 6.10:** Luttinger operation \( \varphi_{\pm}^{-1} : X \mapsto X_{0,1} \)

Remark 6.2. By comparing Figures 6.10 and 6.2 we see that the Luttinger surgery operation is a certain generalization of the Gluck twisting operation (where the 0-framed 2-handle need not represents an imbedded 2-sphere).

Exercise 6.4. Verify that Figure 6.11 describes \( p \) log- transformation operation \( \phi_{\pm} \).

![Log transformation](image2)

**Figure 6.11:** 1-log transformation operation \( \phi_{\pm}^{-1} : X \mapsto X_{1} \)
6.4 Knot surgery

Let $X$ be a smooth 4-manifold, and $T^2 \times B^2 \subset X$ be an imbedded torus with trivial normal bundle, $K \subset S^3$ be a knot and $N(K)$ its tubular neighborhood. The operation (defined in [FS1]) of replacing $T^2 \times B^2$ with $(S^3 - N(K)) \times S^1$, so that the meridian $p \times \partial B^2$ of the torus coincides with the longitude of $K$ is called Fintushel-Stern knot surgery operation.

$$X \hookrightarrow X_K = (X - T^2 \times B^2) \cup (S^3 - N(K)) \times S^1$$

A handlebody of this operation was constructed in [A6]. Notice $(S^3 - N(K)) \times S^1$ is obtained by identifying two ends of $(S^3 - N(K)) \times [0,1]$. Since 4-manifolds are determined by their 1- and 2- handles, it suffices to draw $(B^3 - N(K_0)) \times [0,1]$ with its ends identified, where $K_0 \subset B^3$ is a properly imbedded arc with the knot $K$ tied on it (the rest is a 3-handle). The second picture of Figure 6.12 describes $(B^3 - N(K_0)) \times [0,1]$; identifying its two ends (up to 3-handles) corresponds attaching a new 1-handle, and attaching 2-handles to the knots obtained by connected summing (over the new 1-handle) of the core circles of the 1-handles of two boundary components of $(B^3 - N(K_0)) \times [0,1]$, as shown in the third picture of Figure 6.12 (this is the $M(f)$ technique of Section 3.2, here we can use Section 5.2 for locating the 1-handle core circles of 3-manifolds).

![Figure 6.12](image_url)

Figure 6.12: $(S^3 - K) \times S^1$, where $K$ is the trefoil knot
6.4 Knot surgery

Summarizing: Figure 6.13 gives a recipe of how to modify a 4-manifold handlebody, containing a framed torus $T^2 \times B^2$ to get the handlebody of the knot surgered 4-manifold along this torus, by using a knot $K$ (in the Figure $K$ is taken to be trefoil knot). Figure 6.13 describes how the linking loops $a, b, c$ changes by this operation. For example in this figure, if we attach $-1$ framed 2-handle to either (both) of the loops $a, c$ we get Fishtail (Cusp) on the left, and the knot surgered Fishtail (Cusp) on the right (see Section 3.1).

Figure 6.14 describes the slow evolution of the two pictures of Figure 6.13, from left to right (i.e. description of the boundary diffeomorphism). In Figure 6.14, first we introduce a canceling 2/3 handle pair, and then replace dot with zero in the middle 1-handle turning it to a 2-handle, then do the indicated handle slides, and then put back the dot on the resulting ribbon 2-handle. To go reverse, from right to left, we perform the same operation to $(T^2 \times B^2)_K$, this time by using the dotted arrows in Figure 6.13.

![Figure 6.13: The operation $T^2 \times B^2 \mapsto (T^2 \times B^2)_K$](image)

![Figure 6.14](image)
As an example, let us take the elliptic surface $E(1)$ (see Chapter 7) and apply a knot surgery operation to the cusp inside. The first picture of Figure 6.15 is a handlebody of $E(1)$ (from [A7]), the second picture of this figure is $E(1)_K$ which is obtained by applying the algorithm above.

Theorem 6.3. ([FS1]) Let $[T] \in H_2(X;\mathbb{Z})$ be the homology class of the imbedded torus $T^2 \subset X$ discussed above. Formally write $t = \exp(2[T])$, and let $\Delta_K(t)$ be the (symmetric) Alexander polynomial of the knot $K \subset S^3$. Then the Seiberg-Witten invariant (Chapter 13) $SW_{X_K}$ of $X_K$ is given by $SW_{X_K} = SW_X \cdot \Delta_K(t)$. 

Figure 6.15
6.5 Rational blowdowns

Rational blowing down operation was introduced by Fintushel and Stern as a useful tool in Gauge theory, in [FS2] and [P] this operation is used to construct exotic 4-manifolds. Let $C \subset X^4$ be a negative definite plumbing in a smooth 4-manifold, such that $\partial C \approx \partial B$ for some rational ball $B$. Call $X' = X - \text{int}(C)$, then the operation of replacing $C$ by $B$ 

$$X \mapsto X_p := X' \sim B$$

is called a rational blowing down operation. An important special case is when $C_{p,q}$ is the 4-manifold given by the plumbing Figure 6.16 ([P]), where each $b_i \geq 2$, and

$$p^2/pq - 1 = [b_k, b_{k-1}, ..., b_1]$$

![Figure 6.16: $C_{p,q}$](image)

The boundary is the Lens space: $\partial C_{p,q} = L(p^2, pq - 1)$, which by [CH] bounds a rational ball. In [LM] it was shown that in fact $L(p^2, pq - 1)$ bounds the very simple rational ball $B_{p,q}$ shown in Figure 6.17.

![Figure 6.17: $B_{p,q}$](image)

**Exercise 6.5.** Prove that there is a diffeomorphism $\partial C_{p,q} \approx \partial B_{p,q}$, and describe this diffeomorphism (Hint: First identify $\partial B_{p,q} \approx S^3(U, p^2/pq - 1)$, where $U$ is the unknot) (see Section 5.1, and [Go]).
In the special case of \( q = 1 \) the manifold \( C_p := C_{p,1} \) is the plumbing of chain of 2-spheres \( \{u_0, u_1, \ldots, u_{p-1}\} \) with Euler numbers \( -p-2, -2, -2, \ldots, -2 \). Figure 6.18 demonstrates a concrete diffeomorphism \( \partial B_{p,1} \approx \partial C_p \), which allows us to perform the blowing down operation concretely on handlebodies.

Exercise 6.6. Call \( L := L(p^2, p - 1) \) and \( B_p = B_{p,1} \). Let \( \delta \) to be the dual circle of the 2-handle of \( B_p \), and let \( \gamma_j \) denote the dual circles of the 2-spheres \( u_j \) of \( C_p \), \( 0 \leq j \leq p-2 \). We can consider these dual circles as elements of the 2-dimensional cohomology or (by Poincare duality) 2-dimensional relative homology classes of \( B_p \) and \( C_p \). Show that:

(a) In \( \pi_1(L) = \mathbb{Z}_{p^2} \) the loop \( \gamma_j \) represents \( [1 + j(p + 1)] \gamma_0 \), where \( \gamma_0 \) is a generator.

(b) The map induced by restriction \( H^2(B_p) \to H^2(L) \) is the “multiplication by \( p \)” map \( \mathbb{Z}_p \to \mathbb{Z}_{p^2} \), sending \( \delta \mapsto p \gamma_0 \) (Hint: trace the dual circles in Figure 6.18)

Theorem 6.4. ([FS2]) Let \( k \in H^2(X_p; \mathbb{Z}) \) be a characteristic class (Section 5.5) then the restriction \( k|_{X'} \) lifts to a characteristic class \( \tilde{k} \in H^2(X; \mathbb{Z}) \). Furthermore, for any lifting \( \tilde{k} \) of \( k \) we have:

\[
SW_{X_p}(k) = SW_X(\tilde{k})
\] (6.1)
Proof. : Let \( i^*(k|_{X'}) = k_L \in H^2(L) \) be the restriction induced by inclusion. Since the map \( j^*: H^2(C_p) \to H^2(L) \), induced by inclusion, is onto, we can always choose a class \( k_{C_p} \in H^2(C_p) \) with \( j^*(k_{C_p}) = k_L \). Then \( \tilde{k} = k|_{X'} \oplus k_{C_p} \) defines a class in \( H^2(X) \) by exactness of the Mayer-Vietoris sequence

\[
0 \to H^2(X) \to H^2(X') \oplus H^2(C_p) \xrightarrow{i^*-j^*} H^2(L)
\]

But we need to choose \( k_{C_p} \) carefully so that \( \tilde{k} \) is characteristic in \( X \), i.e. it intersects even classes even times and odd classes odd times. By Exercise 6.6 we have \( k_L = m(p\gamma_0) \) for some \( m \). Since \( B_p \) is an even manifold when \( p \) is odd, and since \( k \) is characteristic \( m \) and \( p \) must have different parity. It is easy to see

\[
(p - 1)\mathbb{CP}^2 = B_p - \partial C_p
\]

Let \( E_1, \ldots, E_{p-1} \) be the 2-sphere generators of \( (p - 1)\mathbb{CP}^2 \), then we see that

\[
u_j = \begin{cases} 
-2E_1 - E_2 - \ldots - E_{p-1} & \text{if } j = 0 \\
E_j - E_{j+1} & \text{if } j \geq 1
\end{cases}
\]

Define \( k_{C_p} = -E_1' - \ldots - E_{(m+p-1)/2}' + E_{(m+p+1)/2}' + \ldots + E_{p-1}' \in H^2(C_p) \)

where \( E_j' = E_j|_{C_p} = \sum(E_j \cdot u_j)u_j \), i.e \( E_1' = 2u_0 - u_1 \), \( E_2' = u_0 + u_1 - u_2 \) etc. Furthermore by Exercise 6.6 (b) each \( E_j' \) restricts to \(-p\gamma_0 \in H^2(L)\) hence \( \lambda_m \) restricts to \( m(p\gamma_0) \).

Since \( \tilde{k}^2 = k^2 + \lambda_m^2 = k^2 - (p - 1) \), the dimensions of Seiberg-Witten moduli spaces coincide \( d_X(\tilde{k}) = d_{X_p}(k) \) (13.23). For the second part \( SW_{X_p}(k) = SW_X(\tilde{k}) \) see [FS3] \( \square \)

Rational blowing down process \( X \sim X_p \) makes the manifold \( X \) smaller, it is used to obtain smaller exotic manifold from larger ones. For this we need to first locate \( C_p \subset X \), and find a characteristic class \( k \) of \( X \), which descends to a characteristic classes \( k \) of \( X \).

Exercise 6.7. For a given \( C_p \subset X \) a characteristic class \( k \) of \( X \) is called taut if \( |k \cdot u_0| \leq p \), and \( k \cdot u_i = 0 \) for all other \( i \geq 1 \). Show that \( \tilde{k} \) is taut if and only it satisfies the adjunction inequality (Section 13.8), and \( \tilde{k} \) descends to a characteristic class \( k \) of \( X_p \) if \( |k \cdot u_0| = \mp p \).
Chapter 7

Lefschetz Fibrations

Let \( \pi : X \to \Sigma \) be a map from an oriented 4-manifold to a surface, we say \( p \in X \) is a Lefschetz singularity of \( \pi \), if we can find charts \((\mathbb{C}^2,0) \leftrightarrow (X,p)\) and \( \mathbb{C} \to \Sigma \), on which \( \pi \) is given by \((z,w) \mapsto zw\). A Lefschetz fibration (LF in short) is a map \( \pi : X \to \Sigma \) which is a submersion in the complement of a finitely many Lefschetz singularities \( \mathcal{P} \), and an injection on \( \mathcal{P} \). Here we will only consider the case \( \Sigma = S^2 \) (and later \( \Sigma = D^2 \)). The set of finite points \( \pi(\mathcal{P}) = \mathcal{C} \) is called the singular values of \( \pi \). Let \( \mathcal{C} = \{p_1, \ldots, p_k\} \).

Clearly over \( S^2 - \mathcal{C} \) the map \( \pi \) is a fiber bundle map with fiber a smooth surface \( F \). Pick \( p_0 \in D \subset X - \mathcal{C} \) where \( D \) is a disk, then we have can identify \( \pi^{-1}(D) \approx D \times F \). Now connect \( p_0 \) to \( p_j \) with a path \( \gamma_j \) in the complement of \( \mathcal{C} \). Let \( D_j = D \cup N(\gamma_j) \), where \( N(\gamma_j) \) is the tubular neighborhood of \( \gamma_j \).

In [Ka] Kas sowed that \( \pi^{-1}(D_j) \) is obtained by attaching a 2-handle \( h_j^2 \) to \( \pi^{-1}(D) \), along a circle \( C_j \subset p \times F \subset S^1 \times F \) (for some \( p \in S^1 \)) with framing one less than the framing induced from the surface \( F \).

\[
\pi^{-1}(D_j) = \pi^{-1}(D) \cup h_j^2
\]

(7.1)

\( h_j^2 \) is called a Lefschetz handle. After attaching \( h_j^2 \), the trivial fibration \( \pi| : S^1 \times F \to \partial D \) becomes a nontrivial fibration over \( \partial D_j \), with monodromy given by the right handed
Dehn twist $\tau_j : F \to F$ along $C_j$. By continuing with this process we obtain Lefschetz handles $h_1^2, \ldots, h_k^2$ which extends the Lefschetz fibration over $D_I = D \cup \cup_{j=1}^k N(\gamma_j)$ with

$$\pi^{-1}(D_I) = \pi^{-1}(D) \cup h_j^2$$

so that the compositions of the monodromies is identity $\tau_k \tau_{k-1} \cdots \tau_1 = id$. Since $S^2 - D_I = D_\infty$ is a disk, we can extend the fibration over $D_\infty$ trivially $D_\infty \times F \to D_\infty$, since the monodomy on the boundary is identity.

Conversely, any collection of imbedded curves $\{C_j\}_{j=1}^k$ in $F$ describes a Lefschetz fibration over $D^2$ with vanishing cycles $\{C_j\}$; furthermore if we have $\tau_{C_k} \tau_{C_{k-1}} \cdots \tau_{C_1} = id$, then they describe a Lefschetz fibration $\pi : X \to S^2$.

**Exercise 7.1.** Let $a, b$ be the two circle generators of $T^2 = S^1 \times S^1$, then show that the vanishing cycles $\{a\}$ and $\{a, b\}$ on $T^2$ give Lefschetz fibration structures (over $D^2$) on the Fishtail and the Cusp, respectively (Figure 7.2) (Hint: Change the 1-handles in the middle of Figure 7.2 to the circle with dot notation).

![Figure 7.2: Making Fishtail and Cusp Lefschetz fibrations over $D^2$](image-url)
7.1 E(n)

Let \(a, b\) be the two circle generators of \(T^2 = S^1 \times S^1\). It is known that \((\tau_a \tau_b)^6 = id\). Let \(E(1)\) be the corresponding Lefschetz fibration. Figure 7.3 describes how to construct a handlebody of \(E(1)\) starting with \(D^2 \times T^2\) (framings of all the middle 2-handles are \(-1\)). In [A7] this handlebody was identified with the handlebody of Figure 7.4, from which it follows that \(E(1) \cong CP^2 \# \overline{CP^2}\). More generally the Lefschetz fibration corresponding to the relation \((\tau_a \tau_b)^{6n} = id\) is called \(E(n)\), which has \(12n\) handles (vanishing cycles), each one after the other attached to \(D^2 \times T^2\) with the same weaving pattern of Figure 7.3.

Exercise 7.2. Show that in Figure 7.3 the small circle with \(-n\) framing corresponds to the 2-handle extending the fibration over \(D_\infty \times T^2 \to D_\infty\).
7.2 Dolgachev surfaces

Let $E(1)_{p,q}$ denote the manifold obtained from the Lefschetz fibration $\pi : E(1) \to S^2$ by performing log transformations of order $p$ and $q$ to two parallel $T^2$-fibers. These manifolds are known as Dolgachev surfaces. In [D1] Donaldson proved that, when $p, q$ relatively prime integers all $E(1)_{p,q}$ are distinct exotic copies of $E(1)$. Following [A13] here we draw a handlebody for $E(1)_{p,q}$. In Figure 7.5 we start with the handlebody of $T^2 \times [0, 1]$ (cf. Figure 4.1), and then by gluing two copies $T^2 \times [0, 1]$ along one of their faces (by Section 3.2) we get a Heegaard picture of $T^2 \times [0, 1]$ so that two parallel copies of $T^2$ inside are visible.

![Figure 7.5](image)

By thickening this Heegaard picture, in Figure 7.6 we obtain a handle picture for $T^2 \times D^2$ where two parallel $T^2 \times D^2$'s inside are visible. By applying the same process to the cusp $C$ (which is $T^2 \times D^2$ with two vanishing cycles) we get Figure 7.7. Now by Section 6.3, we apply log transforms of order $p$ and $q$ to these parallel tori and get $(T^2 \times D^2)_{p,q}$, and also $C_{p,q}$ (Figure 7.8). To apply this to the cusp inside $E(1)$, we first push two parallel tori from the cusp inside $E(1)$ in Figure 7.4 and get Figure 7.9 (where, for emphasis the 1-handles are drawn in pair of balls notation), then we apply the log transforms as in Figure 7.8 obtain Figure 7.10, which is $E(1)_{p,q}$.

![Figure 7.6: $T^2 \times D^2$ with two disjoint tori inside visible](image)
7.2 Dolgachev surfaces

Figure 7.7: The cusp $C$ with two disjoint tori inside visible

Figure 7.8: $(T^2 \times D^2)_{p,q}$ and $C_{p,q}$

Figure 7.9: $E(1)$ with two copies of $T^2$ inside its cusp
**Exercise 7.3.** Verify that converting 1-handles of Figure 7.9 to circle with dot notation, and applying the log transforms as described in Figure 7.8 gives Figure 7.10

![Figure 7.10: $E(1)_{p,q}$](image)

**Exercise 7.4.** By applying the above process to the cusp of $E(n)$ in Figure 12.32 draw a handlebody for $E(n)_{p,q}$
7.3 PALFs

An interesting case of Lefschetz fibration $\pi : X \to D^2$ is when the regular fiber is a smooth surface with boundary, and the base is $D^2$. We will call such Lefschetz fibration a *Positive Lefschetz Fibration* or PLF for short, if every vanishing cycle is a homotopically essential curve on the fiber. Following [AO1], we will call a PLF a *Positive Allowable Lefschetz Fibration* or PALF for short, if every vanishing cycle is a homologically essential curve on the fiber. We will view a PALF $\mathcal{F}$ as an extra structure on the underlying smooth manifold $X$, and express this by denoting $X = |\mathcal{F}|$ (similar to viewing triangulation as an additional structure on a manifold). The boundary $\partial \mathcal{F}$ of a PALF $\mathcal{F}$ defines an a open book structure on the boundary of the underlying manifold $\partial |\mathcal{F}| = \partial X$.

![Figure 7.11: A PALF and its boundary open book](image)

**Definition 7.1.** An open book structure (or decomposition) of a 3-manifold $Y$ is a surjective map $\pi : Y \to D^2$ such that $\pi^{-1}(\text{int}D)$ is a disjoint union of solid tori and

- (a) $\pi$ is a fibration over $S^1 = \partial D^2$.
- (b) On each of the solid torus, $\pi$ is the projection map $S^1 \times D^2 \to D^2$.

The centers of these solid tori are called binding.

So in the complement of the binding $\pi$ is a fibration over $S^1$, with fibers surfaces with boundary. Alternatively, we can define an open book structure on $Y$ just in terms of its pages and the monodromy $(F, \phi)$ of the fibration $\pi| : Y \to S^1$. Here $F$ is an oriented surface with boundary and $\phi : F \to F$ is a diffeomorphism which is identity on $\partial F$. Then $Y$ is the union the mapping torus of $\phi$ and solid tori glued along their boundaries in the obvious way. The monodromy of the open books arising from the boundary of PALF’s are composition of right handed Dehn twists (coming from the vanishing cycles).
Exercise 7.5. Let $\pi : X \to S^2$ be a Lefschetz fibration with closed surface fibers and a section, then show that removing the tubular neighborhood of the section and a regular fiber as indicated in Figure 7.12 gives a PALF ([AO2]).

![Figure 7.12](image)

Remark 7.2. This exercise gives so called concave-convex decomposition of any Lefschetz fibration $\pi : X \to S^2$ with closed fibers and a section: The closed tubular neighborhood of the section union a regular fiber is the concave part, called concave LF; and the closure of its complement is the convex part, which is a PALF. Notice attaching 2-handles to the bindings of a PALF $\pi : Z \to D^2$ gives an LF over $D^2$ with closed fibers, and then by attaching 2-handles to LF (over $D^2$ with closed fibers) along the sections of $\pi : \partial Z \to S^1$ gives the concave part of LF. Notice the boundary of a concave LF is also an open book (pages live in the part of the boundary which is close to the regular fiber).

Just as in the case of Lefschetz fibrations any PALF can be described a collection of imbedded curves $C = \{C_j\}$ (vanishing cycles) lying on a surface with boundary $F$, with framings one less then the framing induced from $F$. A basic example is the standard PALF of $B^4 = \mathcal{F}_0$, which consists of an annulus with the center circle $C$ (Figure 7.13). We can either draw $B^4$ as a canceling pair of 1 and 2-handles with $F$ as a thin cylinder containing the vanishing cycle $C$, or draw $B^4$ as an empty set (after cancellation) with $F$ as looking like a twisted annulus (right picture of Figure 7.13).

More generally a PALF $\mathcal{F} = (F, C)$ can be enlarged to another PALF $\mathcal{F}' = (F', C')$, where $F'$ is the surface obtained from $F$ by attaching a 1-handle to $\partial F$, and $C'$ is the union of $C$ with the new vanishing cycle $C'$ consisting of the core of the 1-handle union any imbedded arc in $F$ connecting the ends of the core (Figure 7.14). The operation $\mathcal{F} \mapsto \mathcal{F}'$ is called positive stabilization of the PALF $\mathcal{F}$. It is clear that $|\mathcal{F}| = |\mathcal{F}'|$ since this is just the operation of introducing canceling 1 and 2-handles. In particular the operation $\partial \mathcal{F} \mapsto \partial \mathcal{F}'$ is called positive stabilization of the corresponding open book.
The second picture of Figure 7.14 is an abstract picture of the PALF $\mathcal{F}'$, the third picture should be thought of $\mathcal{F}'$ being obtained from a PALF $\mathcal{F}_0$ on $B^4$ by introducing vanishing cycles to the pages of the induced open book $S^3 = |\partial\mathcal{F}_0|$. Stabilization had an affect of plumbing with the Hopf band.

In some instances this process can be reversed as follows (cf [AKa3]):

**Exercise 7.6.** Let $C$ be a non-separating curve on a surface $F \subset S^3$, such that the framing on $C$ induced from this surface corresponds to the $-1$ framing. Then $F$ is isotopic rel $C$ to the plumbing of another surface with a Hopf band whose core is $C$ (Hint: find a transverse arc intersecting $C$ at one point with end points on $\partial F$, then do the indicated finger move of Figure 7.15 along $C$).

The $(p, q)$ torus knot $K(p, q)$ can be described as the boundary of the surface $F(p, q)$ in $S^3$, obtained by connecting $p$ long horizontal squares and $q$ vertical squares underneath, by bands as shown in Figure 7.16 (figure drawn for $(p, q) = (4, 4)$), [Ly]. A repeated application of Exercise 7.6 shows that $K(p, q)$ is obtained from $S^3$ by a sequence plumbings by Hopf bands. So the Seifert surface $F(p, q)$ (fiber) of $K(p, q)$ represents a PALF of $B^4$. For future references we will denote this PALF of the 4-ball by $\mathcal{F}_{p,q}$.
7 Lefschetz Fibrations

7.4 ALFs

Let $\pi : X \to \Sigma$ be a map from an oriented 4-manifold to a surface with $\Sigma = S^2$ or $D^2$. We say $p \in X$ is an achiral Lefschetz singularity of $\pi$, if we can find charts $(\mathbb{C}^2, 0) \hookrightarrow (X, p)$ and $\mathbb{C} \to \Sigma$, on which $\pi$ is given by $(z, w) \mapsto zw$. An achiral Lefschetz fibration (ALF in short) is a map $\pi : X \to \Sigma$ which is a submersion in the complement of a finitely many Lefschetz singularities $\mathcal{P}$, some of which are achiral, and $\pi$ is injection on $\mathcal{P}$. So basically an ALF is a general version of a PALF's, where we allow achiral singularities. Similar to the description of Lefschetz singularities in (7.2), any ALF is obtained from the trivial fibration $D^2 \times F \to D^2$ ($F$ being the regular fiber) by attaching 2-handles $\{h_j\}$ to a collection of curves $\mathcal{C} = \{C_j\}$ in $p \times F \subset S^1 \times F$, with framing one less or one more than the page framing, corresponding to Lefschetz or achiral Lefschetz singularities of $\pi$, respectively. So any ALF is represented by such a pair $(F, \mathcal{C})$. We call any curve $C_j$ with framing one more than the page framing a defective handle. A defective 2-handle $h_j$ has the affect of introducing a left handed Dehn twist $\bar{\tau}_j : F \to F$ (i.e. the left handed version of Figure 5.1) to the monodromy of the boundary fibration over the circle, starting with $S^1 \times F \to S^1$. For example, Figure 7.17 describes an ALF structure of $B^4$. We call the operation of Figure 7.18 negative stabilization; which introduces left handed Hopf band. Similar to PALF, the boundary of an ALF’s is an open book $(F, \phi)$, where the monodromy $\phi$ is composition of is right and also left Dehn twists.

![Figure 7.17: An ALF on $B^4$](image1)

![Figure 7.18: Negative stabilization](image2)

**Proposition 7.3.** ([Ha]) Every 4-manifold $X$ without 3 and 4-handles, admits an ALF structure $\pi : X \to D^2$, with fibers having nonempty boundary.

**Proof.** ([AO1]) We are given $X = X_\Lambda$, where $\Lambda = \{K_1^n, ..., K_m^n, C_1, ..., C_s\}$. We will first prove the claim when there are no 1-handles $C_j$ present. First by an isotopy we can put the link $\mathcal{L} = \{K_1, ..., K_n\}$ in a square bridge position (why?).
This means that the link consists of vertical and horizontal line segments glued along their corners, and at the crossings the horizontal segments go over the vertical segments. Figure 7.21 demonstrates this for the trefoil knot. This implies that we can put the link $L$ on the surface $F(p,q)$ as demonstrated in Figure 7.19. If the framings $\{k_1, k_2, \ldots\}$ were one more, or one less then the page framings of the surface $F(p,q)$ we would be done, i.e. $X$ would be the ALF obtained by attaching vanishing cycles to the PALF $\mathcal{F}_{p,q}$ of $B^4$. If not, by positively (or negatively) stabilizing the PALF, as shown in Figure 7.20, we can increase (or decrease) page framing (why?) and hence put ourselves in this case. These operations has the affect of attaching canceling pair of 1 and 2-handles to $X$ to create the ALF.

Now we do the case when there are 1-handles present (i.e. $C_j$’s of $\Lambda$). First by treating each unknotted circle $C$ with dot as part of the framed link $L$, by the same process we put $C$ on the surface $F(p,q)$. Then by an isotopy we make $C$ transversal to the pages, so that it intersects each page of the open book $\partial F_{p,q}$ once, as indicated in Figure 7.22 (see the last paragraph). Then as explained in Section 1.1, to create the 1-handle which $C$ represents, we push the interior of the disk $D$, which $C$ bounds, into $B^4$ then excise the tubular neighborhood of $D$ from $B^4$. This has the affect of puncturing each fiber of the ALF given by $L$ once. Clearly this makes $X$ an ALF.
Finally, here is an explanation how we make $C$ transversal all the pages: $C$ being an unknot, when putting the link $L$ in a square bridge position, we can make sure that $C$ looks like Figure 7.23 on the surface $F(p,q)$. Hence in the handle notation of the PALF $\mathcal{F}_{p,q}$ (as in Figure 7.13) $C$ will look like in Figure 7.24; note that the second equivalent picture of this figure (after the indicated handle slides) makes it clear that $C$ meets each fiber of $\mathcal{F}_{p,q}$ once, since it is isotopic to the small linking circle of $\partial F(p,q)$.

Figure 7.23: $C$

Figure 7.24: $\mathcal{F}_{p,q}$

Let $Y = \partial X_\Lambda$ with $\Lambda = \{K_1^{r_1}, \ldots, K_n^{r_n}, C_1, \ldots, C_s\}$. By replacing the circles with dots $C_j$ (1-handles) with zero framed circles, we can assume that $Y$ bounds a 4-manifold with no 1-handles. Hence $Y$ is the boundary of the ALF obtained by stabilizing the PALF $\mathcal{F}_{p,q}$ on $B^4$, and then attaching vanishing cycles it. We will call such an ALF an extension of a PALF on $B^4$. More generally we will call an ALF $\mathcal{F}$ an extension of another ALF $\mathcal{F}'$, if it is obtained from $\mathcal{F}'$ by adding vanishing cycles.

Remark 7.4. Since any diffeomorphism $\phi : F \to F$ can be written as a composition of (right or left) Dehn twists along imbedded curves of $F$, for any given open book $(F, \phi)$ on $Y$, there is an ALF $\mathcal{F}$ such that $\partial \mathcal{F}$ gives this open book. Furthermore we can assume $\mathcal{F}$ is an extension of a PALF on $B^4$.

To justify the last claim of this remark we first compose the monodromy $\phi$ by pair of right and left Dehn twists, along two parallel copies of each of the core 1-handle curves of $F$. This does not change the isotopy class of $\phi$. Then by using the right handed monodromy curve ($-1$ curve) we cancel all the corresponding 1-handles of the ALF of $(F, \phi)$. We are left with a PALF on $B^4$ obtained from the trivial PALF of $B^4 = B^2 \times B^2$ by positive stablizations, along with (right or left) vanishing cycles on it.

Exercise 7.7. Given $X = X_\Lambda$ with $\Lambda = \{K_1^{r_1}, \ldots, K_n^{r_n}, C_1, \ldots, C_s\}$, put an ALF structure $\mathcal{F}$ on $X$ which is an extension of an ALF structure $\mathcal{F}'$ on the 1-handles $\#_s(S^1 \times B^3)$ (Hint: first put the 2-handles in a square bridge position in $\#_s(S^1 \times B^3)$, and put them on the surface of Figure 7.26. Then construct the required ALF $\mathcal{F}'$ on $\#_s(S^1 \times B^3)$ by first stabilizing $\mathcal{F}_{p,q}$ in $S^3$ then taking away vanishing cycles).
Exercise 7.8. Let $K \subset \mathbb{R}^3$ be an oriented knot in a square bridge position, sitting on the surface $F(p,q)$. Show that the framing of $K$ induced from the surface $F(p,q)$ is given by the so called Thurston-Bennequin number:

$$TB(K) = w(K) - c(K)$$

where $w(K)$ is the writhe (Exercise 1.1), and $c(K)$ is the number of south-west corners.

For example, the unknot of Figure 7.23 has $TB(C) = -1$, and the trefoil knot $K$ of Figure 7.21 has $TB(K) = 3 - 2 = 1$. Also for a given an oriented Knot $K \subset \mathbb{R}^3$ in square bridge position, we define a corner of $K$ an up-corner (or a down-corner) if it is either south-west or north-east corner, and the orientation arrow turns up (or turns down) as it traverses around this corner.
Definition 7.5. The rotational number of an oriented knot $K$ in square bridge position:

$$r(K) = \frac{1}{2} (\text{number of down corners} - \text{number of up corners})$$

For example, the knot $K$ of Figure 7.27 we have $r(K) = \frac{1}{2}(1 - 3) = -1$.

![Figure 7.27: $r(K) = 1$](image)

Definition 7.6. If an ALF $\mathcal{F}$ on $X$ has no 1-handles, and it is in square bridge position (i.e. an extension of $\mathcal{F}_{p,q}$), we define the characteristic class $c = c(\mathcal{F}) \in H^2(X; \mathbb{Z})$ by

$$c(\mathcal{F}) = \sum_{j=1}^{r} r(K_j) K_j^*$$

where $K_j^* \in H^2(X)$ are the hom duals of the 2-dimensional homology classes represented by the framed knots $K_j$ of $X$, i.e. $< K_j^*, K_i > = \delta_{i,j}$. If $\mathcal{F}$ has 1-handles, by turning its 1-handles (dotted circles) to zero-framed circles, while twisting everything going through them $-1$ times (cf. Figure 7.26), we transform $\mathcal{F}$ to an ALF $\mathcal{F}'$ without 1-handles then we define $c(\mathcal{F}) = c(\mathcal{F}')$ (relate this to the process in Remark 7.4).

Exercise 7.9. Show that the ALF modification in Figure 7.28 doesn’t change the underlying smooth manifold $X$, while changing the ALF so that $r(K)$ changes by $\pm 2$.

![Figure 7.28](image)
7.5 BLFs

A Broken Lefschetz fibration (BLF in short) is a singular version of LF, it is a Lefschetz fibration $X^4 \to S^2$ with closed surfaces as regular fibers, where we allow circle singularities away from the Lefschetz singularities (so called fold singularities). This means on neighborhoods of some imbedded circles in $X$ we can choose coordinates such that map $\pi$ looks like the map $S^1 \times B^3 \to \mathbb{R}^2$ is given by $(t, x_1, x_2, x_3) \to (t, x_1^2 + x_2^2 - x_3^2)$, otherwise outside of these circles $\pi$ is a Lefschetz fibration. Obviously these imbedded circles (along with LF singularities) are the singular points of $\pi$. Similarly we can define BLF over a disk $\pi : X \to D^2$, where the fold singularities are in the interior of $X$. So a BLF $\pi : X \to S^2$ is a LF with varying fiber genus, that is there are imbedded circles $C \subset S^2$ where the genus of the fiber $\pi^{-1}(t)$ changes as $t$ moves across $C$. The circles $C$ are the images of the singular circles of $\pi$ mentioned above. BLF’s first introduced in [ADK2]. In [GK2] it was observed that by enlarging PALF handlebodies by round handle attachments we can construct BLF’s; this helps to produce examples.

Definition 7.7. Let $X$ be a 4-manifold with boundary. A round 1-handle is a copy of $S^1 \times D^1 \times D^2$ attached to $\partial X$ along $S^1 \times \{-1\} \times D^2 \sim S^1 \times \{1\} \times D^2$. A round 2-handle is a $S^1 \times D^2 \times D^1$ attached to $\partial X$ along $S^1 \times S^1 \times D^1$.

Figure 3.10 gives a handlebody description of a round 1-handle attached to $X$ along pair of circles $\{C_1, C_2\}$ in $\partial X$.

Exercise 7.10. Show that any BLF $\pi : X \to D^2$ extends to another BLF after performing the following operations:

(a) Attaching a round 1-handle along pair of circles, which are sections of the boundary fibration $\pi| : \partial X \to S^1$. Also show that this operation increases the fiber genus, introducing a fold singularity along $S^1 \times 0 \times 0$.

(b) Attaching a round 2-handle along a (page framed) circle lying on a fiber of the fibration $\pi| : \partial X \to S^1$ (that is lying on a page of the open book). Show that this move decreases the fiber genus if the attaching circle is essential on the page, introducing a fold singularity along $S^1 \times 0 \times 0$.

Broken Lefschetz fibrations can be used to convert convex LF’s to concave LF’s by introducing broken singularity. Following are constructions from [GK2], as presented in [AKa3] (also see [Ba1]).
**Proposition 7.8.** Let $F$ be a closed oriented surface, then $F \times D^2$ can be made a concave BLF over $D^2$.

**Proof.** To turn the fibration $\pi : F \times D^2 \to D^2$ to a concave fibration we need to attach a 2-handle along a section of $\pi| : F \times S^1 \to S^1$ (Remark 7.2), but we have to do this without changing the topology of $F \times D^2$. For this, we attach a canceling pair of 2 and 3-handles, and notice that this is equivalent a $(-1$ framed) section 2-handle and a round handle as shown in Figure 7.29.

A **concave (or convex) BLF** is a concave (or convex) LF where we allow fold singularities. Introducing fold singularities allows us to perform some operations otherwise impossible. The following exercise is a good example.

**Exercise 7.11.** Given a concave BLF $X$, show that we can modify the BLF structure on $X$ so that the corresponding open book on $\partial X$ gets positively stabilized, and also gets negatively stabilized (Hint: First introduce a canceling 2/3 handle pair, then as in Figures 7.30 and 7.31 describe them with round handles. Check that $-1$ framed handles on the right side of the both figures live on the page, and round handles are attached along the sections. Keep in mind that handles can pass through the bindings since they are not handles).
Exercise 7.12. Let $X$ be a closed smooth 4-manifold. Show that we can find a closed imbedded surface with trivial normal bundle $F \times D^2 \subset X$, such that its complement has no 3 and 4-handles (Hint: As in Figure 7.32, introduce canceling 2/3 handle pairs to turn the 4-ball with 1-handles to $F \times D^2$, also recall Figure 2.3, then push the excess 2 and 3-handles to the other side as 1 and 2 handles).

Figure 7.32
Chapter 8

Symplectic Manifolds

A symplectic structure on a closed oriented smooth 4-manifold $X^4$ is a non-degenerate closed 2-form $\omega$ (non-degenerate means $\omega \wedge \omega > 0$ pointwise). It is known that any symplectic manifold $(X, \omega)$ admits a compatible almost complex structure $J$, which is a bundle map $J : TX \to TX$ satisfying the following (e.g. [MSa]):

(a) $J^2 = -id$.

(b) $\omega(Ju, Jv) = \omega(u, v)$.

(c) $\omega(u, Ju) > 0$, for all $u \neq 0$.

(c) implies that $g(u, v) := \omega(u, Jv)$ is a metric on $X$. The set of compatible complex structures is contractible. In particular, all such almost complex structures are homotopic to each other. Hence we can talk about Chern classes $c_i(X) = c_i(X, \omega) \in H^{2i}(X; \mathbb{Z})$ and splitting $\Lambda^{p,q}(T^*X)$, and hence the canonical line bundle $K \to X$.

Definition 8.1. The canonical line bundle of $(X, \omega)$ is defined to be

$$K = \Lambda^{2,0}(T^*X) := \Lambda^2(T^{1,0}X) \to X$$

An interesting class of symplectic 4-manifolds are Kähler surfaces $(X, \omega)$, which are complex surfaces $X \subset \mathbb{CP}^N$ (for some $N$), where $\omega$ is induced from the Kähler form of $\mathbb{CP}^N$. A celebrated theorem of Taubes (Theorem 13.15) states that symplectic manifolds have nonzero Seiberg-Witten invariants.

Theorem 8.2. ([Ta]) Let $(X, \omega)$ be a closed symplectic 4-manifold with $b_2^+(X) > 1$, then $SW_X(K) = \pm 1$. 

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Theorem 8.3. ([KM], [MST]) If $X$ is a closed smooth 4-manifold with $b_2^+(X) > 1$ with a nonzero Seiberg-Witten invariant corresponding to a line bundle $L \to X$, and $\Sigma \subset X$ be a closed oriented surface with $\Sigma . \Sigma \geq 0$. Then if $S$ is not a sphere we have

$$2g(\Sigma) - 2 \geq \Sigma . \Sigma + |(c_1(L), \Sigma)|$$

Theorem 8.4. ([B], [MF], [FS3]) A minimal Kahler surface $X$ satisfying $b_2^+(X) > 1$, can not contain a smooth 2-sphere $S \subset X$ with $S . S \geq -1$.

The proofs of Theorems 8.2 and 8.3 will be given in Sections 13.6 and 13.8.

8.1 Contact Manifolds

A contact structure $\xi$ on an oriented 3-manifold $Y^3$ is a totally non-integrable 2-plane field on its tangent bundle. The 2-plane field $\xi$ can be given by the kernel of a 1-form $\alpha \in \Omega^1(Y)$, and totally non-integrable condition means that $\alpha \wedge d\alpha > 0$ pointwise. We call a contact structure overtwisted if $\xi$ can be made transversal to the boundary of and imbedded 2-disk In $Y$, otherwise it is called tight. For example, Figure 8.1 is the picture of the tight contact structure $(\mathbb{R}^3, dz + xdy)$, given by the 2-plane field on Euclidean 3-space (along lines parallel to $x$-axis planes rotate 90° uniformly from $-\infty$ to $\infty$).

Exercise 8.1. Show that restriction of the complex lines defines a contact structure on $S^3 \subset \mathbb{C}^2$, given by the kernel of the 1-form $\alpha_0 = i/4 \sum_{j=1}^{2} (z_j \bar{z}_j - \bar{z}_j z_j)$

We say that two contact structures $\xi_1$ and $\xi_2$ on $Y^3$ are isomorphic if $\xi_2 = f^*(\xi_1)$ for some diffeomorphism $f : Y \to Y$, if furthermore $f$ is isotopic to identity then we say that $\xi_1$ and $\xi_2$ are isotopic contact structures.
Definition 8.5. We say a contact structure $\xi = \ker(\alpha)$ on $Y^3$ is carried by (or supported by) the open book $(F, \phi)$ structure on $Y$, if $d\alpha$ is a positive area form on each page of the open book, and $\alpha > 0$ on the binding $\partial F$.

It is known that every closed oriented 3-manifold $Y^3$ admits an open book structure, and every open book on $Y^3$ supports a contact structure, conversely Giroux proved that every contact structure comes from (supported by) an open book, and furthermore:

Theorem 8.6. ([Gi]) Two open book structures on an oriented 3-manifold $Y^3$ have common positive stabilization if and only if they support isotopic contact structures.

Classifying contact structures $\xi$ on $Y^3$ is a hard problem, but classifying the homotopy classes of the underlying oriented 2-plane fields of $\xi$ is easier. They are determined by two basic invariants: $c_1(\xi) \in H^2(Y; \mathbb{Z})$ and $d(\xi) \in \mathbb{Q}$ ([G2]). The first one is the Chern class of the complex line bundle $\xi \to Y$. Note that $c_1(\xi)$ is always an even homology class (why?), and when it is a torsion homology class, the second invariant is defined by

$$d(\xi) = \frac{1}{4}(c_1(X,J)^2 - 2\chi(X) - 3\sigma(X))$$

where $(X^4, J)$ is an almost complex manifold bounding $(Y^3, \xi)$, extending $\xi$ as complex lines in $X$ i.e. $\xi = TY \cap J(TY)$, and $\chi(X)$ and $\sigma(X)$ are the Euler characteristic and the signature of $X$, respectively. To make $c_1(X,J)^2 \in \mathbb{Q}$ a well defined quantity, we needed $c_1(\xi) = c_1(X,J)|_Y$ to be torsion. To compute these we choose an open book $(F, \phi)$ supporting the contact structure $\xi$, then by using Remark 7.4 find an ALF $\mathcal{F}$ on $X$ which induces this open book on its boundary $Y = \partial X$, and given by a framed link in a square bridge position $\mathcal{F} = \{K_1, \ldots, K_r\}$, such that each component has framing $TB(K_j) \pm 1$. Then even though $X^4$ may not be almost complex we can compute ([St]):

$$d(\xi) = \frac{1}{4}(c^2 - 2\chi(X) - 3\sigma(X)) + q$$

where $c \in H^2(X)$ is the characteristic homology class of the ALF $\mathcal{F}$ as defined in Definition 7.6, and $q$ is the number of defective 2-handles, i.e. the number of $K_j$’s with framing $TB(K_j) + 1$. Also we can compute $c_1(\xi)$ by restricting the homology class $c_1(\xi) := c|_Y$. The two invariants $c_1(\xi)$ and $d(\xi)$ determine the homotopy class of the 2-plane fields on $Y$ when $c_1(\xi)$ is torsion (e.g. [OSt]).

Theorem 8.7. ([E3]) Two overtwisted contact structures on a 3-manifold have homotopic 2-plane fields if and only if they are isotopic.
8.2 Stein Manifolds

Let \((X^4, J)\) be an almost complex manifold, then a map \(f : X \to \mathbb{R}\) is called a \(J\)-convex function if the symmetric 2-form

\[(u, v) \mapsto g_f(u, v) := -d(J^* df)(u, Jv)\]

defines a metric on \(X\) (e.g. [CE]). This implies \(\omega := -d(J^* df)\) is a symplectic form on \(X\). Also if we call \(\alpha = J^* df\), by solving the equation \(i_u \omega = \alpha\) for \(u\), we get a vector field \(u\). Since \(df(u) = \omega(u, Ju)\) it is easy to check that this vector field has the properties \(u(f) \geq 0\) (it is gradient like), and \(\mathcal{L}_u \omega = \omega\) (it is \(\omega\)-convex). Here we can perturb \(f\) to a Morse function still satisfying these properties. These properties imply that \(X^4\) is collared at infinity \(Y^3 \times [0, \infty)\), and \(u\) is an outward pointing vector field, and \(\alpha\) induces a contact structure \((Y, \xi)\) on \(Y\), where \(Y = f^{-1}(c)\) for some regular value \(c\) of \(f\).

![Figure 8.2](image-url)

Grauert proved that all properly imbedded complex submanifolds \(X \subset \mathbb{C}^N\) admit \(J\)-convex functions. We define any proper complex submanifold \(X \subset \mathbb{C}^N\) to be a Stein manifold, and using this we define a compact Stein manifold to be a region \(M \subset X\) cut out from \(X\) by \(f \leq c\), where \(f : X \to \mathbb{R}\) is a \(J\)-convex proper Morse function, where \(c\) is a regular value of \(f\). Then \(\omega = \frac{1}{2} \partial \bar{\partial} f\).

**Definition 8.8.** We say that \((M, \omega)\) is a Stein filling of the boundary contact manifold \((Y, \alpha) = \partial (M, \omega)\). The contact 2-plane field \(\xi = \ker(\alpha)\) can be identified with the restriction of the dual \(\mathbb{K}^*\) of the canonical line bundle \(\mathbb{K} \to M\) (see Definition 8.1).

For example, the unit ball \(B^4 \subset \mathbb{C}^2\) is a Stein manifold with the \(J\)-convex function \(f(z_1, z_2) = |z_1|^2 + |z_2|^2\), and it induces the so called “standard tight” contact structure \((S^3, \xi)\), where the contact planes are restrictions of the complex lines in \(\mathbb{C}^2\). This is the contact structure of Example 8.1.
8.3 Eliashberg’s characterization of Stein

Let \((Y^3, \xi)\) be a contact 3-manifold. We call a curve \(K \subset Y\) a Legendrian knot if it is tangent to the contact plane filed \(\xi\). Every knot in \((Y, \xi)\) can be isotoped to a Legendrian position (knot). For example in \((\mathbb{R}^3, dz + xdy)\), the Legendrian curves appear as the knots with left handed crossings and cusps (southeast-northwest line going over the southwest-northeast line), as shown in the middle picture of Figure 8.3. Of course rotating Legendrian knot counterclockwise \(45^\circ\) turns it to square bridge position. By attaching a 1-handle to \(B^4\), we can extend the Stein structure on \(B^4\) to a Stein structure on \(S^1 \times B^3\) (see Theorem 8.9). In particular this extends the contact structure \((S^3, \xi)\) to a contact structure on \(S^1 \times S^2\), where the Legendrian knots look like curves with left handed crossings and cusps, that are placed between the attaching balls of the 1-handles, as in the last picture of Figure 8.3. By repeating this process we get a contact structure on \(\#_k(S^1 \times S^2)\) with the similar description of the Legendrian knots inside \(([G2])\).

\[
\text{square bridge position} \quad \text{TB}(K) = 0 \quad \text{rot}(K) = 1 \quad \text{TB}(L) = 2 \quad \text{rot}(L) = 0
\]

Figure 8.3

The framing of a Legendrian knot \(K \subset (Y, \xi)\) induced by the contact planes is called the Thurston-Bennequin framing, denoted by \(TB(K)\). In case \((Y, \xi) = (\mathbb{R}^3, dz + xdy)\), or more generally the induced contact structure on \(\#_k(S^1 \times S^2)\), this framing can be computed by the integer:

\[
TB(K) = w(K) - c(K) \tag{8.3}
\]

where the first term is the writhe (Exercise 1.1) and the second term is the number of right (or left) cusps. Notice that by introducing zig-zags to \(K\) we can always decrease \(TB(K)\) as much as we want. Notice also this is the surface framing of the knot when it is rotated to square bridge position and put on the surface \(F(p, q)\) (Exercise 7.8). The following is the 4-dimensional version of the Stein characterization theorem ([E2], [G2]).
Theorem 8.9. ([E2]) Let $X^4$ be a 4-manifold consisting of $B^4$ with 1 and 2-handles, i.e. $X = X_\Lambda$, where $\Lambda = \{K_1^{r_1},...,K_n^{r_n},C_1,..,C_s\}$. Then

(i) The standard Stein structure of $B^4$ extends to a Stein structure over the 1-handles $X_{\Lambda'}$, where $\Lambda' = \{C_1,..,C_s\}$.

(ii) If $\{K_1^{r_1},...,K_n^{r_n}\}$ is a Legendrian framed link in $\#_{i=1}^s(S^1 \times S^2)$ with $r_j = TB(K_j) - 1$ for all $j$, then the Stein structure extends over $X$.

Furthermore, every Stein manifold admits such a handlebody structure.

Exercise 8.2. Show that the manifold $B_{p,q}$ of Figure 6.17 is Stein.

Let $(X,\omega)$ be a Stein manifold and $(Y,\xi)$ be the induced contact structure on its boundary $Y = \partial X$. Let $K \subset (Y,\xi)$ be an oriented Legendrian knot bounding an imbedded oriented surface $F$ in $X$. We define the rotational number as the evaluation of the relative Chern class of the restriction $\mathbb{K}^* \to F$ of the canonical line bundle $\mathbb{K}^* \to X$

$$\text{rot}(K, F) = \langle c_1(\mathbb{K}^*, v), F \rangle$$

with respect to the tangent vector field $v$ of $K$ (the obstruction to extending $v$ to a section of $\mathbb{K}^*$ over $F$). In case $X = B^4$, $\text{rot}(K, F)$ of an oriented Legendrian knot $K \subset S^3$ does not depend on the choice of the Seifert surface $F$ it bounds in $S^3$, and it can be calculated as $r(K)$ of Definition 7.5 (after rotating $K$ into square bridge position):

$$\text{rot}(K, F) = 1/2 \text{ (number of down cusps} - \text{number of up cusps)}$$

(e.g [AM1], [G2]). Let $X = X_\Lambda$ with $\Lambda = \{K_1^{r_1},...,K_n^{r_n},C_1,..,C_s\}$ be the Stein manifold obtained by Theorem 8.9. As in Section 2.4 by viewing each $K_j$ as a 2-chain, then we can compute $\langle c_1(X), K_j \rangle = \text{rot}(K_j)$, hence we have (compare Definition 7.6)

$$c_1(\mathbb{K}^*) = c_1(X) = \sum_j \text{rot}(K_j)K_j^*$$

where $K_j^* \in H^2(M)$ are the hom duals of $K_j$, i.e. $\langle K_j^*, K_i \rangle = \delta_{ij}$. 
8.4 Convex decomposition of 4-manifolds

The following theorem says that every closed smooth manifold is a union of two Stein manifolds glued along their common boundaries. It is a good application of the carving technique discussed in Section 1.1.

**Theorem 8.10.** ([AM3]) Given any decomposition of a closed smooth 4-manifold $M = N_1 \cup \partial N_2$ by codimension zero submanifolds, such that each piece consists of 1- and 2-handles, after altering pieces by a homotopy inside $M$, we can get a similar decomposition $M = N_1' \cup N_2'$, where both pieces $N_1'$ and $N_2'$ are Stein manifolds, which are homotopy equivalent to $N_1$ and $N_2$, respectively.

**Proof.** Theorem 8.9 (and remarks before) says that manifolds with 1- and 2-handles are Stein if the attaching framings of the 2-handles are sufficiently negative (say admissible), that is any 2-handle $h$ has to be attached along a Legendrian knot $K$ with framing less than $TB(K)$. Call $\Sigma = \partial N_1$. The idea is by local handle exchanges near $h$ (but away from $h$) to alter the boundary $(\Sigma, K) \sim (\Sigma', K')$ by homotopy so that it results an increase of the Thurston-Bennequin framing:

$$TB(K) \rightsquigarrow TB(K') = TB(K) + 3$$

This is done by first carving $N_1$ (which means expanding $N_2$), then homotopically undoing this carving, as demonstrated by Figure 8.4.

For example, as indicated in Figure 8.5, by carving out a tubular neighborhood of a properly imbedded 2-discs $a$ from the interior of $N_1$ increases $TB(K)$ by 3. Carving in the $N_1$ side corresponds to attaching a 2-handle $A$ from the other $N_2$ side, which itself might be attached with a “bad” framing. To prevent this, we also attach a 2-handle $B$ to $N_1$ linking $a$, which corresponds to carving a 2-disc from the $N_2$ side. This makes the
framing of the 2-handle $A$ in the $N_2$ side admissible, also $B$ itself is admissible. So by carving a 2-disc $a$ and attaching a 2-handle $B$ we improved the framing of the attaching circle $K$ of the 2-handle $h$ to $K'$, without changing other handles (we only moved $\Sigma$ by a homotopy).

\[
\begin{array}{c}
\text{View from } N_1: \\
\text{View from } N_2:
\end{array}
\]

Figure 8.5: Making $N_1$ and $N_2$ Stein

**Exercise 8.3.** Show that by using Section 8.6, and by positive and negative stabilizing (Sections 7.3, and 7.4) we can arrange so that:

(a) The induced contact 2-plane fields in the middle level $\Sigma := \partial N'_1 = \partial N'_2$ are homotopic (Hint: Equate the two contact invariants defined in Section 8.1).

(b) By applying Theorems 8.6 and 8.7, we can make the induced open books in the middle level $\Sigma$ match, up to orientation; and furthermore after vanishing cycle exchanges from one side to other we can make both $N'_1$ and $N'_2$ PALF’s ([Ba2]).

**Definition 8.11.** A scarfed 2-handle is a 2-handle attached along a Legendrian knot $K$, plus a small circle (scarf) trivially linking $K$ positively several times (e.g. Figure 8.6)

![Figure 8.6](image)

**Exercise 8.4.** Show that carving a Stein 4-manifold $X$, along the scarf of the 2-handle in Figure 8.6 (as in the proof of Theorem 8.10) increases $TB(K)$ by 2.
8.5 $M^4 = |BLF|$ 

In [GK2] it was shown that any closed smooth 4-manifold admits a weaker version of BLF, which is a BLF $\pi : X \to S^2$ with achiral fibers, which they named BALF. In [Le] and [AKa3] two independent proofs that all closed 4-manifolds are BLF have been given; the first is a proof by singularity theory, and the second is a proof by handlebody. Also independent, in [Ba3] another singularity theory proof was given, with slightly weaker conclusion, where on the circle singularities $\pi$ is not required to be injective.

**Theorem 8.12.** Every closed smooth 4-manifold admits a BLF structure.

*Proof.* ([AKa3]) First by Exercise 7.12 we can decompose $X$ as a union of two codimension two submanifolds along their common boundaries $X = W \cup - (F \times D^2)$. Then by Propositions 7.3 and 7.8 we can make $W$ an ALF, and $(F \times D^2)$ a concave BLF over $D^2$. Now we need to match the induced open books in the middle: Using Exercise 7.9 and (8.2) by positively and negatively stabilizing both sides we can first match the two contact invariants (note that positively stabilizing one sides corresponds to negatively stabilizing the other side), and then by Theorems 8.6 and 8.7 we can make the induced open books (induced from $W$ and $F \times D^2$ sides) agree. We know how to stabilize ALF $W$, and by Exercise 7.11 know how to stabilize the concave BLF $(F^2 \times D^2)$ side, and furthermore know that negative stabilizations on the concave side can be gotten rid of by introducing more folds there. So we are left with a convex ALF and a concave BLF over $D^2$, with matching open books on their common boundary. Now during all these processes we carry a scarf (maybe several scarfs, maybe none) for each 2-handle on the $W$ side (Defintion 8.11). While moving things into square bridge position (during the process of turning $W$ into ALF), we treat scarfs as part of the framed link of $W$ and make them lie on the page of the surface $F(p,q)$ as well. Then we make them transverse to all the pages of the boundary open book (just as we treated circles with dots in the proof of Proposition 7.3), and carve $W$ side along these scarfs, turning the ALF structure on $W$ to a PALF structure (carving makes achiral 2-handles of $W$ chiral as in the proof of the Theorem 8.10). By Remark 7.2 this keeps the other side concave BLF.

**Remark 8.13.** (Making achiral singularity chiral by carving) The proof in [Le] starts with the result of [GK2], and then by using singularity theory turns the achiral fibers to chiral fibers. The magic of carving can also handle this approach, e.g. as in Figure 8.7 first by carving make the achiral singularity chiral, then by attaching a round 2-handle to keep its topology unchanged (check that the zero framed knot in the third picture can be made to lie on the page of the open book, then recall Exercise 7.10).
8.6 Stein = \(|P\text{ALF}|\)

Here we prove that for every Stein manifold \(X\), we can choose an underlying PALF structure \(X = |\mathcal{F}|\). Existence of this structure on Stein manifolds was first proven in [LoP], later in [AO1] an algorithmic topological proof along with its converse was given. More generally, consider the inclusion from (Section 7.3) \(\{\text{PALF}'s\} \subset \{\text{PLF}'s\}\), then

**Theorem 8.14.** There is a surjection \(\Psi : \{\text{PLF}'s\} \rightarrow \{\text{Stein Manifolds}\}\), given by the map \(\mathcal{F} \mapsto \mathcal{F}\). Moreover, the restriction \(\Psi\) to \(\{\text{PALF}'s\}\) is a surjection.

**Proof.** ([AO1]) The proof that a PALF \(\mathcal{F}\) defines a Stein manifold \(|\mathcal{F}|\) proceeds as follows: By Chapter 7 (and Section 7.3) a PALF \(\mathcal{F}\) is obtained by starting with the trivial fibration \(X_0 = F \times B^2 \rightarrow B^2\) and attaching sequence of 2-handles to some curves \(k_i \subset F\), \(i = 1, 2, ..\) with framing one less than the page framing: \(X_0 \sim X_1 \sim .. \sim X_n = \mathcal{F}\). We start with the standard Stein structure on \(F \times B^2\) and assume that on the contact boundary \(F\) is a convex surface with the divider \(c = \partial F\) ([To]). By applying the Legendrian realization principle “LRP” ([Ho]), by an isotopy we make the surface framings of \(k_i \subset F\) agree with Thurston-Bennequin framings, then result follows from Theorem 8.9.

In case \(\mathcal{F}\) is only PLF proof proceeds exactly the same with a small tweak (this was remarked in [AO1] without proof). The only part of the proof we need to be careful is when applying LPR to \(k\), because to apply it we need each \(k \subset F\) to be nonisolating (i.e. the components of \(F - k\) must touch the divider curves). For this we choose any nonisolating curve \(\alpha \subset F\) and isotope to a Legendrian, then by folding principle of Giroux, turn \(\alpha\) to a divider curve ([Ho]), this makes \(k\) nonisolating, hence LRP applies.
Conversely, the proof that a Stein manifold $X$ is a PALF, can be obtained by an easier version of the proof of Proposition 7.3: When there are no 1-handles by Theorem 8.9 $X$ is a handlebody consisting of 2-handles, attached along Legendrian framed link $L$ in $S^3$, such that each of its components $K$ is framed with $TB(K) - 1$ framing. Then by isotoping $L$ to square bridge position (by turning each component counterclockwise 45 degrees), we put the framed link $F$ on the surface $F(p,q)$ as in Figure 7.19. Now by Exercise 7.8 the framings induced from $F(p,q)$ of each component $K$ coincides with $TB(K)$. Hence $F$ gives a PALF which is extension of the PALF $F_{p,q}$ on $B^4$. When there are 1-handles the proof is the same as the the proof of Proposition 7.3 (by carving). In particular this time we use Figure 7.26 instead of Figure 7.19. An improved version of this theorem was proven in [Ar].

Remark 8.15. One Stein manifold can have many PALF structures. For example $B^4$ has a unique Stein structure, whereas it has infinitely many PALF structures coming from different fibered links (why?). This means in some sense PALF’s are finer structure then Stein structures.

### 8.7 Imbedding Stein to Symplectic via PALF

Choosing an underlying PALF structure on a Stein manifold can be a useful tool for solving problems about Stein manifolds. A good application of this principle was the approach of [AO2] to the compactification problem of Stein manifolds, which was later generalized to symplectic category in [E1] and [Et]. The problem of compactifying a Stein manifold $W$ into a minimal Kahler surfaces was first solved in [LMa], then in [AO2] by using PALF’s an algorithmic solution of the symplectic version was given.

**Theorem 8.16. ([AO2])** Any PALF more generally PLF $F$ determines a compactification of the corresponding Stein manifold $X = |F|$ into a closed symplectic manifold.

**Proof.** Given $X = |F|$, by attaching a 2-handle to the binding of the open book on the boundary (Figure 8.9) we get a closed surface bundle over the 2-disk with monodromy consisting of product of positive Dehn twists $\alpha_1, \alpha_2, \ldots, \alpha_k$. Let $F^*$ be the new closed fiber.

Next we extend this fibration by adding new monodromies $\alpha_1, \alpha_2, \ldots, \alpha_k \alpha_k^{-1} \ldots \alpha_1^{-1}$ (i.e. attaching corresponding 2-handles) and capping off with $F^* \times B^2$ on the other side (the outside monodromy is identity). We do this after converting each negative Dehn twist $\alpha_i^{-1}$ in this expression, into products of positive Dehn twists by using the relation $(a_1 b_1 \ldots a_g b_g)^{4g+2} = 1$ of the standard Dehn twist generators of the genus $g$ surface $F^*$.

In particular this imbeds $X$ into a Lefschetz Fibration $Z \to S^2$ with closed fibers, which is a symplectic manifold by [GS].
Exercise 8.5. Show that we can always choose $b_2^+(Z) > 1$ (Hint: First expand $X$ to a larger Stein manifold by introducing extra vanishing cycles to $\mathcal{F}$ so that $b_2^+(X) > 1$, then continue with the compactification process).

After choosing a PALF $\mathcal{F}$ on $X$, compactifying $X$ becomes an algorithmic process. Different PALF’s on $X$ could give different compactifications. Even in the case of Stein ball $B^4$, different PALF’s on $B^4$ we can get different symplectic compactifications.

Exercise 8.6. Show that this process compactifies $B^4 = |\text{Unknot}|$ to $S^2 \times S^2$, and compactifies $B^4 = |\text{Trefoil}|$ into the $K3$ Surface.
Chapter 9
Exotic 4-manifolds

9.1 Constructing small exotic manifolds

Theorem 9.1. ([AM1]) Let $X^4$ be a compact Stein manifold, and $F^2 \subset X$ be a properly imbedded surface, such that $K = \partial F \subset \partial X$ is Legendrian with respect to the induced contact structure. Let $n$ be the framing of $K$ induced from trivialization of the normal bundle of $F$, then

$$-\chi(F) \geq (TB(K) - n) + |\text{rot}(K)|$$

Proof. Attach 2-handle $h^2$ to $M$ with the framing $TB(K) - 1$, by Theorem 8.9 we can extend the Stein structure of $X$ to $X \sim h^2$. By Theorem 8.6 there is a PALF $\mathcal{F}$ with $X \sim h^2 = |\mathcal{F}|$. By Theorem 8.16 and Exercise 8.5, we can compactify $|\mathcal{F}|$ to a closed symplectic manifold $S$ with $b_2^+ > 1$. Then applying Theorems 8.2, 13.17 to $S$ and the closed surface $F' = F \cup D$ (where $D$ is the core of the 2-handle $h^2$) gives the result.

Theorem 9.2. ([A11]) Let $W$ be the contractible manifold given in Figure 9.1, and let $f : \partial W \rightarrow \partial W$ be the involution induced from the obvious involution of the symmetric link in $S^3$, with $f(\gamma) = \gamma'$ where $\gamma, \gamma'$ are the circles in $\partial W$ as shown in the figure. Then $f : \partial W \rightarrow \partial W$ does not extend to a diffeomorphism $F : W \rightarrow W$ (but it does extends to a homeomorphism).

Proof. ([AM1]) From the Theorem 8.9, we see that $W$ is Stein (for this we use the description of $W$ given in the second picture of Figure 9.1). Since $\gamma'$ is slice and $f(\gamma') = \gamma$, if $f$ extended to a diffeomorphism $F : W \rightarrow W$ then $\gamma$ would also be slice in $W$. But this violates the inequality of Theorem 9.1 (here $F = D^2$, $n = 0$ and $TB(\gamma) = 0$). The fact that $f$ extends to a homeomorphism of $W$ follows from the Freedman’s theorem [F].
Remark 9.3. If $F : W \to W$ is a homeomorphism extending $f$, it pulls back the smooth structure of $W$ to a smooth structure which is exotic copy of $W$ rel boundary.

Theorem 9.4. ([A12]) Let $Q_1$ and $Q_2$ be the manifolds obtained by attaching 2-handles along the knots $K_1$ and $K_2$ with $-1$ framing respectively, as shown in Figure 9.2. Then $Q_1$ and $Q_2$ are homomorphic but not diffeomorphic to each other, even their interiors are not diffeomorphic to each other.

Proof. ([AM1]) By Theorem 8.9, $Q_1$ is a Stein manifold. By [LMa] $Q_1$ is a domain in a minimal Kahler surface $X$ with $b_2^+(X) > 1$. If $Q_1$ and $Q_2$ were diffeomorphic to each other the generator of $H_2(Q_1; \mathbb{Z})$ would be represented by a smoothly imbedded 2-sphere with $-1$ self intersection (since $K_2$ is a slice knot, the generator of $H_2(Q_2; \mathbb{Z})$ is represented by a smoothly imbedded sphere). This violates Theorem 8.4.

Exercise 9.1. By using the alternative descriptions of $Q_1$ and $Q_2$ given in Figure 9.3, show that $Q_1$ and $Q_2$ both contain a copy of $W$, and they are obtained from each other by removing this copy of $W$ and regluing with the involution $f$ of Theorem 9.2.
9.2 Iterated 0-Whitehead doubles are non-slice

Such contractible manifolds with involution on their boundaries \((W, f)\) are called *corks*, which will be studied more formally in Chapter 10. Recently it was shown that corks can be used to construct absolutely exotic manifolds (i.e. no condition on their boundaries).

**Theorem 9.5.** \([AR]\) There are compact contractible smooth 4-manifolds \(V\) and \(V'\) with diffeomorphic boundaries, such that they are homeomorphic but not diffeomorphic to each other. Similarly, there are absolutely exotic smooth manifolds which are homotopy equivalent to \(S^1\).

9.2 Iterated 0-Whitehead doubles are non-slice

The inequality of Theorem 9.1 can give obstruction to sliceness of some knots as follows. Let \(K \subset S^3\) be a Legendrian knot, and let \(K'_0\) be a 0-framing push off of \(K\), then:

**Proposition 9.6.** \([AM1]\) We can move \(K'_0\) to a Legendrian representative \(K_0\) by an isotopy which fixes \(K\), such that \(TB(K_0) = -|TB(K)|\).

**Proof.** Figure 9.4 shows 0-framing push offs of \(K\), when \(w(K) \leq 0\) and \(w(K) \geq 0\), respectively. Clearly in the first case of \(w(K) \leq 0\) we have \(w(K_0) = w(K)\) and \(c(K_0) = c(K)\), hence \(TB(K_0) = TB(K)\) (see Section 8.3 to recall definitions).

![Figure 9.4: w(K) ≤ 0 and w(K) ≥ 0 cases](image-url)
When \( w(K) > 0 \) and \( TB(K) \leq 0 \), the \( c(K) \geq w(K) \), hence there are enough (right and left) cusps to accommodate \( 2w(K) \) half twist between two strands as shown in Figure 9.5. Hence \( TB(K_0) = TB(K) \).

\[
\text{Figure 9.5: The case } w(K) > 0 \text{ and } TB(K) \leq 0
\]

If \( w(K) > 0 \) and \( TB(K) > 0 \), then \( w(K) > c(K) \). Place \( 2c(K) \) half twist near the cusps, and leave the remaining \( 2w(K) - 2c(K) \) half twist with zig-zags (Figure 9.6) to make it Legendrian.. Hence \( TB(K_0) = TB(K) - [2w(K) - 2c(K)] = TB(K) - 2TB(K) = -TB(K) \).

\[
\text{Figure 9.6: The case } w(K) > 0 \text{ and } TB(K) > 0
\]

The \( n = 1 \) case of the following theorem was first proved by the author (unpublished) via branched covers (see Exercise 11.4), and the general case was proven by Rudolph [R]. The following is a strengthening of this from [AM1], which is an easy consequence of Theorem 9.1.

**Theorem 9.7.** If \( K \subset S^3 \) a Legendrian knot with \( TB(K) \geq 0 \), then all iterated positive 0-Whithead doubles \( Wh_n(K) \) are not slice. In fact if \( Q^n_r(K) \) is the manifold obtained by attaching a 2-handle to \( B^4 \) along \( Wh_n(K) \) with framing \( -1 \leq r \leq 0 \), then there is no smoothly imbedded 2-sphere in \( Q^n_r(K) \) representing the generator of \( H_2(Q^n_r(K); \mathbb{Z}) \).

**Proof.** Since \( TB(K) \geq 0 \), \( TB(K_0) = -TB(K) \) (Proposition 9.6). First Whitehead double \( Wh(K) \) is obtained by connecting \( K \) and \( K_0 \) by a left handed cusp as in Figure 9.7 which contributes \(+1\) to \( TB \), hence

\[
TB(Wh(K)) = TB(K) + TB(K_0) + 1 = TB(K) - TB(K) + 1 = 1
\]

By iterating this process we get \( TB(Wh_n(K)) = 1 \). But Theorem 9.1 (applied to the case \( X = B^4 \), hence \( n = 0 \)) says that any slice knot \( L \) must have \( TB(L) \leq -1 \)
For the second part, observe that since $TB(Wh_n(K)) = 1$, the manifold $Q^r_n(K)$ is Stein when $r \leq 0$, hence it can be imbedded into a minimal Kahler surface $S$. Then by Theorem 8.4 we can not have a smoothly imbeded 2-sphere $\Sigma \subset Q^r_n(K) \subset S$ representing $H_2(Q^r_n(K); \mathbb{Z})$, since $\Sigma.\Sigma = r \geq -1$. (why can we assume $b^+_2(S) > 1$?)

**Remark 9.8.** The 0-Whitehead double $Wh(K)$ of any knot $K$ may not be slice in $B^4$ as we have seen above, but it is slice in some contractible manifold, when it is imbedded on its boundary [Lit]. Let $W$ be the contractible manifold obtained blowing down $B^4$ along the obvious ribbon $D_-$ which $K\#-K$ bounds (see Section 6.2), and let $Wh(K) \subset W$ be the imbedding as shown in the first picture of Figure 9.8. Then the moving pictures of Figure 9.8 demonstrates how $Wh(K)$ bounds an imbedded disk in $W$. More specifically, the ribbon move performed in the second picture results two disjoint circles in $W$ which we can cap by disjoint disks. The two disjoint circles first appear as a linked Hopf circles in fourth picture, but they get separated in the final picture after blowing down $D_-$. 

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**Figure 9.7**

**Figure 9.8**

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111
9.3 A Solution of a conjecture of Zeeman

Every knot in $K \subset S^3$ bounds a PL disk in $B^4$, which is the disk obtained by simply coning $K$ from the center of $B^4$. In [Z] Zeeman conjectured that the analogue of this should not hold for contractible manifolds, more specifically he conjectured the knot $\gamma \subset \partial(W)$ of Figure 9.1 should not bound a PL disk in $W$. This conjecture was settled in [A2] as follows:

We already know that from the above discussion, the knot $\gamma$ can not bound a smooth disk in $W$, but after moving it by the involution $f$ the image $\gamma' = f(\gamma)$ bounds a smooth disk in $D_2 \subset W$. But now if $\gamma$ bounds a PL disk in $W$, by pushing the singularities together we can assume that it bounds a PL disk $D_1 \subset W$ with one singularity, around which it looks like a cone on a knot $K$ in a small 4-ball $B_1 \subset int(W)$. Furthermore, we know that $-W \sim W = S^4$, hence the knot $K$ bounds a smooth disk in the outside 4-ball $B_2 := S^4 - int(B_1)$ namely $D_3 = (D_1 - B_1) \cup D_2$. Now by replacing the ball $B_1$ with a copy $B_2'$ of the outside ball $B_2$ (i.e. inverting outside with inside) we see that $\gamma$ bounds a smooth disk in $W$, namely $(D_1 - B_1) \cup D_3'$ where $D_3'$ is a copy of $D_3$; a contradiction!
9.4 An exotic non-orientable closed manifold

An exotic copy of $\mathbb{RP}^4$ was constructed in [CS]. There is also an exotic copy $Q^4$ of the nonorientable manifold $M^4 := S^3 \tilde{\times} S^1 \# S^2 \times S^2$ [A15]. An interesting feature of $Q^4$ is that it can be obtained from $M$ by a Gluck twisting (Section 6.1). We construct $Q$ by identifying the two boundary components of a compact manifold homotopy equivalent to $S^3 \times [0,1] \# S^2 \times S^2$. Each boundary component of this manifold is the homology sphere $\Sigma(2,3,7)$ (Section 2.5). We construct an upside down handlebody of $Q$ starting with the handlebody of $\Sigma(2,3,7) \times [0,1]$ Figure 9.10 (Section 6.4), then attaching an orientation reversing 1-handle (Section 1.5), a pair of 2-handles and a 3 and a 4-handle as in Figure 9.11 (3 and 4-handles are not drawn since they are attached in unique way).

Exercise 9.2. By isotoping the attaching circles across the orientation reversing 1-handle identify the manifold in Figure 9.11 with Figure 9.12 then check that its boundary is $S^2 \tilde{\times} S^1$ (Hint: By circle with dot and 0-framed circle exchanges, turn the two linking 0-framed circles in the middle a canceling 1 and 2-handle pairs then cancel them).
Proposition 9.9. \( Q \) is an exotic copy of \( M \).

Proof. By composing with the collapse map, any homotopy equivalence \( Q \rightarrow M \) gives rise to a degree 1 normal map \( f : Q \rightarrow S^3 \times S^1 \) (normal means \( f \) pulls back the stable normal bundle to the stable normal bundle [Br]). Let \( S(M) \) be the set of equivalence classes of pairs \( (Q^4, f) \), where \( Q^4 \) is a closed smooth 4-manifold and \( f : Q \rightarrow S^3 \times S^1 \) is a degree 1 normal map. We define the equivalence \( (Q_1, f_1) \sim (Q_2, f_2) \) as a compact smooth cobordism \( Z^5 \) between \( Q_1 \) and \( Q_2 \), and a normal map \( F : Z \rightarrow S^3 \times S^1 \) which restricts to \( f_1 \cup f_2 \) on the boundary. Define \( \lambda : S(M) \rightarrow \mathbb{Z} \) (mod 32) as follows: For \( (Q, f) \in S(M) \), make \( f \) transverse to \( S^3 \) and let \( \Sigma \) be the framed 3-manifold \( f^{-1}(S^3) \)

\[
\lambda(Q, f) = 2\mu(\Sigma) - \sigma(W) \pmod{32}
\]

where \( W = Q - \Sigma \). This is a well defined invariant of \((Q, f)\), this is because if we cut open \( Z \) along a transverse inverse image \( N = F^{-1}(S^3) \), we get an oriented 5-manifold \( H = Z - N \) with boundary \( \partial H = -W_1 \cup N \cup W_2 \cup N \) (Figure 9.13) for some spin 4-manifold \( N \) (why?) with \( \partial N = \Sigma_1 - \Sigma_2 \), and hence the signature of its boundary is zero, and by definition \( \mu(\Sigma_1) - \mu(\Sigma_2) = \sigma(N) \) (mod 16) (Section 5.5).

![Figure 9.13](image)

Now we can compute \( \lambda(Q, f) \) for the specific manifold we constructed above:

\[
\lambda(Q, f) = 2\mu(\Sigma(2, 3, 7)) - \sigma(S^3 \times [0, 1] \# S^2 \times S^2) = 2\mu(\Sigma(2, 3, 7)) = -16 \pmod{32}
\]

We claim \( Q \) can not even be \( h \)-cobordant to \( S^3 \times S^1 \# S^2 \times S^2 \), because if it is, by attaching a 3-handle to this cobordism we can extend \( f \) across the cobordism and get \((Q, f) \sim (S^3 \times S^1, g)\), where \( g : S^3 \times S^1 \rightarrow S^3 \times S^1 \) is a homotopy equivalence. Therefore \( \lambda(Q, f) = \lambda(S^3 \times S^1, g) = 0 \) (why?). Hence \( Q \) can not be standard. \( \square \).
Exercise 9.3. Show that $Q$ is obtained from $S^3 \times S^1 \# S^2 \times S^2$ by a Gluck twisting across an imbedded 2-sphere (Hint: observe that the $-1$ framed circle of Figure 9.12 is isotopic to the trivially linked $-1$ framed circle of Figure 9.14 (see Section 6.2))

![Figure 9.14](image1)

Exercise 9.4. Show that $Q$ is also obtained from $(S^3 \times S^1) \# \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ by a Gluck twisting across an imbedded 2-sphere (Hint: Slide $-1$ framed circle of Figure 9.14 across the orientation reversing 1-handle, then slide it over the 0-framed 2-handle, and then slide the other 0-framed handle over it to obtain Figure 9.15)

![Figure 9.15](image2)
9 Exotic 4-manifolds
Chapter 10

Cork decomposition

In higher dimensions 1-handles of simply connected manifolds can be eliminated (this is the first step of the proof of the $h$-cobordism theorem). In dimension 4 it is not clear if this can be done, in fact when the manifold has boundary, this can not be done by the Example 7.5 of [AY2]. Nevertheless in dimension 4 there is a useful weaker analogue of this (e.g. [Ma] and [GS]). That is, we can decompose $X^4 = W \sim_N N$ as a union of contractible manifold $W$ and a manifold $N$ without 1-handles glued top of $W$ along its boundary, and furthermore by applying Theorem 8.10 we can make each piece Stein.

![Figure 10.1: $X^4$](image)

**Proposition 10.1.** Let $X$ be a smooth 1-connected 4-manifold given as a handlebody with $X^{(2)} = X_\Lambda$ where $\Lambda = \{K_1^{r_1},...,K_n^{r_n},C_1,...,C_p\}$, where $X^{(i)}$ denotes the subhandlebody of $X$ consisting of handles of index $\leq i$. Then by introducing pairs of canceling 2 and 3-handles, we can obtain a new enlarged handlebody for $X$ with $X^{(2)} = X_{\Lambda'}$, where

$$\Lambda' = \{K_1^{r_1},...,K_n^{r_n},L_1^{s_1},...,L_p^{s_p},C_1,...,C_p\}$$

such that the attaching circle of each 2-handle $L_j$ is homotopic in $X^{(1)}$ to the core of the 1-handle $C_j$. Hence this gives the decomposition of $X = W \cup N$ as in Figure 10.1 such that $W = W_{\Lambda''}$ with $\Lambda'' = \{L_1^{s_1},...,L_p^{s_p},C_1,...,C_p\}$, and $N = N_{\Lambda''}$ with $\Lambda''' = \{K_1^{r_1},...,K_n^{r_n}\}$. 

Proof. Let $\gamma$ be the core circle of a 1-handle $C$ (the linking circle of the circle with dot). If $\gamma$ bounded a disk in $\partial X^{(2)}$, by attaching a canceling pair of 2- and 3-handles, along $\gamma$, we could create canceling 2-handle for $C$. But we don’t know this. To prove the proposition we only have to create a 2-handle in $X$ whose attaching circle $L$ is homotopic to $\gamma$ in $X^{(1)}$. Since $X$ is simply connected we can find an immersed cylinder $H$ in $X^{(2)}$ connecting $\gamma$ to an unknotted trivial circle $c \subset \partial X^{(2)}$. By general position, $H$ misses the cores of the 1-handles of $X$, but might meet 2-handles of $X$ along points. By piping these points to $\gamma$ as shown in Figure 10.2, we obtain a circle $\gamma' \subset \partial X^{(2)}$, which is homotopic to $c$ in the complements of the cores of 1- and 2-handles (and also homotopic to $\gamma$ in $X^{(1)}$). Hence this homotopy $H$ can be pushed into the 3-manifold $\partial X^{(2)}$, where it can be viewed as moving pictures starting at $\gamma'$ occasionally self crossing and ending at $c$. So by replacing $\gamma'$ by connected summing with the small dual circles of itself (when self crossings occur) we obtain $L$ which is isotopic to $c$, and still homotopic to $\gamma$ in $X^{(1)}$. So after attaching a canceling 2/3-handle pair to $c$, we see that $L$ bounds an imbedded disk, which we can viewed as the attaching frame circle of a 2-handle. By repeating this process all the 1-handles we get the required frame link $\{L_{s_{1}},...,L_{s_{p}}\}$. $\square$

Remark 10.2. When $\text{dim}(X) \geq 5$ we can get a stronger conclusion with less work. In the beginning of the proof, we can make the cylinder $H$ miss all the 2-handles by general position, and imbed it into $\partial X^{(2)}$, which we can use to construct the 2-handle canceling the 1-handle. By this process we can cancel all the 1-handles of $X$. 

![Figure 10.2](image_url)
10.1 Corks

Definition 10.3. A Cork is a pair \((W, f)\), where \(W\) is a compact contractible Stein manifold, and \(f : \partial W \to \partial W\) is an involution, which extends to a self-homeomorphism of \(W\), but it does not extend to a self-diffeomorphism of \(W\). We say \((W, f)\) is a cork of \(M\), if \(W \subset M\) and cutting \(W\) out of \(M\) and re-glueing with \(f\) produces an exotic copy \(M'\) of \(M\) (a smooth manifold homeomorphic but not diffeomorphic to \(M\)). This means that we have the following decomposition: (where \(N = M - \text{int}(W)\))

\[
M = N \cup_{id} W, \quad M' = N \cup_f W
\]  

A Cork of 4-manifold first appeared in [A11] (Theorem 9.2 and Exercise 9.1). More generally in [Ma] and [CFHS] it was shown that any exotic copy \(M'\) of a closed simply connected 4-manifold \(M\) differs from its original copy by a cork (this was also discussed in [K2]). In [AM3], by using the convex decomposition Theorem 8.10, it was shown that in the cork decompositions (10.1), each \(W\) and \(N\) pieces can be made Stein manifolds. Since then the Stein condition has become a part of the definition of cork. Note that a cork is a fake copy of itself relative to its boundary. There are some natural infinite families of corks: \(W_n, \bar{W}_n, n \geq 1\) as shown in Figure 10.3 ([AM3], [AY2]), where their involutions \(f\) are induced by the zero and dot exchanges on their underlying symmetric links. \(W_1\) is a cork of \(E(2)\#\mathbb{CP}^2\) ([A11]), and \(W_1\) is a a cork of \(E(1)_{2,3}\) ([A7]). We call \(W_n\)'s Mazur corks, since \(W_1\) is the Mazur manifold. In [AM3] \(\bar{W}_n\)'s are called positrons.

Exercise 10.1. Prove \(W_n\) and \(\bar{W}_n\) are corks (Hint: By attaching \(-1\) framed 2-handles to \(\gamma\) or \(f(\gamma)\), where \(\gamma\) is the linking circle of the 1-handle, obtain two manifolds \(Q_1\) and \(Q_2\), such that \(Q_1\) is Stein. Then by compactifying \(Q_1\) to a Kahler surface conclude that, if \(Q_1 \approx Q_2\) we get violation to Theorem 8.4).

Next we outline R. Matveyev’s proof of the general Cork theorem, here we will present a weaker version where two different contractible manifolds \(W\) are allowed on two sides of the decomposition Definition 10.3.
Proof. ([Ma]) Let \( f : M \to M' \) be a homotopy equivalence between closed smooth simply connected 4-manifolds. This implies that there is a 5-dimensional \( h \)-cobordism \( Z^5 \) between \( M \) and \( M' \) ([Wa2]). We can assume that \( Z \) has no 1 and 4-handles (Remark 10.2). So \( M' \) is obtained from \( M \) by attaching by 2 and 3-handles. Upside down 3-handles are 2-handles. Let \( N \) be the middle level between 2 and 3-handles.

In the middle level we have following diffeomorphisms. This because surgering a circle in a 1-connected 4-manifold has an affect of connected summing \( S^2 \times S^2 \) (why?).

\[
\phi_1 : M \ # (S^2 \times S^2) \to N
\]
\[
\phi_2 : M' \ # (S^2 \times S^2) \to N
\]

After introducing canceling 2 and 3-handle pairs we can assume:

\[
(\phi_2^{-1} \circ \phi_1)_* \mid_{H_2(M)} = f_*
\]

and assume that the 2-spheres are mapped to homotopic copies of the corresponding 2-spheres by \( \phi_2^{-1} \circ \phi_1 \) in the connected sum decomposition. For simplicity assume \( k = 1 \).
So we have two homotopic imbeddings \( c_i : S^2_i \vee S^2_i \to N, \ i = 1, 2 \), giving rise to an algebraically dual imbedded 2-spheres \( \{S, P\} \) in \( N \), where \( S = c_1(S^2_i) \) and \( P = c_2(S^2_i) \), with the property that surgering \( N \) along \( S \) gives \( M \), and surgering \( N \) along \( P \) gives \( M' \). Take the tubular neighborhood of the union of these two spheres 

\[
V_0 = N(S \cup P)
\]

Now we need to expand \( V_0 \) a little bit so that surgering \( S \) and \( P \) inside would give the two contractible manifolds \( W_1 \) and \( W_2 \) to be used in (10.1). If \( S \) and \( P \) intersect more than one point, we can take a pair of oppositely signed intersection points, and take the corresponding Whitney circle \( \lambda \). Clearly \( \lambda \) is null homotopic in the complements of either \( S \) or \( P \) (because surgering \( N \) along them gives simply connected manifolds \( M \) and \( M' \)). Let \( D \) and \( E \) be two immersed disks which \( \lambda \) bounds in the complements of \( S \) and \( P \), respectively. Then as homology we have

\[
[D \cup -E] \in H_2(M) \oplus \langle S, P \rangle
\]

So we can decompose \([D \cup -E] = a + \alpha[S] + \beta[P]\). After possibly by connected summing \( D \) (and \( E \)) by immersed 2-spheres lying in the complement of \( S \) (and in the complement of \( P \)), we can assume that \( D \) and \( E \) are homotopic to each other relative to the boundary \( \lambda \). We can put this homotopy \( \varphi_1 : B^2 \to N \) in general position so that the middle level \( K = \varphi_{1/2}(B^2) \) might meet both \( S \) and \( P \), but \( D \) and \( E \) are obtained from \( K \) by a series of Whitney moves. This means that there are imbedded Whitney disks \( X = \{X_i\} \) between \( K \) and \( S \), and \( Y = \{Y_j\} \) between \( K \) and \( P \), with interiors disjoint from everything else (but their boundaries might meet in \( K \)). Whitney moves across them produces the disks \( D \) and \( E \). Now take the union of \( V_0 \) with the tubular neighborhood of \( K \cup X \cup Y \)

\[
V_1 = V_0 \cup N(K \cup X \cup Y)
\]

We have the following properties (first two follow from the construction):

Figure 10.5
(a) $\pi_1(V_1)$ is a free group.

(b) $H_2(V_1) = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\{S, P\}$

(c) $S, P$ have geometric duals $S^\perp, P^\perp \subset V_1$
   (immersed spheres intersecting $S$ and $P$ at one point)

To prove (c), by using disks $X$ we push $K$ to a new disk $K'$, whose interior is disjoint from $S$. Then by using $K'$ we do the immersed Whitney trick to $P$ eliminate the pair of points of opposite sign in $S \cap P$. By continuing this way we obtain $S^\perp$. Similarly we construct $P^\perp$.

Finally, by first representing the free generators of $\pi_1(V_1)$ by 1-handles, and then enlarging $V_1$ by attaching the 2-handles algebraically canceling each one of these 1-handles (provided by Proposition 10.1), we obtain a simply connected manifold $V_2$ satisfying (b) and (c). Then surgering $S$ (or $P$) from $V_2$ we get the required $W_1$ and $W_2$. \qed
10.2 Anticorks

When the involution $f$ of a cork $(W, f)$ fixes the boundary of a properly imbedded disk $D \subset W$ up to isotopy, $f(\partial D)$ bounds a disk in $W$ as well (isotopy in the collar union $D$), hence we can extend $f$ across the tubular neighborhood of $N(D)$ of $D$ by the carving process of Section 2.6. This results a manifold $Q = W - N(D)$ homology equivalent to $B^3 \times S^1$, and an involution on its boundary $\tau : \partial Q \to \partial Q$ which doesn’t extend to $Q$ as a diffeomorphism (otherwise $f$ would extend to a self diffeomorphism of $W$). Clearly when $\tau$ takes a null-homotopic loop $\delta \subset Q$ in $Q$ to an essential loop in $Q$, it can not extend to $Q$ by homotopical reason. Of course the interesting case is when $\tau$ extends to a self homeomorphism of $Q$, in which case we can conclude that $\tau$ gives an exotic structure to $Q$ relative to its boundary (just as in cork case). This happens when $\pi_1(Q) = \mathbb{Z}$ by [F]. For example, when we apply this process to the Mazur cork we get the manifold $Q$ shown in Figure 10.8 (cf [A2]). This leads us to the following definition.

**Definition 10.4.** An Anticork is a pair $(Q, \tau)$, where $Q$ is a compact manifold which is homotopy equivalent to $S^1$, and an involution $\tau : \partial Q \to \partial Q$, which extends to a self-homeomorphism of $Q$, but it does not extend to a self-diffeomorphism of $Q$.

![Diagram of Anticorks](image-url)
The reason we call \((Q, \tau)\) an anticork, is because when it comes from a cork \((W, f)\) as above, twisting \(Q\) by the involution \(\tau\) undoes the effect of twisting \(W\) by \(f\). Notice that the loop \(\gamma = \partial D\) bounds two different disks in \(B^4\) with the same complement \(Q\) (where the identity map between their boundaries does not extend to a diffeomorphism inside), they are described by the two different ribbon moves indicated in the last picture of Figure 10.8. These two disks correspond to the obvious disks which \(\gamma\) bounds in the next to last picture of Figure 10.8, before and after zero and dot exchanges in the figure.

**Remark 10.5.** Let \(M\) be the 4-manifold obtained by attaching a 2-handle to \(B^4\) along the ribbon knot \(\gamma\) of Figure 10.8, with +1 framing. Clearly \(M\) has two imbedded 2-spheres \(S_i, i = 1, 2\) of self intersection +1 representing \(H_2(M) \cong \mathbb{Z}\), corresponding to the two different 2-disks which \(\gamma\) bounds in \(B^4\). Blowing down either \(S_1\) or \(S_2\) turns \(M\) into the (positron) cork \(\overline{W}_1\) of Figure 10.3, and the two different blowings down process turns the identity map \(\partial M \to \partial M\) to the cork involution \(f : \partial \overline{W}_1 \to \partial \overline{W}_1\), i.e. the maps in Figure 10.9 commute (this can be seen by blowing down \(\gamma\) of Figure 10.8 by using the two different disks). In particular since \(M = \overline{W}_1 \# \mathbb{CP}^2\), this shows that the blowing up \(\mathbb{CP}^2\) operation undoes the cork twisting operation of the cork \((\overline{W}_1, f)\) (cf \([A20]\)).

![Figure 10.9](image-url)

**Exercise 10.2.** Explain why 2-spheres \(S_1, S_2 \subset M\) above are not isotopic to each other in \(M\), but they become isotopic in \(M \# S^2 \times S^2\) (Hint use next to last picture of Figure 10.8).
10.3 Knotting corks

Even though corks are contractible objects, any one cork in a 4-manifold \( f : W \hookrightarrow X^4 \) should be thought of analogous to a knot in \( \mathbb{R}^3 \). The imbedding of \( W \) can be knotted with different isotopy types, detected by different exotic smooth structures it induces on \( X \). The question whether all different exotic smooth structures of \( X \) can be detected by a fixed cork is an intriguing one.

**Theorem 10.6 ([AY3]).** There exists a closed smooth 4-manifold \( X^4 \), and infinitely many non-isotopic imbeddings \( f_k : W \hookrightarrow X \) of a cork \( W \), such that by cork twisting \( X \) along \( f_k(W) \) for different \( k = 1, 2, \ldots \) gives infinitely many different exotic copies of \( X \).

**Proof.** ([AY6]) Let \( S \) be the Stein manifold in Figure 10.10 (consider as a PALF). Compactify \( S \) to a symplectic manifold \( \tilde{S} \) (Theorem 8.16). Notice that \( S \) contains the cork \( W := W_1 \) and the cusp \( C \), imbedded disjointly. Then commutativity of the diagram in Figure 10.11, along with the facts (a), (b), and (c) gives the proof.

![Figure 10.10](image1)

![Figure 10.11](image2)
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(a) Knot surgery of $\tilde{S} \mapsto \tilde{S}_K$ along $C$, by using knots $K$ with different Alexander polynomials, produces different exotic manifolds (Section 6.4). This is indicated by the left vertical arrow.

(b) Cork twisting $\tilde{S}$ along $W_1$ splits an $S^2 \times S^2$ (top horizontal arrow).

(c) Knot surgery operation gets undone when connected summing with $S^2 \times S^2$, hence we have a diffeomorphism $f_K : X \to X_K$ ([A14], [Au]).

If the involution $f : \partial W_{m,n} \to \partial W_{m,n}$ extended to a homeomorphism inside, we would get homeomorphic manifolds $W_{m,n}^1$ and $W_{m,n}^2$, obtained by attaching 2-handles to $\alpha$ and $f(\alpha)$ with $-1$ framings, respectively. But $W_{m,n}^2$ and $W_{m,n}^1$ have non-isomorphic intersection forms (Show that one of them does not have an element of square $-1$)

$$
\begin{pmatrix}
-2n - mn^2 & 1 \\
1 & -1
\end{pmatrix},
\begin{pmatrix}
-2n - mn^2 & -1 - mn \\
-1 - mn & -1 - m
\end{pmatrix},
$$

10.4 Plugs

Similar to corks there are different behaving codimension zero submanifolds appearing naturally, which are also responsible for exoticness of 4-manifolds, they first appeared in [AY2], where they are named plugs. Plugs generalize the Gluck twisting operation.

Definition 10.7. A plug is a pair $(W, f)$, where $W$ is a compact Stein manifold, and $f : \partial W \to \partial W$ is an involution satisfying the conditions: (a) $f$ does not extend to a self-homeomorphism of $W$, (b) $W$ imbeds into some smooth manifold $W \subset M$ such that the operation $M = N \cup_{id} W \mapsto M' = N \cup_f W$ produces an exotic copy $M'$ of $M$.

Manifold $W_{m,n}$ of Figure 10.12, with $m \geq 1, n \geq 2$, is an example of a plug, where $f$ is the involution induced by $180^\circ$ rotation.

Figure 10.12: $W_{m,n}$
Exercise 10.3. Verify that \((W_{m,n}, f)\) satisfies the second condition of being plug (Hint: Use the hint of Exercise 10.1).

Exercise 10.4. By canceling the 1-handle and the \(-m\) framed 2-handle, show that \(W_{m,n}\) can be obtained from \(B^4\) by attaching a 2-handle to a knot with \(-2n - n^2m^2\) framing.

The degenerate case of plug twisting \((W_{1,0}, f)\) corresponds to the Gluck twisting operation (Exercise 6.1). Recall that there is an example of an exotic non-orientable manifold which is obtained from the standard one by the Gluck operation (Exercises 9.3). Corks and plugs can also be used to construct exotic Stein manifold pairs as follows:

**Theorem 10.8.** ([AY2]) The manifold pairs \(\{Q_1, Q_2\}\) and \(\{M_1, M_2\}\) of Figures 10.13 and 10.14 are simply connected Stein manifolds which are exotic copies of each other.

![Figure 10.13: Q1 and Q2](image1)

![Figure 10.14: M1 and M2](image2)

**Proof.** Let us show \(\{M_1, M_2\}\) are exotic Stein pairs (the proof of the other case is similar). By Theorem 8.9 both \(M_1\) and \(M_2\) are Stein manifolds. Then we compactify \(M_1\) into a symplectic manifold \(S\) (Theorem 8.16). From the Legendrian handle picture of \(M_1\) in Figure 10.15, we can compute \(\mathbb{K}_S \cdot \gamma = rot(\gamma) = \pm 2\) (Section 8.3). Then if \(\gamma\) is represented by a surface \(\Sigma\) of genus \(g\) in \(S\), by the adjunction formula \(2g - 2 \geq \Sigma \cdot \Sigma + |\mathbb{K}_S \cdot \Sigma| = 2\). 
Hence \( g \geq 2 \). But on the other hand, we see from the picture of \( M_2 \) that the homology class \( \gamma' \) is represented by a torus. So \( M_1 \) and \( M_2 \) can not be diffeomorphic.

Notice from the pictures that, there are transformations \( M_1 \sim M_2 \), and \( Q_1 \sim Q_2 \) which are obtained by twisting along the cork \( W \), and twisting along the plug \( W_{1,3} \), respectively. It is easy to check that the boundary diffeomorphisms extends to a homotopy equivalence inside, since they have isomorphic intersection forms (e.g \([Bo]\)). Hence by Freedman’s theorem they are homeomorphic.

The existence of infinitely many simply connected Stein manifolds which are exotic copies of each other was established in \([AEMS]\), where they produced such manifolds as fillings of a fixed contact manifold. Here we prove the same result in a more concrete way, by drawing small 4-dimensional handlebodies (of Betti number 2) with this property.

**Theorem 10.9. ([AY5])** There are infinitely many simply connected Stein manifolds (with Betti number 2), which are exotic copies of each other, furthermore they are Stein fillings of the same contact 3-manifold.

**Proof.** Let \( X_p \) be the 1-connected Stein manifold drawn in Figure 10.16 (the second picture is the same manifold drawn as a Stein handlebody). First of the boundaries of all \( X_p \) are diffeomorphic to each other, we can see this by replacing the dotted circle with 0-framed circle and blowing up a \(-1\) circle on it and sliding this \(-1\) circle over the 0-framed handle and blowing it down (this describes a diffeomorphism \( \partial X_p \approx \partial X_{p+1} \)). Note that \( X_p \mapsto X_{p+1} \) is just the Luttinger surgery operation performed along the obvious torus inside \( X_p \), which is discussed in Section 6.3.1. Luttinger surgery preserves symplectic structure inducing isomorphic contact structures on the boundaries of \( X_p \) and \( X_{p+1} \) (cf \([EP]\)). Now it follows that all \( X_p \) are homotopy equivalent to each other, and hence by \([F]\) are homeomorphic to each other. Next we will show that for infinitely many different values of \( p \) these manifolds are exotic copies of each other.
First notice that $\partial X_p$ is a homology sphere. Compactify $X_p$ into a closed Symplectic manifold $S$ (Theorem 8.16). Let $\alpha$, $\beta$, and $T$ be the homology classes of $S$ with self intersections $p - 3$, $-2$, and $0$, respectively. By computing rotational numbers from the picture, we get $K_S.(\alpha - p\beta) = 1 \mp p$ (depending upon how we draw the zig-zag). Also $K_S.T = 0$ (this follows from the adjunction inequality since $T$ is represented by a self intersection zero torus). Then by taking $p$ odd we can assume that the self intersection number $(\alpha - p\beta)^2 = p - 3 - 2p^2$ is some even integer $2k$. Then the class $\gamma = \alpha - p\beta - kT$ has self intersection zero and $K_S.\gamma = 1 \pm p$, hence if $\gamma$ is represented by a surface $\Sigma$ of genus $g$, then $2g - 2 \geq |1 \pm p|$. Hence $g \mapsto \infty$ as $p \mapsto \infty$. So as $p \mapsto \infty$ infinitely many $X_p'$s are not mutually diffeomorphic. \hfill \qed

Exercise 10.5. ([A13]) By using the description of $E(1)_{p,q}$ in Figure 7.10, show that $E(1)_{p,q} \# 5\mathbb{CP}^2$ contains the Stein submanifold $M_{p,q}$ of Figure 10.18 inducing the decomposition $E(1)_{p,q} \leftarrow M_{p,q} \times \partial N$ for some codimension zero submanifold $N \subset E(1)_{p,q}$ (Hint: Identify Figure 10.18 with the second picture of Figure 10.17).

Remark 10.10. ([A13]) It turns out that the infinite family of exotic manifolds $E(1)_{p,2}$ with $p$ odd, differ from each other by codimension zero Stein submanifolds $Z_p$ with the same property of the manifolds of Theorem 10.9, i.e. as $p \to \infty$ they become distinct exotic Stein manifolds filling the same contact 3-manifold on their boundary.
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Figure 10.17

Figure 10.18: $M_{p,q}$
Chapter 11

Covering spaces

11.1 Handlebody of coverings

A quick way to draw a handlebody of a covering space of a 4-manifold is to first draw the part of the covering space lying over its 1-handles, which is a wedge of thickened circles $\#_k(S^1 \times B^3)$, then extend the covering space over its 2-handles. By uniqueness extending over the 3-handles is automatic. 1-handles of a 4-manifold $X^4$ give generators $x_1, x_2, ..., x_n$, and the 2-handles $h_1, ..., h_k$ give the relations of the fundamental group $\pi_1(X) = \langle x_1, x_2, ..., x_n | h_1(x_1, ..., x_n) = 1, ..., h_k(x_1, ..., x_n) = 1 \rangle$. Let $\pi_1(X) \to G$ be an epimorphism mapping $x_1, x_2, ..., x_k$ to the generators of $G$ ($k \leq n$). To draw the covering corresponding to the subgroup given by kernel of this map, we first draw the Cayley graph $K_G$ of the group $G$, which is a 1-complex with vertices consisting of elements of $G$, and any two vertex $a$ and $b$ are connected by a 1 simplex whenever $a = xb$ for some generator $x^{\pm 1} \in G$. Then thicken this 1-complex to get the 1-handles of the covering space, then over them naturally extend it to covering of the 2-handles. The other 1-handles $x_{k+1}, ..., x_n$ just lift trivially to $|G|$ copies of 1-handles upstairs. For example Figure 11.1 describes the 4-fold cyclic covering $\pi : \tilde{X} \to X$ of the Fishtail $X$.

![Figure 11.1](image-url)
Figure 11.3 describes the 60-fold covering space $\tilde{W} \to W$ corresponding to the symmetric group $A_5 = \langle x, y \mid x^5 = y^3 = (xy)^2 = 1 \rangle$. Here the handlebody of $W$ consists of one 0-handle, two 1-handles $x, y$, and one 2-handle attached along the contractible $-1$ framed knot $K$ as shown in the Figure 11.3. Hence $\pi_1(W)$ is the free group $\langle x, y \rangle$ in two generators with the obvious epimorphism $\pi_1(W) = \langle x, y \rangle \to A_5$. Figure 11.2 is the Cayley graph of $A_5$ which describes the covering space of the wedge of two circles. By thickening this, we first get a 60-fold covering space of $\#2(S^1 \times B^3)$, then by lifting the attaching circle $K$ of the 2-handle to the 60 framed circles, as shown in Figure 11.3, we get the handlebody of the covering space $\pi : \tilde{W} \to W ([A18])$. 

![Figure 11.2](image-url)
Exercise 11.1. In the examples in Figure 11.1 and Figure 11.3, by lifting the normal vector fields of the attaching circles of the 2-handles of $X$ and $W$ to $\tilde{X}$ and $\tilde{W}$, verify that the framings on the lifted 2-handles are as indicated in the figures.
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11.2 Handlebody of branched coverings

Let $F \subset B^4$ a properly imbedded surface obtained by pushing the interior of an imbedded surface $F \subset S^3$ into $B^4$ by a small isotopy. We want to draw a handlebody picture of the cyclic branched covering $\pi : \tilde{B}^4 \to B^4$ branched along $F$ ([AK3]). First of all, when $F = D^2$ the $n$-fold cyclic branched covering of $B^4$ is $B^4$. We can construct this by first cutting $B^4$ open along the trace of this small isotopy of $D^2$, creating a ball with a small codimension zero “lip” $A \cup A'$ on $\partial B^4$, then by cyclically gluing $n$-copies of balls pairs $(B^4_i, A_i \cup A'_i), i = 1, \ldots, n$ with identification $A'_i \leftrightarrow A_{i+1}, B^4 = \bigcup B^4_i$ (where $A'_{n+1} = A_1$).

![Figure 11.4](image)

This process easily generalizes to the case $D^2 \subset M^4 = B^4 + 2 - handles$, provided the attaching circles of the 2-handles of $M$ link the circle $\partial D^2$ zero (mod $n$) times. In this case, to form the $n$-fold cyclic branched covering $\pi : \tilde{M} \to M$ we first take the branched covering $B^4$ along $D$, then simply lift $n$ copies of each handle (i.e. a framed knot) upstairs as a framed knot in the most natural way with appropriate framing. Figure 11.5 demonstrates this when $M = \mathbb{C}P^2 - B^4$.

![Figure 11.5](image)

**Exercise 11.2.** In Figure 11.5 by lifting normal vector field of the $-1$ framed attaching circle of the 2-handle verify that two lifted 2-handles are attached with framings $-2$. 

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One useful application of this is, if \( K \subset S^3 \) is a knot and we are only interested in finding the 3-manifold which is the cyclic branched covering of \( S^3 \) along \( K \), we simply blow up the interior of \( B^4 \) to \( M = B^4 \# \pm k\mathbb{C}P^2 \) for some \( k \), where the knot \( K \subset \partial M \) looks like an unknot \( K = \partial D \) for some properly imbedded \( D^2 \subset B^4 \) and the handles of \( M \) link \( \partial D \) zero times, we then apply this technique (e.g. the example in Figure 11.6).

![Figure 11.6: \( \Sigma(2, 3, 5) \) as the 5-fold branched cover of the trefoil knot \( K \)](image)

**Exercise 11.3.** By the steps in Figure 11.7 show that the homology sphere \( \Sigma(2, 3, 11) \) (Exercise 2.6) is the 2-fold branched covering of the knot \( K \) in the figure. \( \Sigma(2, 3, 11) \) is also the 2-fold branched covering space of the \((3, 11)\)-torus knot (Section 12.1).

![Figure 11.7](image)
Exercise 11.4. By the steps of Figure 11.8 and blowing up the last figure once, show that 2-fold branched covering of the 0-Whitehead double \( Wh(K) \) of the trefoil knot (Section 9.2) bounds a compact smooth manifold \( X^4 \) with intersection form \( E_8 \oplus (-1) \oplus (-1) \). Show that this implies \( Wh(K) \) is not slice (Hint: If \( Wh(K) \) was slice, branched covering of \( B^4 \) branched along the slice disk would give a rational ball, together with \( X \) you get a closed 4-manifold realizing a non-diagonizable definite intersection form, violating \([D3]\))

Now we can construct the \( n \)-fold cyclic branched covering \( \pi : \tilde{M}^4 \to B^4 \) branched along a properly imbedded connected surface \( F \subset B^4 \) (isotoped from \( S^3 \)). As described in Figure 11.4, \( \tilde{M} \) is obtained by cutting \( B^4 \) along \( F \) then gluing \( n \) copies of the resulting “balls with lips” along their lips \( \cong F \times I \). By writing \( F \) as a 2-disk with handles \( D^2 \cup h \) we obtain a ball with 1-handles \( F \times I = A \cup a \). We do the gluings in two steps: First glue along the balls, then extend it to the identification of the 1-handles. By cutting \( B^4 \) open along \( F \) we create lips which are the two halves of a ball with 1-handles \( (A \cup a) \cup (A' \cup a') \). Then as before, we glue 4-balls with lips \( (B^4_1, (A_i \cup a_i) \cup (A'_i \cup a'_i)) \), \( i = 1, \ldots, n \), along the balls to get \( B^4 = \bigcup_{A'_i=A_{i+1}} B^4_1 \), then identify \( a'_i \leftrightarrow a_{i+1} \) by attaching \( n-1 \) 2-handles to \( B^4 \).
This process is demonstrated in Figure 11.9. By sliding the 2-handles over each other we can make all the half arcs $K'_i$ of the attaching circles $K_i = K'_i \cup \partial K''_i$, $i = 1, 2, \ldots, n - 1$ of the 2-handles go over one copy of $h$ of the surface $F$, the other halves $K''_i$ are just doubling of $K'_i$ s where they pile over each other as if lying on different pages of a book. Framings of $K_i$ are twice the page framing induced from $F$ (e.g. Figure 11.10).

![Figure 11.10](image)

For example, the right and left pictures of Figure 11.11 give the handlebodies of the 2- and 3-fold branched coverings of $B^4$ branched along the surface $F$ shown in the middle, respectively (also compare to Figure 11.14). Also the case $F \subset M^4 = B^4 + handles$, where the attaching circles of the 2-handles of $M$ link the circle $\partial F$ zero (mod $n$), proceeds as before, that is we lift the handles as the most natural way.

![Figure 11.11](image)

Next we discuss the case of branched covering $B^4$ along disconnected surface $F$. As before, we first deal the simple case of when the surface is a disjoint union disks $F = \cup_{i=1}^{\mu} D_j$. Surprisingly, even in this case there are variety of very different branched coverings. To see these, as before we cut $B^4$ along the disks of $F$ create n lips on $B^4$ (Figure 11.12). We codify this picture of $B^4$ by denoting it with a star shaped graph,
where the outer edges are indexed by integers \( k_1, \ldots, k_n \), call this graph a \((k_1, \ldots, k_n)\) switch or just simply a switch.

To construct various (regular or irregular) branch coverings of \( B^4 \) branched along \( \sqcup_{j=1}^n D_j \) we glue many copies of these switches (i.e. balls) along their outer vertices, with the rule that \( k \) copies of index \( k \) vertices of switches meet at a vertex (index 1 switches are unmatched). Imagine an index \( k \) vertex of a switch as a lip of a 4-ball with \( 2\pi/k \) solid angle, so that \( k \) of them nicely fit together creating a \( k \)-fold branching around that disk. So this process constructs a branched covering \( \pi : \#_k S^1 \times B^3 \to B^4 \) as indicated in Figure 11.13 (to emphasize various \( S^1 \times B^3 \)'s we put circle with dots in the figure).

Now again by the gluing process of Figure 11.9, we can extend the handlebody of branched covers \( \#_k S^1 \times B^3 \to B^4 \) of \( B^4 \) branched along \( \sqcup_j D_j \) to the branched covers branched along the (disconnected) surfaces \( F \subset B^4 \) which are obtained from \( \sqcup_j D_j \) by attaching 1-handles (bands). In fact as before, we can go one step further and extend this to the case of disconnected surfaces \( F \subset M^4 = B^4 + \text{handles} \), provided the attaching circles of the 2-handles of \( M \) link the \( \partial(\sqcup_j D_j) \) zero (mod \( n \)) times.
Exercise 11.5. By first taking the branched covering of $B^4$ branched over the disjoint union of two disks $D = D_1 \sqcup D_2$, and then extending this over the surface $F$ consisting of $D$ with two 1-handles, show that the picture on the right in Figure 11.14 gives the handlebody of the 3-fold cyclic branched cover of $B^4$ branched over $F$.

![Figure 11.14: 3-fold cyclic branched cover](image)

Exercise 11.6. By first taking the branched covering of $B^4$ along $\sqcup_{j=1}^{3} D_j$ by the switches indicated in Figure 11.15, and extending over the handles of the surface $F = D + \text{bands}$, and extending over the $-2$ framed 2-handle, show that figure on the right gives the irregular 3-fold covering space of the manifold on the left branched along the surface $F$.

![Figure 11.15: 3-fold irregular branched cover](image)
11.3 Branched covers along ribbon surfaces

In the previous section we described the handlebodies of various branched covering spaces of 4-manifolds $\pi : \tilde{M} \to M$ branched along properly imbedded surface $F \subset M$, which is an isotopic copy of a surfaces lying on the boundary of the 0-handle $F \subset \partial B^4 \subset M$. Now we generalize this to the case of when the surfaces $F \subset M$ is an isotopic copy of a ribbon-immersed surface $f : F \looparrowright \partial B^4$. This case can be dealt similar to the imbedded case ([Mo]), because a ribbon surface $F = \sqcup D_j + \sum h_i$ is just collection of disks $D = \sqcup D_j$ connected with ribbon 1-handles $\{h_i\}$, rather then imbedded 1-handles. Hence once we put the ribbons into standard position as shown in the second picture of Figure 11.16, we extend the branched covering space of $B^4$ branched over $D$ to $F$ by the same way as before (the third picture of Figure 11.16).

![Figure 11.16: Mazur manifold as the 2-fold branched cover of $B^4$](image)

Now it is clear from Figure 11.16 that this process can be reversed, if the 2-handles of the 4-manifold exhibit some symmetry in the following sense:

**Definition 11.1.** A link $\{K_1,..,K_k\}$ in $\#_k S^1 \times S^2$ is called strongly invertible if the “standard” involution of $\#_k S^1 \times S^2$ (induced by $180^\circ$ degree rotation) induces an involution on each of the knots $K_j$ with two fixed points.

![Figure 11.17: X as a 2-fold branched cover over $B^4$](image)
11.3 Branched covers along ribbon surfaces

Theorem 11.2. ([Mo]) If a compact smooth 4-manifold has only handles of index \( \leq 2 \), and its 2-handles are attached along a strongly invertible link in \( \#_k S^1 \times S^2 \), then \( M \) is a 2-fold branched covering space of \( B^4 \) branched along a properly imbedded surface \( F \subset B^4 \).

Corollary 11.3. If the 2-handles of any closed smooth 4-manifold \( M \) are attached along a strongly invertible link in \( \#_k S^1 \times S^2 \), then \( M \) is a 2-fold branched covering space of \( S^4 \) branched along a closed smooth 2-manifold \( F \subset S^4 \).

Proof. Let \( M^{(2)} \) be the sub-handlebody of \( M \) consiting of handles of index \( \leq 2 \), we have \( M - int(M^{(2)}) = \#_k S^1 \times B^3 \). By Theorem 11.2 we have a branched covering \( \pi: M^{(2)} \to B^4 \) branched along properly imbedded \( F \subset B^4 \), and by [KT] the induced branched covering on the boundary (by restriction) \( \pi_\partial: \#_k S^1 \times S^2 \to S^3 \) has to be standard with branch set \( S^1 \sqcup \ldots \sqcup S^1 \). We can then extend \( \pi_\partial \) trivially to another branched covering map \( \pi': \#_k S^1 \times B^3 \to B^4 \). By putting \( \pi \) and \( \pi' \) together we get the desired branched covering \( M \to S^4 \).

Exercise 11.7. Show that there is a 2-fold branched covering \( T^3 \to S^3 \) branched along the link \( L \) of Figure 11.18 (with branching index 2 each components).

Exercise 11.8. Generalize the above process of taking 2-fold branched covering of \( B^4 \) along a ribbon disk (of Figure 11.16) to taking \( k \)-fold cyclic branched covering of \( B^4 \) along a ribbon disk \( D \subset B^4 \) (Hint: imitate the Seifert surface case).
Exercise 11.9. Show that the \( k \)-fold cyclic branched covering of the manifold \( M \) of Figure 11.19 branched along 2-spheres \( A \sqcup B \sqcup C \sqcup D \) is given by the plumbing \( \tilde{M} \), where the small case letters indicate Euler classes of the corresponding spheres (Hint: Compute how branched covering changes the Euler class of each 2-sphere. While branched covering along \( A, B, C, D \), the spheres \( e, f, g \) spheres get branched covered along two points).
Chapter 12

Complex surfaces

A rich source of 4-manifolds are complex surfaces (2-dimensional complex algebraic varieties). Next we will construct handlebodies complex surfaces in $\mathbb{C}^3$ and $\mathbb{CP}^3$ ([AK3]).

12.1 Milnor fibers of isolated singularities

Let $F : \mathbb{C}^3 \to \mathbb{C}$ be a complex polynomial map with isolated singularity, that is $dF(z)$ is nonzero on $U - \{0\}$, where $U$ is some connected open subset of $\mathbb{C}^3$ containing the origin. Let $z_0 \in intF(U)$ be a regular value, then the compact smooth manifold with boundary $M_F = F^{-1}(w_0) \cap B^6$, where $B^6 \subset U$ is a ball centered at the origin, is called the Milnor Fiber of $F$. When $F(x, y, z) = f(x, y) + z^d$, the manifold $M_F$ is the $d$-fold cyclic branched covering of the unit ball $B^4$ branched over the Milnor fiber $M_f$ of $f : \mathbb{C}^2 \to \mathbb{C}$.

Figure 12.1
An interesting special case is when $F(x, y, z) = x^a + y^b + z^c$. Denote the Milnor fiber by $M(a, b, c) = M_F$. So $M(a, b, c)$ is $c$-fold cyclic branched covering space of $B^4$ branched along the Milnor fiber $M(a, b)$ of $x^a + y^b$. The boundary $\partial M(a, b, c) := \Sigma(a, b, c)$ is called Brieskorn manifold. The Milnor fiber $M(a, b)$ is the Seifert surface of the $(a, b)$ torus link whose interior is pushed into $B^4$ ([M3] p. 53). Hence $M(a, b)$ is the fiber $(a, b)$ torus link obtained by by stacking $a$ plates connected by $b$ half twisted bands.

Figure 12.2: $M(4, 4)$  
Figure 12.3: $M(3, 5)$

Figure 12.4 is $M(2, 3, 5)$ as 5-fold branched cover of $B^4$ branched along $M(2, 3)$.

Figure 12.4: $M(2, 3, k)$, when $k = 5$

We can easily identify $M(2, 3, 5)$ with $E_8$ by taking the 2-fold branched covering of $B^4$ branched along $M(3, 5)$ (Figure 12.3), and by doing the indicated handle slides.

Figure 12.5: $M(2, 3, 5) = E(1)$
**Proposition 12.1.** $M(2,3,k)$ is given by Figure 12.6 when $k = 2m + 1$, and given by Figure 12.7 when $k = 2m$.

![Figure 12.6](image1)

![Figure 12.7](image2)

**Proof.** Let us do the case $k = 5$ (the general case is similar). The first picture is the 5-fold branched covering of $M(2,3)$ (check), the subsequent steps establishes the proof.

![Figure 12.8](image3)

**Exercise 12.1.** *By using the description given in Proposition 12.1 identify $M(2,3,5)$ with $E_8$ (Hint: Figure 12.9).*
12 Complex surfaces

12.2 Hypersurfaces that are branched covers of \(\mathbb{CP}^2\)

A degree \(d\) hypersurface in \(\mathbb{CP}^n\) is the set \(V_d \subset \mathbb{CP}^n\) induced by the locus of any degree \(d\) homogeneous polynomial \(f(z_0, z_1, \ldots, z_n) = 0\). We say \(V_d\) is nonsingular if \(df(z)\) nonzero on \(Z\). Let \(\{m_j(x)\}_{j=0}^{N}\) be the collection of degree \(d\) monomials on \(\mathbb{C}^{n+1}\), hence any \(V_d\) is the locus of \(\sum a_j m_j(x) = 0\) for some \(a = [a_1, \ldots, a_N]\). All degree \(d\) nonsingular hypersurfaces in \(\mathbb{CP}^n\) are diffeomorphic to each other, because they are inverse images of the regular values of \(\pi\) (the singular values has codimension 2 since it is a complex algebraic set).

\[
Z = \{(x, a) \mid \sum a_j m_j(x) = 0\} \subset \mathbb{CP}^n \times \mathbb{CP}^N \xrightarrow{\pi} \mathbb{CP}^N
\]

When \(n \geq 3\), \(V_d\) are simply connected, because we can describe \(V_d\) as a hyperplane section \(\lambda(\mathbb{CP}^n) \cap H\) of the imbedding, given by \(\lambda: \mathbb{CP}^n \to \mathbb{CP}^N\), \(\lambda(x) = [m_0(x), \ldots, m_N(x)]\), and by Lefschetz hyperplane theorem (e.g. [M1]) \(\pi_r(\lambda(\mathbb{CP}^n), \lambda(\mathbb{CP}^n) \cap H) = 0\) for \(k \leq n - 1\).

By taking \(f(z) = z_0^d + z_1^d + \ldots + z_n^d\), we see that the restriction of the projection gives a branched covering map \(\pi: V_d \to \mathbb{CP}^{n-1}\), branched along a nonsingular degree \(d\) hypersurface in \(\mathbb{CP}^{n-1}\). Since \(V_d \subset \mathbb{CP}^3\) is simply connected its homotopy type is determined by its intersection form \(I(V_d)\) [M2]. In turn the intersection form is determined by 3 data: signature \(\sigma(V_d)\), second Betti number \(b_2(V_d)\), and the parity of \(I(V_d)\) (i.e. the knowledge of whether the intersection form is even or odd).
Proposition 12.2. \( V_d \subset \mathbb{CP}^3 \) is spin if and only if \( d \) is even and

- \( b_2(V_d) = d(6 - 4d + d^2) - 2 \)
- \( \sigma(V_d) = (4 - d^2)d/3 \)
- If \( d \) is odd \( I(V_d) = \lambda_d(1) \oplus \mu_d(-1) \), where
  \[ \lambda_d = (d^3 - 6d^2 + 11d - 3)/3, \]
  \[ \mu_d = (2d^3 - 6d^2 + 7d - 3)/3, \]
- If \( d \) is even \( I(V_d) = m_d(E_S) \oplus l_d \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \), where
  \[ m_d = (d^2 - 4)d/24, \]

Lemma 12.3. If \( S \) is a nonsingular compact complex surface then

- \( c_2(S) = c(S) \)
- \( w_2(S) = c_1(S) \mod(2) \)
- \( c_1^2(S) = 3\sigma(S) + 2e(S) \)

\((c_i(S), p_i(S), e(S) \) denote Chern, Pontryagin and Euler class respectively).

Proof. (a), (b) are standard facts, and \( p_i(S) = 3\sigma(S) \) is the index theorem. To prove (c) we use the splitting principle (e.g. [Sh]), which says that it is sufficient to prove the case when \( T(S) = L_1 \oplus L_2 \) is sum of line bundles. By definition \( p_4(S) = (-1)^ic_{2i}(S) \) so:

\[
p_1(L_1 \oplus L_2) = -c_2(L_1 \oplus \bar{L}_1 \oplus L_2 \oplus \bar{L}_2)
= -c_2((-1 - x_1)(1 + x_1)(1 - x_2)(1 + x_2))
= -c_2(1 - (x_1 + x_2)^2 + 2x_1x_2 + x_1^2x_2^2)
= (x_1 + x_2)^2 - 2x_1x_2
= c_1^2 - 2c_2
\]

Proof. (of Proposition). Recall \( H^*(\mathbb{CP}^3) = \mathbb{Z}[h]/(h^4) \), where \( h \) is the hyperplane class. Let \( j : V_d \hookrightarrow \mathbb{CP}^3 \) be the inclusion, then \( \deg(V_d) = d \Rightarrow j_*[V_d] \) is the Poincare dual of \( dh \), so the Euler class of the normal bundle \( \nu(V_d) \) is \( j^*(dh) \). Also \( j^*h^2 = d[V_d]^* \) since

\[
\langle j^*h^2, [V_d] \rangle = \langle h^2, j_*[V_d] \rangle = \langle h^2, dh \cap [\mathbb{CP}^3] \rangle = d\langle h^2, h \cap [\mathbb{CP}^3] \rangle = d\langle h^3, [\mathbb{CP}^3] \rangle = d
\]
By taking the Chern classes of both sides of $j^*T(\mathbb{CP}^3) = T(V_d) \oplus \nu(V_d)$ we get

$$j^*(1 + h)^4 = (1 + c_1 + c_2)(1 + d^*h)$$

Hence $c_1(V_d) = (4 - dj^*h)$, and $c_2(V_d) = (6 - 4d + d^2)j^*h^2 = (6 - 4d + d^2)d[V_d]$. So by Lemma 12.3 $w_2(V_d) = 0 \iff d$ is even, and $b_2(V_d) = e(V_d) - 2 = d(6 - 4d + d^2) - 2$. By using the expression of $c_1(V_d)$ and by (c) of Lemma 12.3 we calculate $\sigma(V_d) = (4 - d^2)d/3$. Clearly $\lambda_d = (b_2 + \sigma)/2$ and $\mu_d = (b_2 - \sigma)/2$ computes the expressions of $\lambda_d$ and $\mu_d$.

### 12.2.1 Handlebody descriptions of $V_d$

By using the techniques of Section 11.2 we can draw handlebody pictures of all $V_d$'s. As discussed before, since the smooth manifold structure of $V_d$ is independent of which polynomial equation we choose to describe it, we choose the following simplest one:

$$f(z_0, z_1, z_2, z_3) = z_0^d + z_1^d + z_2^d + z_3^d = 0$$

Though the projection $[z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2]$ is not a well defined map at all points of $\mathbb{CP}^3$, its restriction to $V_d$ gives a well defined $d$-fold cyclic branched covering map $\pi : V_d \to \mathbb{CP}^2$, branched over the degree $d$ complex curve $L_d \subset \mathbb{CP}^2$ given by the equation $z_0^d + z_1^d + z_2^d = 0$. By decomposing $\mathbb{CP}^2 = N \cup B^4$, where $N$ is the tubular neighborhood of $\mathbb{CP}^1$ (Euler class $+1$ disk bundle over $S^2$) and its complement $B^4$, we can describe $L_d \cap N$ as the Milnor fiber $M(d,d)$ (Section 12.1) union $L_d \cap B^4$ which is $d$ disjoint copies of trivial disks. Figure 12.12 demonstrates how $L_d$ sits in $\mathbb{CP}^2 = N \cup B^4$.

![Figure 12.11](image)

Figure 12.11

Now by a small isotopy of $L_d$ we can move $d$ half bands of $M(d,d)$ from $N$ side to $B^4$ side, so that $L_d \cap N$ appears as the Milnor fiber $M(d, d - 1)$ union $L_d \cap B^4$ which is a single trivial disks $D \subset B^4$ (Figure 12.13). Now by using the technique of Section 11.2
12.2 Hypersurfaces that are branched covers of $\mathbb{CP}^2$

we can take the branched cover of $N$ branched along $L_d \cap N$, the branched cover of $B^4$ along $D$ is just $B^4$ (the 4-handle) so we don’t need to draw that part of it. To apply this process of course we first need to draw $M(d, d - 1)$ as disk with handles then proceed.

**Figure 12.12:** $L_d = M(4, 4) \cup \bigcup_{j=1}^{d+1} D_j$

**Figure 12.13:** $L_d = M(4, 3) \cup D$

**Figure 12.14:** $M(5, 4)$

**Exercise 12.2.** Show that $V_3 = \mathbb{CP}^2 \# 6 \mathbb{CP}^2$ (Hint Figure 12.15).

**Figure 12.15:** Constructing $V_3$
For example by this process we get the following handlebody of $V_5$

![Figure 12.16: $V_5$](image)

More generally we can construct complex surfaces by taking $k$-fold branched coverings $\pi : V_k(d) \to \mathbb{CP}^2$ branched along $L_d$, where $k$ divides $d$. For example, it is known that Kummer surface $V_4$ is $V_2(6)$ (Figure 12.17).

![Figure 12.17: $V_4$](image)
12.3 $\Sigma(a, b, c)$

One way to study Brieskorn manifold $\Sigma(a, b, c)$ is to resolve the singularity of the complex hypersurface $V(a, b, c) = \{(x, y, z) \in \mathbb{C}^3 \mid x^a + y^b + z^c = 0\}$ at the origin, to get a smooth manifold $M(a, b, c)$ which $\Sigma(a, b, c)$ bounds. The topology of resolutions is a rich subject, which have not yet been completely understood (e.g. [AKi]), but in this special case there are concrete ways of doing this. A nice approach is explained in [HNK]: The idea is to uniformize the algebraic function $f(x, y) = \sqrt[3]{x^a + y^b}$ by successive blowing ups $\mathbb{C}^2$ at the origin $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$, until $V(a, b) = \{(x, y) \mid x^a + y^b = 0\}$ becomes a smooth curve $\Gamma$, with $\pi^{-1}(0)$ consisting of a union of 2-spheres $P(a, b)$, all transversal to each other.

![Figure 12.18](image)

This process is continued until the 2-spheres $\alpha_1, \alpha_2, \ldots$ of $P(a, b)$, on which $f \circ \pi$ has multiplicity 0 (mod $c$), are disjoint from each other. Also the other spheres $\beta_1, \beta_2, \ldots$ of $P(a, b)$ and $\Gamma$ are disjoint from each other, and the framings of $\beta_j$ are divisible by $c$.

![Figure 12.19](image)

This means that the function $z \mapsto \sqrt[3]{z}$ is uniformized on $\tilde{\mathbb{C}}^3$, and we can take the $c$-fold branched covering of $\tilde{\mathbb{C}}^3$ along $\bigcup \beta_j \cup \Gamma$. So $M(a, b, c)$ is the branched covering of the plumbed 4-manifold $\bigcup \alpha_i \cup \beta_j$ along $\bigcup \beta_j$ (done as in the Exercise 11.9).
$M(a,b,c)$ is the plumbing obtained from the plumbing of the spheres $P(a,b)$ by simply dividing the framings of $\beta_j$ spheres by $c$, and multiplying the framings of $\alpha_j$ spheres by $c$. The idea basically is: $V(a,b,c)$ is the $c$-fold branched covering space of $\mathbb{C}^2$ branched along $V(a,b)$, and throughout the blowing ups this branched covering information is carried along so that $M(a,b,c)$ is the branched covering space of $\mathbb{C}^2$, branched along $\cup \beta_j \cup \Gamma$.

Let us do the example $\Sigma(2,3,5)$ by hand using the above recipe. Recall that blowing $(y,z)$ chart gives two new charts $(u,v)$ and $(u',v')$, they are related to the old chart by $(y,z) = (uv,u)$ and $(y,z) = (u',u'v')$ (where $u = 0$ and $u' = 0$ is the equation of the new $-1$ sphere appeared as the result of the blowup). Here we will only trace the pictures on selected charts and guess the other parts. (cf. [Mu], [Lau], [AKi])

Figure 12.20: $(y,z) = (uv,u)$
$u^3(v^3 + u^2) = 0$

Figure 12.21: $(u,v) = (ab,a)$
$a^5b^3(a + b^2) = 0$

Figure 12.22: $(a,b) = (cd,c)$
$c^9d^5(d + c) = 0$

Figure 12.23: $(c,d) = (e,ef)$
e$^{15}f^5(1 + f) = 0$

Figure 12.24: P(3,5)

Figure 12.25: M(2,3,5)
We first blow up the curve $y^3 + z^5 = 0$ at the origin to get Figure 12.20, then by successive blowups at the indicated points we get Figures 12.20, through 12.23. The self intersections of spheres are indicated by integers, and their multiplicities (which can be read from the equations) are denoted by circled integers. The algebraic curve itself is assigned the multiplicity 1 (as a rule a blow up at a point $p$ results a new $-1$ sphere, whose multiplicity is the sum of multiplicities of the spheres meeting at that point $p$). Then finally to separate the odd multiplicity spheres from the even ones we perform the indicated blowups to get Figure 12.24. Then applying the above mentioned branched covering process, from Figure 12.24 we get Figure 12.25, which is $M(2, 3, 5)$.

**Exercise 12.3.** By resolving singularities verify the following identifications.

<table>
<thead>
<tr>
<th>Figure 12.26</th>
<th>Figure 12.27</th>
<th>Figure 12.28</th>
<th>Figure 12.29</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M(3, 4, 5)$</td>
<td>$M(2, 5, 7)$</td>
<td>$M(2, 3, 11)$</td>
<td>$M(2, 3, 13)$</td>
</tr>
</tbody>
</table>

In fact by [HJ], when $a_j$’s pairwise coprime $M(a_1, a_2, a_3)$ is the following plumbing:

![Image of plumbing](image.png)

where $a_j/b_j = [b_{j1}, ..., b_{js_j}]$, with $b_{ji} \geq 2$, and $ba_1a_2a_3 = 1 + b_1a_2a_3 + a_1b_2a_3 + a_1a_2b_3$, where $b_k$ are given by $0 < b_k < a_k$ and $a_i a_j b_k = -1 \pmod{a_k}$ for $\{i, j, k\} = \{1, 2, 3\}$.

**Exercise 12.4.** By using Exercise 12.3 verify the the identifications of Figure 2.12.
Exercise 12.5. By using Exercise 12.3 show that we have the following identifications with the boundaries of the indicated contractible manifolds defined in Figure 2.11 (Hint: Figure 12.31 indicates a diffeomorphism \( \Sigma(2, 5, 7) \approx \partial W^-(0, 3) \) then use Exercise 2.2. The others are similar cf [AK2]).

\[
\begin{align*}
\Sigma(2, 5, 7) & \approx \partial W^+(0, 0) \\
\Sigma(3, 4, 5) & \approx \partial W^+(-1, 0) \\
\Sigma(2, 3, 13) & \approx \partial W^+(1, 0)
\end{align*}
\]

![Figure 12.31: \( \Sigma(2, 5, 7) \approx \partial W^-(0, 3) \)](image)

Remark 12.4. Proposition 12.2 is a special case of the following ([Hi]): If \( X \) is a closed orientable simply connected 4-manifold, and \( Z^4 \to X^4 \) is a \( d \)-fold cyclic branched covering map branched along an orientable surface \( F \subset X \) of genus \( g(F) \), then

\[
\begin{align*}
b_2(Z) &= db_2(X) + 2(d - 1)g(F) \\
\sigma(Z) &= d\sigma(Z) - \left(\frac{d^2 - 1}{3d}\right)F.F
\end{align*}
\]
Exercise 12.6. By identifying the handlebody of $E(n)$ in Figure 7.3 as the handlebody $M(2, 3, 6n - 1)$ (Figure 12.4) plus a pair of 2-handles, and then by imitating the steps of the proof of Proposition 12.1 show that Figure 12.32 gives a handlebody of $E(n)$ (this is done in [GS]).
Seiberg-Witten invariants

Seiberg-Witten invariants are smooth 4-manifold invariants discovered by N. Seiberg and E. Witten, they help to distinguish smooth manifolds. Here we will give an introduction to these invariants by mostly following the exposition in [A8]. Let $X$ be a closed smooth 4-manifold with $b_2^+(X) \neq 0$. A Spin$^c$ structure (or a characteristic class) on $X$ is an integral lift of its second Steifel-Whitney class $w_2(X) \in H^2(X; \mathbb{Z}_2)$ under the reduction map $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}_2)$, given by $a \mapsto a \mod 2$. Then the Seiberg-Witten invariant is a function $SW_X : S(X) \to \mathbb{Z}$, defined on the set of Spin$^c$ structures on $X$:

$$S(X) = \{ L \in H^2(X; \mathbb{Z}) \mid w_2(X) = c_1(L) \ (\text{mod} \ 2) \}$$

It turns out that the set of so called basic classes $B(X) := \{ L \in S(X) \mid SW_X(L) \neq 0 \}$ is a finite set. Also, every compact oriented smooth 4-manifold has a Spin$^c$ structure. This is because the map $x \to x^2 \ (\text{mod} \ 2)$ defines a homomorphism $H_2(X; \mathbb{Z}) \to \mathbb{Z}_2$, hence there is a $a \in H_2(X; \mathbb{Z})$ with $x.a = x^2 \ (\text{mod} \ 2)$ for all $x$. Recall that the class $w_2 := w_2(X)$ also satisfies $x.w_2 = x^2$ for all $x \in H_2(X; \mathbb{Z}_2)$ (Wu relation [MS]). So if $\alpha$ is the Poincare dual of $a$, the class $b = w_2 - \rho(\alpha)$ lies in the kernel of the evaluation map $h$ in the universal coefficient sequence (here $\rho$ denotes the $\mathbb{Z}_2$ reduction map):

$$0 \to \ Ext(H_1(X); \mathbb{Z}_2) \to H^2(X; \mathbb{Z}_2) \xrightarrow{h} Hom(H_2(X); \mathbb{Z}_2) \to 0$$

$$0 \to \ Ext(H_1(X); \mathbb{Z}) \to H^2(X; \mathbb{Z}) \to Hom(H_2(X); \mathbb{Z}) \to 0$$

So $b$ comes from $Ext(H_1(X); \mathbb{Z}_2)$. The universal coefficient sequence is natural with respect to coefficient homomorphism and the first vertical arrow is onto, hence we get $b = \rho(\beta)$ for some $\beta$, which implies that $w_2$ comes from an integral class.
13.1 Representations

Let us recall some basic identifications: $SU(2) = Spin(3) = S^3 \subset \mathbb{H}$, where $\mathbb{H}$ denotes the quaternions and $S^3$ are the unit quaternions, and we have the following identifications.

$$
\begin{align*}
Spin(4) &= SU(2) \times SU(2) \\
Spin^c(4) &= (SU(2) \times SU(2) \times S^1)/\mathbb{Z}_2 = (Spin(4) \times S^1)/\mathbb{Z}_2 \\
SO(4) &= (SU(2) \times SU(2))/\mathbb{Z}_2 \\
Spin^c(3) &= (SU(2) \times S^1)/\mathbb{Z}_2 = U(2)
\end{align*}
$$

There is also the idetification $\{(A, B) \in U_2 \times U_2 \mid \text{det}(A) = \text{det}(B)\} = Spin^c(4)$ via:

$$(A, B) \mapsto (A.(\text{det}A)^{-1/2} I, B.(\text{det}B)^{-1/2} I, (\text{det}A)^{1/2})$$

By projecting the above descriptions to various factors we get fibrations:

$$
S^1 \to Spin^c(4) \to SO(4)
$$

$$
\mathbb{Z}_2 \to Spin^c(4) \to SO(4) \times S^1
$$

The fibrations above naturally extend to fibrations:

$$
S^1 \to Spin^c(4) \to SO(4) \to K(\mathbb{Z}, 2) \to BSpin^c(4) \to BSO(4) \to K(\mathbb{Z}, 3) \quad (13.1)
$$

The last map in the sequence is given by the Bokstein of the universal second Steifel-Whitney class $\delta(w_2)$, which is the obstruction to lifting $w_2$ to an integral class. This explains why this corresponds to a $Spin^c(4)$ structure. We also have the fibration:

$$
\mathbb{Z}_2 \to Spin^c(4) \to SO(4) \times S^1 \to K(\mathbb{Z}_2, 1) \to BSpin^c(4) \to BSO(4) \times BS^1 \to K(\mathbb{Z}_2, 2)
$$

The last map is given by $w_2 \times 1 + 1 \times \rho(c_1)$ which clearly vanishes when $\delta(w_2) = 0$. This says that $Spin^c$ can be thought of a complex line bundle $L \to X$ whose first Chern class reduces to $w_2(X)$. Finally we have the fibration:

$$
\mathbb{Z}_2 \to Spin(4) \times S^1 \to Spin^c(4) \to K(\mathbb{Z}_2, 1) \to BSpin(4) \times BS^1 \to BSpin^c(4) \to K(\mathbb{Z}_2, 2)
$$

The last map is given by $w_2$. This sequence says that locally a $Spin^c(4)$ bundle consists a pair of a $Spin(4)$ bundle and a complex line bundle. Also recall we have identifications:

$$
H^2(X; \mathbb{Z}) = [X, K(\mathbb{Z}, 2)] = [X, BS^1] = \{\text{Complex line bundles } L \to X\} \quad (13.2)
$$
Definition 13.1. A $\text{Spin}^c(4)$ structure on $X^4$ is a complex line bundle $L \to X$ with $w_2(TX) = c_1(L) \pmod{2}$ (i.e. $L$ is a characteristic line bundle). This means a principal $\text{Spin}^c(4)$-bundle $P \to X$ such that the associated framed bundles of $TX$ and $L$ satisfy:

$$P_{SO(4)}(TX) = P \times_{\rho_0} SO(4)$$
$$P_{S^1}(L) = P \times_{\rho_1} S^1$$

where $(\rho_0, \rho_1) : \text{Spin}^c(4) \to SO(4) \times S^1$ are the obvious projections.

So we have the following natural projections:

$$\begin{array}{ccc}
\text{Spin}(4) \times S^1 & \downarrow & \pi \\
\text{Spin}^c(4) & \overset{\rho_1}{\longrightarrow} & S^1 \\
\end{array}$$

$$\begin{array}{ccc}
\not{\rho}_+, \downarrow \rho_0 & \rho_- \not{\rho} \\
U(2) & SO(4) & U(2) \\
Ad \downarrow & \not{p}_+ & p_- \not{\rho} \downarrow Ad \\
SO(3) & SO(3) \\
\end{array}$$

Let $\tilde{\rho}_\pm = Ad \circ \rho_\pm$, also call $\tilde{\rho}_\pm = \rho_\pm \circ \pi$. For $x \in \mathbb{H} = \mathbb{R}^4$ we have

$$\begin{align*}
\rho_1[ q_+, q_-, \lambda ] &= \lambda^2 \\
\rho_0[ q_+, q_-, \lambda ] &= [ q_+, q_- ] \text{ i.e. } x \mapsto q_+ x q_-^{-1} \\
\rho_\pm[ q_+, q_-, \lambda ] &= [ q_\pm, \lambda ] \text{ i.e. } x \mapsto q_+ x \lambda^{-1} \\
\tilde{\rho}_\pm[ q_+, q_-, \lambda ] &= Ad \circ q_\pm \text{ i.e. } x \mapsto q_\pm x q_\pm^{-1} \\
\tilde{\rho}_\pm( q_+, q_-, \lambda ) &= \lambda q_\pm \\
\end{align*}$$

Besides $TX$ and $L$, a $\text{Spin}^c(4)$ bundle $P \to X$ induces a pair of $U(2)$ bundles:

$$W^\pm = P \times_{\rho_\pm} \mathbb{C}^2 \longrightarrow X$$
In general these are very different $\mathbb{C}^2$-bundles (e.g. $[A22]$), for example

\[
\begin{align*}
    c_2(W^+)[M] &= (c_1^2(L) - 3\sigma(M) - 2\chi(M))/4 \\
    c_2(W^-)[M] &= (c_1^2(L) - 3\sigma(M) + 2\chi(M))/4
\end{align*}
\]

Let $\Lambda^p(X) = \Lambda^pT^*(X)$ be the bundle of exterior $p$ forms over a Riemannian manifold $(X, g)$ ($g$ is metric), we can construct the bundle of self (antiself)-dual 2-forms $\Lambda^2(X)$ which we abbreviate by $\Lambda^\pm(X)$. We can identify $\Lambda^2(X)$ by the Lie algebra $so(4)$-bundle

\[
\Lambda^2(X) = P(T^*X) \times_{ad} so(4) \quad \text{by} \quad \sum a_{ij} \, dx^i \wedge dx^j \leftrightarrow (a_{ij})
\]

where $ad : SO(4) \to so(4)$ is the adjoint representation. The adjoint action preserves the two summands of $so(4) = spin(4) = so(3) \times so(3) = \mathbb{R}^3 \oplus \mathbb{R}^3$. By above identification it is easy to see that the $\pm 1$ eigenspaces $\Lambda^\pm(X)$ of the star operator $*: \Lambda(X) \to \Lambda(X)$ corresponds to these two $\mathbb{R}^3$-bundles: this gives:

\[
\Lambda^\pm_\pm(X) := \Lambda^\pm(X) = P \times_{\tilde{\rho}_\pm} \mathbb{R}^3 = P_{SO(4)}(TX) \times_{\tilde{\rho}_\pm} \mathbb{R}^3
\]

The Chern classes are given by $c_2(\Lambda^\pm(X)) = 4c_2(W^\pm)$. Also if the $Spin^c(4)$ bundle $P \to X$ lifts to $Spin(4)$ bundle $\tilde{P} \to X$ (i.e. when $w_2(X) = 0$), corresponding to the obvious projections $r_\pm : Spin(4) \to SU(2)$, $r_\pm(q_+, q_-) = q_\pm$ we get a pair of $SU(2)$ bundles:

\[
V^\pm = P \times_{r_\pm} \mathbb{C}^2
\]

Clearly since $x \mapsto q_\pm x \lambda^{-1} = q_\pm x \,(\lambda^2)^{-1/2}$ in this case we have:

\[
W^\pm = V^\pm \otimes L^{-1/2}
\]

### 13.2 Action of $\Lambda^\pm(X)$ on $W^\pm$

From the definition of $Spin^c(4)$ structure above we can identify ($\mathbb{H}$ denotes quaternions):

\[
T^* (X) = P \times \mathbb{H}/(p, v) \sim (\tilde{p}, q_+, q_-), \quad \text{where} \quad \tilde{p} = p[\, q_+, q_-, \lambda]
\]

We define left actions (Clifford multiplications), which is well defined by

\[
\begin{align*}
    T^*(X) \otimes W^+ &\to W^-, \quad \text{by} \quad [\, p, v \,] \otimes [\, p, x \,] \mapsto [\, p, -\bar{v}x \,] \\
    T^*(X) \otimes W^- &\to W^+, \quad \text{by} \quad [\, p, v \,] \otimes [\, p, x \,] \mapsto [\, p, vx \,]
\end{align*}
\]

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From identifications, we can check the well definedness of these actions, e.g.:

\[(p,v) \otimes (p,x) \sim (\tilde{p}, q, x\lambda^{-1}) \rightarrow (\tilde{p}, q, (-\bar{v}x)\lambda^{-1}) \sim (p, -\bar{v}x)\]

Alternatively we can describe this actions by using only complex numbers with the aid of the following obvious fact:

**Lemma 13.2.** Let \(P \rightarrow X\) be a principal \(G\) bundle and \(E_{\rho_i} \rightarrow X\) be vector bundles associated to representations \(\rho_i : G \rightarrow \text{Aut}(V_i), i = 1, 2, 3\). Then a map \(\lambda : V_1 \times V_2 \rightarrow V_3\) induces an action \(E_{\rho_1} \times E_{\rho_2} \rightarrow E_{\rho_3}\) if \(\lambda(\rho_1(g)v_1, \rho_2(g)v_2) = \rho_3(g)\lambda(v_1, v_2)\)

By dimension reason complexification of these representation give

\[\rho : T^*(X)_C \rightarrow Hom(W^*, W^*) \cong W^* \otimes W^*\]

We can put them together as a single representation (which we still call \(\rho\))

\[\rho : T^*(X) \rightarrow Hom(W^* \oplus W^-) , \text{ by } v \mapsto \rho(v) = \begin{pmatrix} 0 & v \\ -\bar{v} & 0 \end{pmatrix}\]

We have \(\rho(v) \circ \rho(v) = -|v|^2 I\). By universal property of the Clifford algebra this representation extends to the Clifford algebra \(C(X) = \Lambda^*(X)\) (exterior algebra)

\[\Lambda^*(X)\]

\[\downarrow \quad \eta\]

\[T^*(X) \rightarrow Hom(W^* \oplus W^-)\]

One can construct this extension without the aid of the universal property of the Clifford algebra, for example since

\[\Lambda^2(X) = \left\{ v_1 \wedge v_2 = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \mid v_1, v_2 \in T^*(X) \right\}\]

The action of \(T^*(X)\) on \(W^*\) determines the action of \(\Lambda^2(X) = \Lambda^+(X) \otimes \Lambda^-(X)\), and since \(2Im(v_2\bar{v}_1) = -v_1\bar{v}_2 + v_2\bar{v}_1\) we have the action \(\rho\) with property:

\[\Lambda^+(X) \otimes W^+ \rightarrow W^+ \text{ to be } [p, v_1 \wedge v_2] \otimes [p, x] \rightarrow [p, Im(v_2\bar{v}_1)x]\]

\[\rho : \Lambda^+ \rightarrow Hom(W^*, W^*)\]
\[ \rho(v_1 \wedge v_2) = \frac{1}{2} [ \rho(v_1), \rho(v_2) ] \]  

(13.3)

Let us write the local descriptions of these representations: We first choose a local orthonormal basis of covectors \( \{ e^1, e^2, e^3, e^4 \} \) for \( T^*(X) \), then we can take\[
\{ f_1 = \frac{1}{2} (e^1 \wedge e^2 \pm e^3 \wedge e^4), f_2 = \frac{1}{2} (e^1 \wedge e^3 \pm e^4 \wedge e^2), f_3 = \frac{1}{2} (e^1 \wedge e^4 \pm e^2 \wedge e^3) \}
\]
to be a basis for \( \Lambda^*(X) \). After the local identification \( T^*(X) = \mathbb{H} \) we can take \( e^1 = 1, e^2 = i, e^3 = j, e^4 = k \). Let us identify \( W^\pm = \mathbb{C}^2 = \{ z + jw \mid z, w \in \mathbb{C} \} \), then the multiplication by \( 1, i, j, k \) (action on \( \mathbb{C}^2 \) as multiplication on left) induce the representations \( \rho(e^i), i = 1, 2, 3, 4 \). From this we see that \( \Lambda^+(X) \) acts trivially on \( W^- \); and the basis \( f_1, f_2, f_3 \) of \( \Lambda^+(X) \) acts on \( W^+ \) as multiplication by \( i, j, k \), respectively (these are called Pauli matrices).

\[
\begin{align*}
\rho(e^1) &= \begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & 1
\end{pmatrix} & \rho(e^2) &= \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix} \\
\rho(e^3) &= \begin{pmatrix}
0 & -1 \\
1 & 0 \\
0 & 1
\end{pmatrix} & \rho(e^4) &= \begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix} \\
\rho(f_1) &= \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix} & \rho(f_2) &= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} & \rho(f_3) &= \begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix}
\end{align*}
\]

In particular we get an isomorphism \( \rho : \Lambda^+(X) \longrightarrow su(W^+) \) (traceless skew adjoint endomorphism of \( W^+ \)). After complexifying this isomorphism extends to an isomorphism \( \rho : \Lambda^+(X)_\mathbb{C} \cong sl(W^+) \) (traceless endomorphism of \( W^+ \)). We will also definde a map \( \sigma : \)

\[
\begin{align*}
\Lambda^+(X) &\underset{\cong}{\longrightarrow} su(W^+) \\
\cap &\quad \cap \\
W^+ &\underset{\sigma}{\longrightarrow} \Lambda^+(X)_\mathbb{C} \underset{\rho}{\longrightarrow} sl(W^+)
\end{align*}
\]
Remark 13.3. \( \text{Hom}(W^+, W^+) \cong W^+ \otimes (W^+)^* \), and the dual space \((W^+)^* \) can be identified with \( W^+ \) (\( \cong W^+ \) with scalar multiplication \( c.v = \overline{c}v \)) by the pairing \( W^+ \otimes W^+ \to \mathbb{C} \) given by \( z \otimes w \to z \overline{w} \). Usually \( \text{sl}(W^+) \) is denoted by \( (W^+ \otimes \overline{W}^+) \) \( 0 \) and the trace map gives the identification:

\[
W^+ \otimes \overline{W}^+ = (W^+ \otimes \overline{W}^+) \_0 \oplus \mathbb{C}
\]

Now define \( \sigma : W^+ \to \Lambda^+(X) \) by \( \left[ p, x \right] \mapsto \left[ p, \frac{1}{2} (x + \overline{x}) \right] \) (multiplications in \( \mathbb{H} \)). By local identification as above \( W^+ = \mathbb{C}^2 \) and \( \Lambda^+(X) = \mathbb{R} \oplus \mathbb{C} \), we see \( \sigma \) corresponds to

\[
\sigma(z, w) = i \left( \frac{|z|^2 - |w|^2}{2} \right) - k \text{Re}(z \overline{w}) + j \text{Im}(z \overline{w})
\]  \( (13.4) \)

Here \( i, j, k \) correspond to the basis elements \( f_1, f_2, f_3 \) of \( \Lambda^+(X) \). By applying the isomorphism \( \rho \) we make the composition \( \sigma := \rho \circ \sigma \) take values in \( \text{sl}(W^+) \)

\[
\sigma(z, w) = \rho \left[ \frac{|z|^2 - |w|^2}{2} f_1 + \text{Im}(z \overline{w}) f_2 - \text{Re}(z \overline{w}) f_3 \right]
\]

By plugging in the values of the Pauli matrices \( \rho(f_1), \rho(f_2), \rho(f_3) \) we calculate

\[
(z, w) \mapsto \sigma(z, w) = i \left( \frac{|z|^2 - |w|^2}{2} \right) z \overline{w} \left( \frac{|w|^2 - |z|^2}{2} \right)
\]  \( (13.5) \)

Put another way, \( \sigma \) is the projection of the diagonal elements of \( W^+ \otimes \overline{W}^+ \) onto the subspace \( (W^+ \otimes \overline{W}^+) \_0 \), that is if we write \( x = z + jw \) we get \( \sigma(z, w) = i(z \overline{w}) \_0 \). We get

\[
|\rho(u)| = \sqrt{2} |u| \quad (13.6)
\]

\[
|\sigma(\psi)| = \frac{1}{2} |\psi|^2 \quad (13.7)
\]

\[
\langle \sigma(\psi), \psi \rangle = \frac{i}{2} |\psi|^4 \quad (13.8)
\]

\[
\langle \rho(u) \psi, \psi \rangle = 2i \langle \rho(u) , \sigma(\psi) \rangle \quad (13.9)
\]

Here the norm in \( su(2) \) is induced by the inner product \( \langle A, B \rangle = -\text{trace}(AB) \).
By renaming $\sigma(\psi, \psi) := \sigma(\psi)$ we can extend the definition of $\sigma$ to $W^+ \otimes \bar{W}^+$ by

$$\langle \rho(u) \psi, \varphi \rangle = i \langle \rho(u), \sigma(\psi, \varphi) \rangle$$

$$W^+ \xrightarrow{\sigma} su(W^+) \quad \cap \quad \cap$$

$$W^+ \otimes \bar{W}^+ \xrightarrow{\sigma} sl(W^+) = (W^+ \otimes \bar{W}^+)_0$$

**Remark 13.4.** A $Spin^c(4)$ structure can also be defined as a pair of $U(2)$ bundles:

$$W^+ \to X \text{ with } det(W^+) = det(W^-) \to X \text{ (a complex line bundle),}$$

and an action $c_{\pm} : T^*(X) \to Hom(W^+, W^\mp)$ with $c_{\pm}(v)c_{\pm}(v) = -|v|^2 I$

The first definition clearly implies this, and conversely we can obtain the first definition by letting the principal $Spin^c(4)$ bundle to be:

$$P = \{ (p_+, p_-) \in P(W^+) \times P(W^-) \mid det(p_+) = det(p_-) \}$$

Clearly, $Spin^c(4) = \{ (A, B) \in U_2 \times U_2 \mid det(A) = det(B) \}$ acts on $P$ freely. This definition generalizes and gives way to the following definition:

**Definition 13.5.** A Dirac bundle $W \to X$ is a Riemannian vector bundle with an action $\rho : T^*(X) \to Hom(W, W)$ satisfying $\rho(v) \circ \rho(v) = -|v|^2 I$. $W$ is also equipped with a connection $D$ satisfying:

$$\langle \rho(v)x, \rho(v)y \rangle = \langle x, y \rangle$$

$$D_Y(\rho(v)s) = \rho(\nabla_X v)s + \rho(v)D_Y(s)$$

where $\nabla$ is the Levi-Civita connection on $T^*(X)$, and $Y$ is a vector field on $X$.

An example of a Dirac bundle is $W = W^+ \oplus W^- \to X$ and $D = d + d^*$ with $W^+ = \oplus \Lambda^{2k}(X)$ and $W^- = \oplus \Lambda^{2k+1}(X)$ where $\rho(v) = v \wedge + v \cdot$ (exterior + interior product with $v$). In this case $\rho : W^+ \to W^\mp$. In the next section we will discuss the natural connections $D$ for $Spin^c$ structures $W^\pm$. 

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13.3 Dirac Operator

Let $\mathcal{A}(L)$ denote the space of connections on a $U(1)$ bundle $L \to X$ over a smooth 4-manifold. Note that after choosing a base connection $a_0$ we can identify $\mathcal{A}(L) = \{a_0\} + \Omega^1(X)$. Any $A \in \mathcal{A}(L)$ and the Levi-Civita connection $A_0$ on the tangent bundle coming from Riemannian metric of $X$ defines a product connection on $P_{SO(4)} \times P_{S^1}$. Since $Spin^c(4)$ is the two fold covering of $SO(4) \times S^1$, they have the same Lie algebras $spin^c(4) = so(4) \oplus i \mathbb{R}$. Hence we get a connection $\tilde{A}$ on the $Spin^c(4)$ principle bundle $P \to X$. In particular the connection $\tilde{A}$ defines connections to all the associated bundles of $P$, giving back $A$, $A_0$ on $L$, $T(X)$ respectively, and two new connections $A^\pm$ on bundles $W^\pm$. We denote the corresponding covariant derivative by

$$\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$$

Locally, by choosing orthonormal tangent vector field $e = \{e_i\}_{i=1}^4$ and the dual basis of 1-forms $\{e^i\}_{i=1}^4$ in a neighborhood $U$ of a point $x \in X$ we can write

$$\nabla_A = \sum e^i \otimes \nabla_{e_i}$$

where $\nabla_{e_i} : \Gamma(W^+) \to \Gamma(W^+)$ is the covariant derivative $\nabla_A$ along $e_i$. Composing this map with the map $\Gamma(T^*X \otimes W^+) \to \Gamma(W^-)$ induced by the Clifford multiplication (Section 13.2) gives the Dirac operator $\Psi_A : \Gamma(W^+) \to \Gamma(W^-)$, i.e.

$$\Psi_A = \sum \rho(e^i) \nabla_{e_i}$$

Locally $W^\pm = V^\pm \otimes L^{-1/2}$, and the connection $A$ gives the untwisted Dirac operator

$$\bar{\partial} : \Gamma(V^+) \to \Gamma(V^-)$$

Notice that as in $W^\pm$, forms $\Lambda^*(X)$ act on $V^\pm$. Now let $\omega = (\omega_{ij})$ be the Levi-Civita connection 1-form, which is an $so(4) = su(2) \oplus su(2)$ valued equivariant 1-form on $P_{SO(4)}(X)$. Let $e^*(\omega) := \bar{\omega} = (\bar{\omega}_{ij}) \oplus (\bar{\omega}'_{ij})$ be the pull-back 1-form on $U$. Since $P_{SO(4)}(U) = P_{Spin^c(4)}(U)$ the orthonormal basis $e \in P_{SO(4)}(U)$ determines an orthonormal basis $\sigma = \{\sigma^k\} \in P_{SU_2}(U)$, then (e.g. [LM] pp.110)

$$\bar{\partial}(\sigma^k) = \sum_{i<j} \rho(\bar{\omega}_{ij}) \sigma^k$$

A metric on $T(X)$ give metrics on $W^\pm$ and $T^*(X) \otimes W^\pm$, hence we can define the adjoint $\nabla_A^* : \Gamma(T^*X \otimes W^-) \to \Gamma(W^+)$.

$$\nabla_A^* = -\sum e^i \otimes \nabla_{e_i}$$

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Similarly we define $\mathcal{P}_A : \Gamma ( W^- ) \rightarrow \Gamma ( W^+ )$ which turns out to be the adjoint of the previous $\mathcal{P}_A$ and makes the following commute (vertical maps are Clifford multiplications):

$$
\begin{array}{cccc}
\Gamma ( W^+ ) & \mathcal{P}_A & \Gamma ( W^- ) & \mathcal{P}_A \\
\downarrow & & \downarrow & \\
\Gamma ( W^+ ) & \mathcal{P}_A & \Gamma ( W^- ) & \mathcal{P}_A \\
\end{array}
$$

Let $\mathbf{F}_A = i F_A \in \Lambda^2 ( X ; i \mathbb{R} )$ be the curvature of the connection $A$ on $L$ and $F^+_A \in \Lambda^+ ( X )$ be the self dual part of $F_A$ and $s$ be the scalar curvature of $X$. Weitzenbock formula says:

$$
\mathcal{P}_A^2 ( \psi ) = \nabla_A^* \nabla_A \psi + \frac{s}{4} \psi - \frac{i}{2} \rho ( F_A^+ ) \psi
$$

(13.10)

To check this we can assume $\nabla_{e_i} ( e^j ) = 0$ at the point $x$

$$
\mathcal{P}_A^2 \psi = \sum \rho ( e^i ) . \nabla_{e_i} \left[ \sum \rho ( e^j ) . \nabla_{e_j} \psi \right] = \nabla_A^* \nabla_A \psi + \sum_{i \neq j} \rho ( e^i ) \rho ( e^j ) ( \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} ) \psi
$$

$$
= \nabla_A^* \nabla_A \psi + \frac{1}{2} \sum_{i \neq j} \rho ( e^i ) \rho ( e^j ) \Omega^A_{ij} \psi
$$

$\Omega^A_{ij} = R_{ij} - \frac{i}{2} F_{ij}$ is curvature on $V^+ \otimes L^{-1/2}$ (endomorphism of $W^+$ valued 2-form). $R_{ij}$ is the Riemannian curvature, and the imaginary valued 2-form $\mathbf{F}_A = \sum i F_{ij} e^i \wedge e^j$ is the curvature of $A$ of the complex line bundle $L$. So if $\psi = \sigma \otimes \alpha \in \Gamma ( V^+ \otimes L^{-1/2} )$, then

$$
\frac{1}{2} \sum_{i \neq j} \rho ( e^i ) \rho ( e^j ) \Omega^A_{ij} ( \sigma \otimes \alpha ) = \frac{1}{2} ( \sum \rho ( e^i ) \rho ( e^j ) R_{ij} \sigma ) \otimes \alpha
$$

$$
- \frac{i}{4} \sum \rho ( e^i ) \rho ( e^j ) \sigma \otimes ( F_{ij} \alpha )
$$

$$
= \frac{1}{8} \sum \rho ( e^i ) \rho ( e^j ) \rho ( e^k ) \rho ( e^l ) R_{ijkl} ( \psi )
$$

$$
- \frac{i}{2} \rho ( \sum_{i \neq j} F_{ij} e^i \wedge e^j ) ( \psi )
$$

It is a standard calculation that the first term is $s/4$ (e.g. [LM] pp. 161, 398), and since $\Lambda^- ( X )$ act as zero on $W^+$ the second term can be replaced by

$$
- \frac{i}{2} \rho ( \sum_{i \neq j} F^+_{ij} e^i \wedge e^j ) \psi = - \frac{i}{2} \rho ( F^+_A ) \psi = - \frac{1}{2} \rho ( F^+_A ) \psi
$$
13.4 A Special Calculation

In coming sections we need some a special case 13.10. For this, suppose

\[ V^+ = L^{1/2} \otimes L^{-1/2} \]

where \( L^{1/2} \to X \) is some complex line bundle with \( L^{1/2} \otimes L^{1/2} = L \). Hence \( W^+ = (L^{1/2} \otimes L^{-1/2}) \otimes L^{-1/2} = L^{-1} \otimes \mathbb{C} \). In this case there is a unique connection \( \frac{1}{2}A_0 \) in \( L^{-1/2} \to X \) such that the induced Dirac operator \( D_{A_0} \) on \( W^+ \) restricted to the trivial summand \( \mathbb{C} \to X \) is the exterior derivative \( d \). This is because for \( \sigma_+ \in \Gamma(L^{1/2}) \), the following determines \( \nabla_{A_0}(\sigma_-) : \)

\[ d(\sigma_+ \otimes \sigma_-) = \nabla_{A_0}[(\sigma_+ + 0) \otimes \sigma_-] = \partial(\sigma_+ + 0) \otimes \sigma_- + (\sigma_+ + 0) \otimes \nabla_{A_0}(\sigma_-) \]

The following is essentially the Leibniz formula for Dirac operator, it is a version of the Weitzenbock formula 13.10

**Proposition 13.6.** Let \( A, A_0 \in A(L^{-1}) \) and \( A = A_0 + ia \). Let \( \nabla_a = d + ia \) be the corresponding covariant derivative of the trivial bundle \( \mathbb{C} \to X \) and \( \alpha : X \to \mathbb{C} \). Let \( u_0 \) be a section of the bundle \( W^+ = L^{-1} \otimes \mathbb{C} \) with a constant \( \mathbb{C} \) component and \( \nabla_{A_0}(u_0) = 0 \) then:

\[ \mathcal{D}_A^2(\alpha u_0) = (\nabla_a^2 \alpha)u_0 + \frac{1}{2}\rho(F_0)(\alpha, u_0 - 2 < \nabla_a \alpha, \nabla_{A_0}(u_0) >) \quad (13.11) \]

**Proof:** By abusing notation \( \nabla_A = \nabla^A \) or \( \nabla_a = \nabla^a \) for the sake of not cluttering formulas, and abbreviating \( \nabla_{e_j} = \nabla_j \) and leaving out summation signs for repeated indices (Einstein convention) we calculate:

\[
\nabla^A(\alpha u_0) = (d\alpha)u_0 + \alpha \nabla^A(u_0) \\
= (d\alpha)u_0 + \alpha(\nabla_{A_0}(u_0) + i a u_0) \\
= (d\alpha + i a \alpha)u_0 + \alpha \nabla^{A_0}(u_0) \\
= \nabla^a(\alpha)u_0 + \alpha \nabla^{A_0}(u_0) \Rightarrow \\
\mathcal{D}_A(\alpha u_0) = \rho(e^j) \nabla^a_j(\alpha) u_0 + \alpha \mathcal{D}_{A_0}(u_0) = \rho(e^j) \nabla^a_j(\alpha) u_0 \\
\]

By abbreviating \( \mu = \nabla^a_j(\alpha) \) and recalling \( d\mu = e^k \otimes \nabla_k(\mu) \) we calculate:

\[
\nabla^A(\rho(e^j) \mu u_0) = e^k \otimes \rho(e^j) \nabla_k(\mu) u_0 + e^k \otimes \rho(e^j) \mu (\nabla^{A_0}(u_0) + i a_k u_0) \\
= e^k \otimes \rho(e^j) \nabla^a_k(\mu) u_0 + e^k \otimes \rho(e^j) \mu \nabla^{A_0}_k(u_0) \Rightarrow \\
\]
Now by plugging in the value \( \mu = 0 \) we get
\[
\nabla^a_\mu u_0 = -\nabla^a_j(\mu) u_0 + \frac{1}{2} \sum_{k<j}^\rho(e^k)\rho(e^j)(\nabla^a_k(\mu) - \nabla^a_j(\mu))u_0
\]
\[
-\mu \nabla^A_0 (u_0) = \mu \rho(e^j)\sum_{k<j}^\rho(e^k)\nabla^A_0(u_0)
\]
(13.12)

But since \( 0 = \nabla^A_0(u_0) = \sum_{k<J}^\rho(e^k)\nabla^A_0(u_0) \) the last term of (13.12) is \(-\mu \nabla^A_0(u_0)\).

Now by plugging in the value \( \mu = \nabla^a_\alpha \) in (13.12) and summing over \( j \) we get
\[
\nabla^2_\alpha u_0 = -\nabla^a_j(\nabla^a_j(\alpha)) u_0 + \frac{1}{2} \rho(\sum_{k<j}^F_k \epsilon^k \wedge \epsilon^j) \alpha. u_0 - 2 \sum_{j<k}^\nabla^a_j(\alpha)\nabla^A_0(u_0) \quad \square
\]

Remark 13.7. Since \( u_0 \) has constant \( \mathbb{C} \) component, and \( \nabla^A_0 \) restricts to the usual \( d \) on the \( \mathbb{C} \) component, hence the term \( \langle \nabla^a \alpha, \nabla^A_0(u_0) \rangle \) lies entirely in \( L \) component of \( W^+ \)

### 13.5 Seiberg-Witten invariants

Let \( X \) be a closed oriented Riemannian manifold, and \( L \to X \) a characteristic complex line bundle. Seiberg-Witten equations are defined for \( (A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+) \),

\[
\nabla A (\psi) = 0
\]
(13.13)

\[
F^*_A = \sigma(\psi)
\]
(13.14)

The curvature of \( L \) is an imaginary valued 2-form \( F_A = iF_A \). Sometimes (13.16) is written as \( \rho(F^*_A) = \sigma(\psi) \), as an equation in \( sl(W^+) \).

**Definition 13.8.** Gauge group \( \mathcal{G}(L) = Map(X, S^1) \) acts on \( \hat{\mathcal{B}}(L) = \mathcal{A}(L) \times \Gamma(W^+) \) as follows: for \( s = e^{i\theta} \in \mathcal{G}(L) \)

\[
s^* (A, \psi) = (s^* A, s^{-1} \psi) = (A + s^{-1} ds, s^{-1} \psi) = (A + i df, s^{-1} \psi)
\]
(13.15)

From this, and by using \( ds^{-1} = -idf s^{-1} \) we calculate

\[
\nabla A (s^{-1} \psi) = \nabla A (s^{-1} \psi) + \rho(idf) (s^{-1} \psi)
\]
\[
= \rho(ds^{-1}) \psi + s^{-1} \nabla A (\psi) + \rho(idf) \rho(s^{-1} \psi)
\]
\[
= s^{-1} \nabla A (\psi)
\]
\[
F^*_{s^* A} = s^{-1} F^*_A s = F^*_A = \sigma(\psi) = \sigma(s^{-1} \psi)
\]
Hence the solution set $\tilde{\mathcal{M}}(L) \subset \tilde{B}(L)$ of Seiberg-Witten equations is preserved by the action $(A, \psi) \mapsto s^*(A, \psi)$ of $\mathcal{G}(L)$ on $\tilde{\mathcal{M}}(L)$. Define

$$\mathcal{M}(L) = \tilde{\mathcal{M}}(L)/\mathcal{G}(L) \subset \mathcal{B}(L) = \tilde{\mathcal{B}}(L)/\mathcal{G}(L)$$

We call a solution $(A, \psi)$ of (13.11) and (13.12) an irreducible solution if $\psi \neq 0$. $\mathcal{G}(L)$ acts on the subset $\tilde{\mathcal{M}}^*(L)$ of the irreducible solutions freely, we denote

$$\mathcal{M}^*(L) = \tilde{\mathcal{M}}^*(L)/\mathcal{G}(L)$$

Any solution $(A, \psi)$ of Seiberg-Witten equations satisfies the $C^0$ bound

$$|\psi|^2 \leq \max(0, -s)$$

where $s$ is the scalar curvature function of $X$. This follows by plugging (13.15) in the Weitzenbock formula 13.10 we get

$$\mathcal{D}^2_A(\psi) = \nabla_A^* \nabla_A \psi + \frac{s}{4} \psi - \frac{i}{2} \sigma(\psi) \psi$$

Then at the points of $X^4$ where $|\psi|^2$ is maximum we calculate ($d^* = -\ast d*$):

$$0 \leq \frac{1}{2} \Delta |\psi|^2 = \frac{1}{2} d^* d < \psi, \psi > = \frac{1}{2} d^* ( < \nabla_A \psi, \psi > + < \psi, \nabla_A \psi > )$$

$$= \frac{1}{2} d^* ( \sqrt{ < \psi, \nabla_A \psi > + < \psi, \nabla_A \psi > } ) = d^* < \psi, \nabla_A \psi >_{\mathbb{R}}$$

$$= < \psi, \nabla_A^* \nabla_A \psi > - |\nabla_A \psi|^2 \leq < \psi, \nabla_A \nabla_A \psi >$$

$$\leq - \frac{s}{4} |\psi|^2 - \frac{1}{4} |\psi|^4$$

The last step follows from (13.17) and 13.8.

**Proposition 13.9.** $\mathcal{M}(L)$ is compact

Proof: Given a sequence $[A_n, \psi_n] \in \mathcal{M}(L)$ we claim that there is a convergent subsequence (which we will denote by the same index), i.e. there is a sequence of gauge transformations $g_n \in \mathcal{G}(L)$ such that $g_n^*(A_n, \psi_n)$ converges in $C^\infty$. Let $A_0$ be a base connection. By Hodge theory of the elliptic complex:

$$\Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^*} \Omega^2_+(X)$$
where $\mathcal{H}$ are the harmonic 1-forms. After applying gauge transformation $g_n$ we can assume that $b_n = 0$, i.e. if $b_n = i df_n$ we can let $g_n = e^{if}$. Also
\[
h_n \in \mathcal{H} = H^1(X; \mathbb{R}) \text{ and a component of } \mathcal{G}(L) \text{ is } H^1(X; \mathbb{Z})
\]
Hence after a gauge transformation we can assume $h_n \in H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$ so $h_n$ has convergent subsequence. Consider the first order elliptic operator:
\[
D = d^* \otimes d^*: \Omega^1(X)_{L^p_k} \rightarrow \Omega^0(X)_{L^p_{k-1}} \oplus \Omega^2(X)_{L^p_{k-1}}
\]
The kernel of $D$ consists of harmonic 1-forms, hence by Poincare inequality if $a$ is a 1-form orthogonal to the harmonic forms, then for some constant $C$
\[
||a||_{L^p_{k}} \leq C ||D(a)||_{L^p_{k-1}}
\]
Now $a_n = (d^*)^n \alpha_n$ implies $d^*(a_n) = 0$. Since $a_n$ is orthogonal to harmonic forms, and by calling $A_n = A_0 + a_n$ we see :
\[
||a_n||_{L^p_{k}} \leq C ||D(a_n)||_{L^p} \leq C ||d^*a_n||_{L^p} = C ||F^+_n - F^+_0||_{L^p}
\]
Here we use $C$ for a generic constant. By 13.14, 13.7, and along with the bound on $||\psi_n||$ given by 13.16 there is a $C$ depending only on the scalar curvature $s$ with
\[
||a_n||_{L^p_{k}} \leq C \tag{13.18}
\]
By iterating this process we get $||a_n||_{L^p_{k}} \leq C$ for all $k$, hence $||a_n||_{\infty} \leq C$. From the elliptic estimate and $\mathcal{P}_{A_n}(\psi_n) = 0$ : 
\[
||\psi_n||_{L^p_{k}} \leq C( ||\mathcal{P}_{A_0}\psi_n||_{L^p} + ||\psi_n||_{L^p}) = C( ||a_n\psi_n||_{L^p} + ||\psi_n||_{L^p})
\]
\[
||\psi_n||_{L^p_{k}} \leq C( ||a_n||_{\infty} ||\psi_n||_{L^p} + ||\psi_n||_{L^p}) \leq C \tag{13.19}
\]
By repeating this (boothstrapping) process we get $||\psi_n||_{L^p_{k}} \leq C$, for all $k$, where $C$ depends only on the scalar curvature $s$ and $A_0$. By Rallich theorem we get convergent subsequence of $(a_n, \psi_n)$ in $L^p_{k-1}$ norm for all $k$. So we get this convergence to be $C^\infty$ convergence.

It is not clear that the solution set of Seiberg-Witten equations is a smooth manifold. However we can perturb the Seiberg-Witten equations 13.13, 13.14 by any self dual 2-form $\delta \in \Omega^+(X)$, in a gauge invariant way, to obtain a new set of equations whose solutions set is a smooth manifold.
Definition 13.10. The parametrized Seiberg-Witten equations with \( \delta \in \Omega^2_+(X) \) are:

\[
\begin{align*}
\mathcal{P}_A(\psi) &= 0 \\
F_A^+ &= \sigma(\psi) + i \delta
\end{align*}
\]  
(13.20)

Denote this solution space by \( \tilde{M}_\delta := \tilde{M}_\delta(L) \), and parametrized solution space by

\[
\tilde{M} = \bigcup_{\delta \in \tilde{M}} \tilde{M}_\delta(L) \times \{ \delta \} \subset A(L) \times \Gamma(W^+) \times \Omega^+(X)
\]

\[
M_\delta = \tilde{M}_\delta(L) / G(L) \subset \mathcal{M} = \tilde{M} / G(L)
\]

(13.22)

Let \( \tilde{M}_\delta^* \subset \tilde{M}^* \) be the corresponding irreducible solutions (i.e. \( \psi \neq 0 \) solutions), and also let \( \mathcal{M}_\delta^* \subset \mathcal{M}^* \) be their quotients by Gauge group. We will soon prove that when \( b_2(X) > 1 \) for generic choice of \( \delta \), \( \mathcal{M}_\delta = \mathcal{M}_\delta^* \). But first we will show that for a generic choice of \( \delta \), the set \( \mathcal{M}_\delta^* \) is a closed smooth manifold.

Proposition 13.11. \( \tilde{M} \) is a smooth manifold. The projection \( \pi : \tilde{M} \rightarrow \Omega^+(X) \) is a proper surjection of Fredholm index: (here we identify \( c_1(L)^2 < c_1(L), [X] > \)):

\[
d(L) = \frac{1}{4} [ c_1(L)^2 - (2\chi + 3\sigma) ]
\]

where \( \chi \) and \( \sigma \) are Euler characteristic and the signature of \( X \).

Proof: \( \tilde{M} = \mathcal{P}^{-1}(0) \), where \( \mathcal{P} \) is the map \( (A, \psi, \delta) \mapsto (F_A^+ - \sigma(\psi) - i \delta, \mathcal{P}_A(\psi)) \).

Linearization of \( P \) at \( (A_0, \psi_0, \delta_0) \), for \( \psi_0 \neq 0 \) is given by \( (\sigma(\psi) \) is a quadratic function):

\[
P : \Omega^1(X) \oplus \Gamma(W^+) \oplus \Omega^+(X) \rightarrow \Omega^+(X) \oplus \Gamma(W^-)
\]

\[
P(a, \psi, \epsilon) = (d^+ a - 2\sigma(\psi, \psi_0) - i \epsilon, \mathcal{P}_{A_0} \psi + \rho(a) \psi_0)
\]

To see that this is onto we pick \( (\kappa, \theta) \in \Omega^+(X) \oplus \Gamma(W^-) \), by varying \( \epsilon \) we can see that \( (\kappa, 0) \) is in the image of \( P \). To see \( (0, \theta) \) is in the image of \( P \), we will show that if it is orthogonal to \( \text{image}(P) \) then it has to be \( (0, 0) \); so assume

\[
< \mathcal{P}_{A_0} \psi, \theta > + \rho(a) \psi_0, \theta > = 0
\]

for all \( a \) and \( \psi \). By choosing \( a = 0 \) and by the self adjointness of Dirac operator we get \( \mathcal{P}_{A_0} \theta = 0 \). Then by choosing \( \psi = 0 \) and \( a \) to be a “bump” form supported in \( U \), we see that \( < \rho(a) \psi_0, \theta > = 0 \) for all such 1-forms \( a \). This implies \( \theta = 0 \) pointwise on \( U \). Here we used \( T^* \)(\( X \)) \( \subset W^+ \otimes W^- \). Since we can get \( \theta = 0 \) at any sequence of points in \( U \), by analytic continuation \( \theta = 0 \).
By implicit function theorem $\tilde{M}$ is a smooth manifold, and by the Sard's theorem $\tilde{M}_\delta$ are smooth manifolds, for generic choice of $\delta$'s. Therefore their free quotients $M = \tilde{M} / \mathcal{G}(L)$ and $M_\delta = \tilde{M}_\delta / \mathcal{G}(L)$ are smooth manifolds.

Since the derivative of the gauge group action map $s \mapsto s^*(A_0)$

$$\mathcal{G}(L) = \text{Map}(X, S^1) \to \mathcal{A}(L)$$

is just $d : \Omega^0(X) \to \Omega^1(X)$, the tangent space of $M_\delta = \tilde{M}_\delta / \mathcal{G}(L)$ is obtained by intersecting the tangent space of $\tilde{M}_\delta$ with $\text{im}(d) = \text{kernel}(d^*)$, which is called “gauge fixing”. So after taking gauge fixing into account, the dimension of $M_\delta(L)$ is given by the index of $P + d^*$ (c.f. [DK]). $P + d^*$ is the compact perturbation of

$$S : \Omega^1(X) \oplus \Gamma(W^+) \longrightarrow [\Omega^0(X) \oplus \Omega^2_+(X)] \oplus \Gamma(W^-) \quad (13.23)$$

$$S = \begin{pmatrix} d^* \oplus d^* & 0 \\ 0 & \Phi_{A_0} \end{pmatrix}$$

By Atiyah-Singer index theorem

$$\text{dim } M_\delta(L) = \text{ind}(S) = \text{index}(d^* \oplus d^*) + \text{index}_{\mathbb{R}} \Phi_{A_0}$$

$$= \frac{1}{2}(\chi + \sigma) + \frac{1}{4}(c_1(L)^2 - \sigma)$$

$$= \frac{1}{4} \left[ c_1(L)^2 - (2\chi + 3\sigma) \right] \quad (13.24)$$

where $b^+$ is the dimension of positive definite part $H^2_+ \subset H^2(X; \mathbb{Z})$. Clearly $\text{ind}(S)$ is even, and when $b^+$ is odd, this expression is even, so for the characteristic line bundle $L$ we have $c_1(L)^2 = \sigma \mod 8$. \hfill \Box

Now assume that $H^1(X) = 0$, then $\mathcal{G}(L) = K(\mathbb{Z}, 1)$. Therefore being a free quotient of a contractible space by $\mathcal{G}(L)$ we have

$$\mathcal{B}^*(L) = K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$$

By (13.23) the orientation of $H^2_+$ gives an orientation to $M_\delta(L)$. Now by Proposition 13.11, when $b^+$ is odd $M_\delta(L) \subset \mathcal{B}^*(L)$ is an even $2d = 2d(L)$ dimensional smooth closed oriented submanifold, then we can define Seiberg-Witten invariants as:
\[ SW_X(L) = (\mathcal{M}_\delta(L), [\mathbb{CP}^d]) \]  

(13.25)

Even though \( \mathcal{M}_\delta(L) \) depends on the perturbation term \( \delta \) and on the metric, when \( b^+_2(X) \geq 2 \) the invariant \( SW_L(X) \) is independent of these choices. To see this this let

\[ \pi_+: \Omega^+_2(X) \to \mathcal{H}^2_+ \] be projection to self dual harmonic forms

**Lemma 13.12.** The set of \( \delta \in \Omega^+_2(X) \) such that \( \mathcal{M}_\delta(L) \) contains a reducible solution (\( \psi = 0 \)) is given by \( Z = \{ \delta \in \Omega^+_2(X) \mid \pi_+(\delta) = -2\pi c_1(L)^+ \} \). Furthermore, \( Z \) is a codimension \( b^+_2(X) \) affine subspace of \( \Omega^+_2(X) \), which we can identify \( Z^\perp = \mathcal{H}^2_+ \).

**Proof.** If \( \mathcal{M}_\delta(L) \) contains a reducible solution, then by (13.22) \( F^+_A - i\delta = 0 \) for some \( A \). But since \( c_1(L) = [\frac{1}{2\pi} F_A] \), we get \( \pi_+(\delta) = -2\pi c_1(L)^+ \), which means \( \delta \in Z \). In particular, for any pair of connections \( A_0 \) and \( A \) the form \( F^+_A - F^+_A \in \text{image} \ (d^+) \) since \( [F_{A_0}] = [F_A] \).

Fix a connection \( A_0 \) with \( F^+_{A_0} - i\delta_0 = 0 \), then any other solution \( F^+_A - i\delta = 0 \) we get \( \delta - \delta_0 = i(F^+_{A_0} - F^+_A) = d^+(\sigma) \) for some \( \sigma \in \Omega^1(X) \). This gives an affine identification \( Z = \text{im}(d^+) \). Now we claim \( \Omega^+_2 = \mathcal{H}^2_+ \oplus \text{im}(d^+) \). Clearly this claim implies the result.

To prove the claim we need to show that \( \text{coker}(d^+) = \mathcal{H}^2_+ \) where \( d^+ : \Omega^1 \to \Omega^2_+ \). If \( \langle d^+\beta, \alpha \rangle = 0 \), \( \forall \beta \Rightarrow \langle \beta, d^*\alpha \rangle = 0 \Rightarrow d^*\alpha = 0 \). Also since \( +\alpha = \alpha \Rightarrow d\alpha = 0 \), so \( \alpha \in \mathcal{H}^2_+ \). \( \square \)

Hence when \( b^+_2(X) \geq 1 \) we can choose a generic \( \delta \in Z^\perp \subset \Omega^2_+ \) to make \( \mathcal{M}_\delta(L) = \mathcal{M}_\delta^+(L) \) smooth, and when \( b^+_2(X) \geq 2 \) any two such choices for \( \delta \) can be connected by a path in \( \delta \in Z^\perp \) which has an affect of connecting the corresponding manifolds \( \mathcal{M}_\delta(L) \) by a cobordism. \( \mathcal{M}_\delta(L) \) also depends on the metric on \( X \), similarly by connecting any two choices of \( \delta \) by a path of metrics will result changing \( \mathcal{M}_\delta(L) \) by a cobordism in \( \mathbb{CP}^\infty \). So if \( b^+_2(X) \geq 2 \) Definition 13.25 does not depend on generic choice of \( (g, \delta) \).

Also by 13.16 if \( X \) has nonnegative scalar curvature then all the solutions are reducible, i.e. \( \psi = 0 \). This implies \( \mathcal{H}^2_+ = 0 \). Hence for \( b^+ \geq 1 \) and for a generic metric \( \mathcal{M} = \emptyset \) which implies \( SW_L(X) = 0 \). It is a conjecture that only 0-dimensional moduli spaces \( \mathcal{M}_\delta(L) \) give nonzero invariants \( SW_L(X) \).

As we saw in Definition 13.1 we can identify \( Spin^c \) structures with characteristic line bundles \( \mathcal{C}(X) \), and hence can assume \( SW_X \) as a function defined on \( \mathcal{C}(X) \)

\[ SW_X : \mathcal{C}(X) = \{ L \in H^2(X; \mathbb{Z}) \mid w_2(TX) = c_1(L) \ (\text{mod} \ 2) \} \to \mathbb{Z} \]  

(13.26)

**Lemma 13.13.** For any closed oriented smooth 4-manifold \( X \) with \( b^+_2(X) \geq 2 \), there are finitely many \( L \in \mathcal{C}(X) \) for which \( SW_X(L) \neq 0 \)
Proof. It is enough to show that in order to have nonempty solution set $\mathcal{M}_\delta$ the integral class $c_1(L)$ must lie in a fixed bounded subset of $\mathcal{H}^2$. We know $c_1(L) = \{ F \}$ where $F = \frac{1}{2\pi} F_A \in \Omega^2(X)$ write $F = F^+ + F^-$, hence $\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$, and by 13.24

$$c_1(L)^2 = \|F^+\|^2 - \|F^-\|^2 \geq 2\chi(X) + 3\sigma(X) \quad (13.27)$$

Also by 13.21 and 13.16 $\|F^+\|^2$ is bounded in terms of scalar curvature of $X$, and hence by 13.27, $\|F^-\|^2$ is bounded as well, so $\|F\|$ is bounded \( \square \).

We can define Seiberg-Witten invariants as a single Laurent polynomial $(t_L^k := t_{kL})$

$$SW_X = \sum_{L \in C(X)} SW_X(L)t_L$$

There is a “connected sum theorem” for Seiberg-Witten invariants: If $X_i, i = 1, 2$ are oriented compact smooth manifolds such that $b^+(X_i) > 0$ and with common boundary, which is a 3-manifold with a positive scalar curvature; then gluing these manifolds together along their boundaries produces a manifold $X = X_1 \sim X_2$ with vanishing Seiberg-Witten invariants (cf [FS3]).

### 13.5.1 Seiberg-Witten for $X$ with $b^+_2(X) = 1$

Let $(X^4, g)$ be a closed Riemannian manifold with $b^+_2(X) = 1$. In this case Seiberg-Witten invariant depends on the perturbation terms $SW_X = SW_{X,g,\delta}$. Let $\omega_\delta$ be the unique harmonic form in $\Omega^2$ representing $1 \in \mathcal{H}^2_\delta(X) \cong \mathbb{R}$. By Lemma 13.12 the condition that $\mathcal{M}_\delta(L)$ can not contain any reducible solution $(\psi \neq 0)$ is $(2\pi c_1(L) + \delta)^+ \neq 0$, which can be expressed by $(2\pi c_1(L) + \delta). \omega_\delta \neq 0 \Rightarrow (g, \delta)$ lies in one of the (plus/minus) chambers:

$$C_\pm = \{ (g, \delta) | \pm (2\pi c_1(L) + \delta). \omega_\delta > 0 \}$$

Varying $(g, \delta)$ in the same chamber does not change $SW_{X,g,\delta}$. Let $H = H_g \in H_2(X, \mathbb{R})$ be the Poincare dual of $\omega_\delta$, define $SW^\pm_{X,H} = SW_{X,g,\delta}$ if $(g, \delta) \in C_\pm$. By [KM] and [LL]

$$SW^+_{X,H}(L) - SW^-_{X,H}(L) = (-1)^{1+d(L)}$$

So for any basic class $L$ at least one of $SW^\pm_X(L)$ is nonzero. Also we can reduce the dependence on $\delta$ by taking it to be an arbitrarily small generic perturbation. Let $L$ denotes Poincare dual of $c_1(L)$, and define “small” Seiberg-Witten invariant to be

$$SW^\pm_{X,H}(L) = \begin{cases} SW^+_{X,H}(L) & \text{if } L.H > 0 \\ SW^-_{X,H}(L) & \text{if } L.H < 0 \end{cases} \quad (13.28)$$
This quantity depends on the metric $g$ since $H = H_g$, but in fact it depends only on

$$P = \{H \in H_2(X) \mid H^2 > 0\}$$

**Exercise 13.1.** *(Light cone lemma)* Let $X^4$ be a closed smooth manifold with $b_2^+(X) = 1$, then show that if $H_1, H_2 \in H_2(X) - \{0\}$ with $H_1^2 \geq 0, H_2^2 \geq 0$ and $H_1.H_2 = 0$, then $H_2 = \lambda H_1$ for some $\lambda > 0$ *(Hint: write the terms in terms of a basis and use Schwarz inequality).*

**Remark 13.14.** In case $b_2^-(X) \leq 9$ then $SW_{X,H}^o(L)$ doesn’t even depend on $H$. This is because to have a solution to Seiberg-Witten equations we must have $d(L) \geq 0$, so $c_1(L)^2 - 3\sigma(X) - 2\chi(X) \geq 0 \implies c_1(L)^2 - 9 + b_2^- \geq 0 \implies c_1(L)^2 \geq 0$. The set $P$ is closed under convex linear combinations, so if $L.H_1 > 0$ and $L.H_2 < 0$ for some $H_1, H_2 \in P$, then $L.H_3 = 0$ for some $H_3 \in P$. So by the Light cone lemma $H_3 = \lambda L$ which implies $H_3^2 = 0$, a contradiction. Hence if $L.H_1 > 0$ then $L.H_2 > 0$ for every other element $H_2 \in P$. 
13.6 Almost Complex and Symplectic Structures

Now assume that $X$ has an almost complex structure. This means that there is a principal $\text{GL}(2, \mathbb{C})$-bundle $Q \to X$ such that, as complex bundles, there is an isomorphism

$$T(X) \cong Q \times_{\text{GL}(2, \mathbb{C})} \mathbb{C}^2$$

By choosing Hermitian metric on $T(X)$ we can assume $Q \to X$ is a $U(2)$ bundle, and the tangent frame bundle $P_{SO(4)}(TX)$ comes from $Q$ by the reduction map

$$U(2) = (S^1 \times SU(2))/\mathbb{Z}_2 \to (SU(2) \times SU(2))/\mathbb{Z}_2 = SO(4)$$

Equivalently there is an endomorphism $I \in \Gamma(\text{End}(TX))$ with $I^2 = -\text{Id}$

$$T(X) \xrightarrow{I} T(X) \xrightarrow{\pi} X$$

The $\pm i$ eigenspaces of $I$ splits the complexified tangent space $T(X)_C$

$$T(X)_C \cong T^{1,0}(X) \oplus T^{0,1}(X) = \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$$

This gives us a complex line bundle which is called the canonical line bundle:

$$K = K_X = \Lambda^{2,0}(X) = \Lambda^2(T^{1,0}) \to X$$

Both $K^\pm$ are characteristic, i.e. $c_1(K^\pm) = w_2(X)$ mod 2. Line bundle $K \to X$ defines a canonical $\text{Spin}_c(4)$ structure on $X$, given by the lifting of $f[\lambda, A] = ([\lambda, A], \lambda^2)$

$$\text{Spin}_c(4)$$

$$F \xrightarrow{l}$$

$$U(2) \xrightarrow{f} SO(4) \times S^1$$

where $l$ is the canonical projection and $F[\lambda, A] = [\lambda, A, \lambda]$. Transition function $\lambda^2$ gives $K$, and the corresponding $\mathbb{C}^2$-bundles are given by:

$$W^+ = \Lambda^{0,2}(X) \oplus \Lambda^{0,0}(X) = K^{-1} \oplus \mathbb{C}$$

$$W^- = \Lambda^{0,1}(X)$$
13.6 Almost Complex and Symplectic Structures

We can check this by taking \( x = z + jw \in \mathbb{H} \)

\[
x \mapsto \lambda x \lambda^{-1} = \lambda (z + jw) \lambda = z + jw \lambda \lambda = z + jw \lambda^{-2}
\]

Since we can identify \( \bar{\Lambda}^{0,1}(X) \cong \Lambda^{1,0}(X) \), and \( \Lambda^{0,2}(X) \otimes \Lambda^{1,0}(X) \cong \Lambda^{0,1}(X) \) we readily see the decomposition \( T(X)_C \cong W^+ \otimes W^- \). As real bundles we have

\[
\Lambda^+(X) \cong K \oplus \mathbb{R}
\]

We can verify this by taking \( \{e^1, e^2 = I(e^1), e^3, e^4 = I(e^3)\} \) to be a local orthonormal basis for \( T^+(X) \), then

\[
\Lambda^{1,0}(X) = \langle e^1 - ie^2, e^3 - ie^4 \rangle, \quad \text{and} \quad \Lambda^{0,1}(X) = \langle e^1 + ie^2, e^3 + ie^4 \rangle
\]

\[
\Lambda^+(X) = \langle \omega := f_1 = \frac{1}{2} (e^1 \wedge e^2 + e^3 \wedge e^4), \ f_2 = \frac{1}{2} (e^1 \wedge e^3 + e^4 \wedge e^2), \ f_3 = \frac{1}{2} (e^1 \wedge e^4 + e^2 \wedge e^3) \rangle
\]

\( \omega \) is the global form \( \omega(X, Y) = g(X, IY) \) where \( g \) is the hermitian metric (which makes the basis \( \{e^1, e^2, e^3, e^4\} \) orthogonal). Also since \( f = 2(f_2 - i f_3) \), we see as \( \mathbb{R}^3 \)-bundles \( \Lambda^+(X) \cong K \oplus \mathbb{R}(\omega) \). We can check

\[
W^+ \otimes \bar{W}^+ \cong \mathbb{C} \oplus \mathbb{C} \oplus K \oplus \bar{K} = (K \oplus \mathbb{R})_C \oplus \mathbb{C}
\]

We see that \( \omega, f_2, f_3 \) act as the Pauli matrices \( \rho : \Lambda^+_C \to \text{sl}(W^+) \):

\[
\rho(\omega) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\rho(f) = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - 2i \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \tag{13.29}
\]

\[
\rho(\bar{f}) = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + 2i \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}
\]

If we write a section \( \psi \in \Gamma(W^+) = \Gamma(\mathbb{C} \oplus K^{-1}) \) by \( \psi = (\alpha, \beta) \), then in terms of the basis \( \{\omega, f, \bar{f}\} \) the formula 13.4, the map \( \sigma : W^+ \to \Lambda^+_C \) and the equality \( \sigma(\psi) = F_A \) is:

\[
\sigma(\psi) = \left( \frac{|\alpha|^2 - |\beta|^2}{2} \right) \omega - \frac{i}{4} (\alpha \bar{\beta}) f + \frac{i}{4} (\bar{\alpha} \beta) \bar{f}
\]

\[
F_A = i F_A = i [F_A^{1,1} + F_A^{2,0} + F_A^{0,2}]
\]
From the second Seiberg-Witten equation $\sigma(\psi) = F_A = iF_A$ we get:

$$F_A = -i\sigma(\psi) = \left(\frac{\beta^2 - \alpha^2}{2}\right) i \omega - \frac{1}{4} (\alpha \beta) f + \frac{1}{4} (\bar{\alpha} \bar{\beta}) \bar{f}$$  \hspace{1cm} (13.30)

$$F_{A}^{2,0} = -\frac{1}{4} (\alpha \beta) f$$  \hspace{1cm} (13.31)

$$F_{A}^{0,2} = \frac{1}{4} (\bar{\alpha} \bar{\beta}) \bar{f}$$  \hspace{1cm} (13.32)

$$F_{A}^{1,1} = \left(\frac{\beta^2 - \alpha^2}{2}\right) i \omega$$  \hspace{1cm} (13.33)

In case $X$ is a Kahler surface the Dirac operator is given by (c.f. [LaM])

$$\bar{D}_A = \bar{\partial}_A + \bar{\partial}_A : \Gamma(W^+) \to \Gamma(W^-)$$

Hence from the Dirac part of the Seiberg-Witten equation (13.14) we have

$$\bar{\partial}_A^* (\beta) + \bar{\partial}_A (\alpha) = 0 \implies \bar{\partial}_A \bar{\partial}_A^* (\beta) + \bar{\partial}_A \bar{\partial}_A (\alpha) = 0$$  \hspace{1cm} (13.34)

Notice $\bar{\partial}_A \bar{\partial}_A (\alpha) = F_{A}^{0,2} \alpha = \frac{1}{4} |\alpha|^2 \beta$. By taking inner product of both sides the last equation by $\beta$ and integrating over $X$ we get the $L^2$ norms satisfy

$$||\alpha||^2 ||\beta||^2 + ||\bar{\partial}_A (\beta)||^2 = 0$$  \hspace{1cm} (13.35)

Hence both of the terms must be zero, in particular $\bar{\alpha} \beta = 0$ and $\bar{\partial}_A^* (\beta) = 0$. Then by (13.31) $A$ is a holomorphic connection on $K$, and by (13.33) $\alpha$ is a holomorphic function, and $\beta$ is anti-holomorphic section of $K^{-1}$, hence it is a holomorphic 2-form in $K \to X$. Also we must have at any point one of $\alpha$ or $\beta$ must be zero. Since the following quantity is constant

$$c_1(K) \omega = \int_X c_1(K) \wedge \omega = \int_X \frac{i}{2\pi} F_A \wedge \omega = \frac{1}{4\pi} \int_X (|\beta|^2 - |\alpha|^2) dvol$$

either $\alpha$ or $\beta$ must be zero. This argument eventually computes $SW_X(K) = 1$ ([Wi], [Mor]). We will not repeat the argument here, instead we will review a stronger result of C.Taubes for symplectic manifolds which implies this result.
We call an almost complex manifold with Hermitian metric \( \{X, I, g\} \) symplectic if \( d\omega = 0 \). Clearly a nondegenerate closed form \( \omega \) and a hermitian metric determines the almost complex structure \( I \) by \( g(u, v) = \omega(u, Iv) \). Given \( \omega \), then \( I \) is called an almost complex structure taming the symplectic form \( \omega \).

By Proposition 13.6 there is a unique connection \( A_0 \) in \( \mathcal{K} \to X \) such that the induced Dirac operator \( D_{A_0} \) on \( W^+ \) restricted to the trivial summand \( \mathbb{C} \to X \) is the exterior derivative \( d \). Let \( u_0 = (z, 0) \) be the section of \( W^+ = \mathbb{C} \oplus K^{-1} \) with constant \( \mathbb{C} \) component and \( \|u_0\| = 1 \). Taubes’s first fundamental observation is

\[
\mathcal{P}_{A_0}(u_0) = 0 \quad \text{if and only if} \quad d\omega = 0
\]

This can be seen by applying the Dirac operator to both sides of \( iu_0 = \rho(\omega)u_0 \), and observing that by the choice of \( u_0 \) the term \( \nabla_{A_0}(u_0) \) lies entirely in \( K^{-1} \) component:

\[
i\mathcal{P}_{A_0}(u_0) = \sum \rho(e^i) \nabla_i (\rho(\omega) u_0) = \sum \rho(e^i) [ \nabla_i (\rho(\omega)) u_0 + \rho(\omega) \nabla_i (u_0) ] = \sum \rho(e^i) \nabla_i (\rho(\omega)) u_0 \]

\[
2i \mathcal{P}_{A_0}(u_0) = \rho((d + d^*)\omega) u_0 = \rho((d - * d)\omega) u_0
\]

Last equality holds since \( \omega \in \Lambda^+(X)_{\mathbb{C}} \oplus \mathbb{C} \), and by naturality, the Dirac operator on \( \Lambda^*(X)_{\mathbb{C}} \) is \( d + d^* \), and since \( d = -*d* \) on 2 forms and \( \omega \) is self dual:

\[
2i \mathcal{P}_{A_0}(u_0) = -\rho(*d\omega)u_0 \quad \square
\]

**Theorem 13.15.** (Taubes) Let \((X, \omega)\) be a closed symplectic manifold with \( b_2(X)^+ \geq 2 \), then \( SW_X(L) = \pm 1 \).

**Proof.** Let \( \psi = \alpha u_0 + \beta = (\alpha, \beta) \in \Gamma(W^+) = \Gamma(\mathbb{C} \oplus K^{-1}) \) where \( \alpha : X \to \mathbb{C} \), and \( u_0 \) is a constant section. Consider the perturbed Seiberg-Witten equations:

\[
\mathcal{P}_A(\psi) = 0 \quad (13.36)
\]

\[
F^+_A = F^+_{A_0} + r \left[ -i\sigma(\psi) + i\omega \right] \quad (13.37)
\]

where \((A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+)\). By 13.30 the second equation is:

\[
F^+_A - F^+_{A_0} = r \left[ \frac{|\beta|^2 - |\alpha|^2}{2} + 1 \right] i\omega - \frac{1}{4}(\alpha\bar{\beta})f + \frac{1}{4}(|\alpha|^2 + |\beta|^2) - f \right] \quad (13.38)
\]

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Clearly \((A, \psi) = (A_0, u_0)\) and \(r = 0\) trivially satisfy these. We will show that up to gauge equivalence there is a unique solution to these equations. Write \(A = A_0 + ia\), after a gauge transformation we can assume that \(a\) is co-closed, \(d^*(a) = 0\). It suffices to show that for \(r \mapsto \infty\) these equations admit only \((A_0, u_0)\) as a solution. From Proposition 13.6, Weitzenböck formula 13.10, and abbreviating \(\nabla_{A_0}(u_0) = b\) we get

\[
\mathcal{P}_A^2(\psi) = \mathcal{P}_A^2(\beta) + (\nabla_a^* \nabla_a \alpha) u_0 - 2 < \nabla_a \alpha, b > + \frac{1}{2} \rho(F_A^+ - F_{A_0}^+) \alpha u_0
\]  

(13.39)

\[
\mathcal{P}_A^2(\beta) = (\nabla_A^* \nabla_A \beta) + \frac{s}{4} \beta + \frac{1}{2} \rho(F_A^+) \beta + \frac{1}{2} \rho(F_A^+ - F_{A_0}^+) \beta
\]  

(13.40)

By applying \(\rho\) to 13.38, using 13.29, and recalling \(\psi = \alpha u_0 + b = (\alpha, \beta)\) we get

\[
\frac{1}{2} \rho(F_A^+ - F_{A_0}^+) = \left( \begin{array}{cc}
(\alpha|\beta|^2 - 2) r/4 & \alpha \beta r/2 \\
\bar{\alpha} \beta r/2 & - (\alpha|\beta|^2 - 2) r/4
\end{array} \right)
\]

(13.41)

\[
\frac{1}{2} \rho(F_A^+ - F_{A_0}^+) \alpha u_0 = \frac{r}{4} \alpha (|\alpha|^2 - |\beta|^2 - 2) u_0 + \frac{r}{2} |\alpha|^2 \beta
\]

(13.42)

\[
\frac{1}{2} \rho(F_A^+ - F_{A_0}^+) \beta = - \frac{r}{4} (|\alpha|^2 - |\beta|^2 - 2) \beta + \frac{r}{4} |\alpha| \beta^2 u_0
\]

(13.43)

By substituting (13.41) in (13.38) we get

\[
\mathcal{P}_A^2(\psi - \beta) = [ \nabla_a^* \nabla_a \alpha + \frac{r}{4} \alpha (|\alpha|^2 - |\beta|^2 - 2) ] u_0
\]

\[
- 2 < \nabla_a \alpha, b > + \frac{r}{2} |\alpha|^2 \beta
\]

(13.44)

By substituting (13.42) in (13.39), then substituting (13.39) in (13.43) we obtain:

\[
0 = \mathcal{P}_A^2(\psi) = [ \nabla_a^* \nabla_a \alpha + \frac{r}{4} \alpha (|\alpha|^2 - 2) ] u_0 - 2 < \nabla_a \alpha, b >
\]

\[
+ [ \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2} \rho(F_A^+) + \frac{r}{4} (|\alpha|^2 + |\beta|^2 + 2) ] \beta
\]

(13.45)

By recalling that \(\beta\) and \(u_0\) are orthogonal sections of \(W^+\), we take inner product of both sides of (13.44) with \(\beta\) and integrate over \(X\) and obtain:

\[
\int_X (|\nabla_A \beta|^2 + \frac{r}{4} |\beta|^4 + \frac{r}{2} |\beta|^2 + \frac{r}{4} |\alpha|^2 |\beta|^2 ) =
\]

\[
2 \int_X (\langle \nabla_a \alpha, b \rangle, \beta > - \frac{s}{4} |\beta|^2 - \frac{1}{2} < \rho(F_{A_0}^+) \beta, \beta >
\]

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Hence \( \int_X |\nabla_A \beta|^2 + \frac{r}{4} |\beta|^4 + \frac{r}{2} |\beta|^2 + \frac{r}{4} |\alpha|^2 |\beta|^2 \leq \int_X c_1 |\beta|^2 + c_2 |\beta| |\nabla_a \alpha| \)

where \( c_1 \) and \( c_2 \) are positive constants depending on the Riemannian metric and the base connection \( A_0 \). Choose \( r \gg 1 \), by calling \( c_2 = 2c_3 \) we get:

\[
\int_X \left( |\nabla_A \beta|^2 + \frac{r}{4} |\beta|^4 + \frac{r}{2} |\beta|^2 + \frac{r}{4} |\alpha|^2 |\beta|^2 \right) \leq \int_X (c_1 - \frac{r}{4}) |\beta|^2 + 2c_3 |\beta| |\nabla_a \alpha| = 
\]

\[
- \int_X \left[ (r/4 - c_1)^{1/2} |\beta| - c_3 (r/4 - c_1)^{-1/2} |\nabla_a \alpha| \right]^2 + \frac{c_3^2}{(r/4 - c_1)} |\nabla_a \alpha|^2 \leq \int_X \frac{C}{r} |\nabla_a \alpha|^2
\]

For some \( C \) depending on the metric and \( A_0 \). In particular we have

\[
\int_X r |\beta|^2 - \frac{4C}{r} |\nabla_a \alpha|^2 \leq 0
\]

\[
\int_X 4c_2 |\beta| |\nabla_a \alpha| - \frac{4C}{r} |\nabla_a \alpha|^2 \leq \int_X (r - 4c_1) |\beta|^2
\]

Hence

\[
\int_X c_2 |\beta| |\nabla_a \alpha| - \frac{2C}{r} |\nabla_a \alpha|^2 \leq 0 \quad (13.46)
\]

Now by self adjointness of the Dirac operator, and by \( \alpha u_0 = \psi - \beta \) we get:

\[
< D_A^2 (\psi) , \alpha u_0 > = < D_A^2 (\psi - \beta) , \alpha u_0 > + < D_A^2 (\beta) , \alpha u_0 >
\]

\[
= < D_A^2 (\psi - \beta) , \alpha u_0 > + < \beta , D_A^2 (\psi - \beta) > \quad (13.47)
\]

We can calculate (13.46) by using (13.43) and obtain the inequalities:

\[
0 = < D_A^2 (\psi) , \alpha u_0 > = |\nabla_a \alpha|^2 + \frac{r}{4} |\alpha|^4 - \frac{r}{4} |\alpha|^2 |\beta|^2 - \frac{r}{4} |\alpha|^2
\]

\[
+ \frac{r}{2} |\alpha|^2 |\beta|^2 - 2 < \nabla_a \alpha , b > , \beta >
\]

\[
\int_X |\nabla_a \alpha|^2 + \frac{r}{4} |\alpha|^4 - \frac{r}{2} |\alpha|^2 \leq \int_X 2 < \nabla_a \alpha , b > , \beta > - \frac{r}{4} |\alpha|^2 |\beta|^2
\]

\[
\leq \int_X 2 < \nabla_a \alpha , b > , \beta > \leq \int_X c_2 |\nabla_a \alpha| |\beta| |
\]
By choosing $c_4 = 1 - 2C/r$ and by (13.45), we see
\[
\int_X c_4 |\nabla_a \alpha|^2 + \frac{r}{4}|\alpha|^4 - \frac{r}{2}|\alpha|^2 \leq 0
\] (13.48)

Since for a connection $A$ in $K \rightarrow X$ the class $(i/2\pi)F_A$ represents the Chern class $c_1(K)$, and since $\omega$ is a self dual 2-form we can write:
\[
\int_X \omega \wedge F_A = -2\pi i \omega c_1(K) \quad \int_X \omega \wedge F_A = \int_X \omega \wedge F_A^*
\]
\[
\int_X \omega \wedge (F_A^* - F_{A_0}^*) = 0
\]

By (13.38) this implies:
\[
\frac{r}{2} \int_X (2 - |\alpha|^2 + |\beta|^2) = 0
\] (13.49)

By adding (13.48) to (13.47) we get
\[
\int_X c_4 |\nabla_a \alpha|^2 + \frac{r}{4}|\beta|^2 + r \left(1 - \frac{1}{2}|\alpha|^2\right)^2 \leq 0
\] (13.50)

Assume $r \gg 1$, then $c_4 \geq 0$ and hence $\nabla_a \alpha = 0$ and $\beta = 0$ and $|\alpha| = \sqrt{2}$, hence:
\[
\beta = 0 \quad \alpha = \sqrt{2} e^{i\theta} \quad \nabla_a (e^{i\theta}) = d(e^{i\theta}) + i a e^{i\theta} = 0
\]

Hence $a = d(-\theta)$, recall that we also have $d^*(a) = 0$ which gives
\[
0 = <d^*d(\theta), \theta> = <d(\theta), d(\theta)> = ||d(\theta)||^2
\]

Hence $a = 0$ and $\theta =$constant. So up to a gauge equivalence $(A, \psi) = (A_0, u_0)$.
13.7 Seiberg-Witten equations on $\mathbb{R} \times Y^3$

Let $X^4 = \mathbb{R} \times Y^3$, if a $Spin^c$ structure on $X$ (an integral lifting of $w_2(X) \in H^2(X; \mathbb{Z}_2)$ which is represented by a complex line bundle $L \to X$) comes from a $Spin^c$ structure on $Y$, we can identify both $\mathbb{C}$-bundles $W^* \to \mathbb{R} \times Y^3$ with the pull back of a $\mathbb{C}$-bundle $W \to Y$ by the projection $\mathbb{R} \times Y \to Y$. Clifford multiplication with $dt$ identifies $W^+$ with $W^-$. More specifically, $Spin^c$ structures on $\mathbb{R} \times Y$ are induced from the imbedding:

$$Spin^c(3) = (SU(2) \times S^1)/\mathbb{Z}_2 \hookrightarrow (SU(2) \times SU(2) \times S^1)/\mathbb{Z}_2 = Spin^c(4)$$

given by the map $[q, \lambda] \mapsto [q, q^{-1}, \lambda]$. In the notation of Section 13.1 the isomorphism $W^+ \mapsto W^-$ is induced by $[q, \lambda] \mapsto [q^{-1}, \lambda]$. Also we have the following identification:

$$\Lambda^1(Y) \cong \Lambda^2_c(X) \text{ given by the map } a \mapsto \frac{1}{2} \left[ dt \wedge a + *_3 a \right]$$

For example if $a \in \Omega^1(X)$, we identify $d^+(a) \leftrightarrow da/dt + *_3 d_Y(a) \in \Omega^1(Y)$ since

$$2d^+(a) = da + *_A da = dt \wedge \frac{da}{dt} + d_Y(a) + *_3 \left[ dt \wedge \frac{da}{dt} + d_Y(a) \right]$$
$$= dt \wedge \left[ \frac{da}{dt} + *_3 d_Y(a) \right] + *_3 \left[ \frac{da}{dt} + *_3 d_Y(a) \right]$$

By viewing Seiberg-Witten equations 13.13, 13.14 as equations in the quotient space $\mathcal{B}(L) = \tilde{\mathcal{B}}(L)/\mathcal{G}(L)$, we can apply gauge transformations 13.15 to change the $dt$ component of the connection parameter $A = A_0 + a \in \{A_0\} + \Omega^1(X)$ by any $f(x, t) dt$, hence make it zero. Such a connection is called a connection in temporal gauge. So $A = A(t, x)$ can be viewed as a family of connections on $Y$ parametrized by $t$, hence the Dirac operator on $X$ is $\not\!D_A(\psi) = d\psi/dt + \partial_A(\psi)$, where $\partial_A$ is the Dirac operator on $Y$, which is expressed by $\partial_A \psi = \partial_{A_0} \psi + a.\psi$. For simplicity here we are abbreviating $\rho(a) \psi = a\psi$. Then the Seiberg-Witten equations on $\mathbb{R} \times Y$ become

$$\frac{da}{dt} = -*_3 F_A - i\sigma(\psi)$$  \hspace{1cm} (13.51)
$$\frac{d\psi}{dt} = -\partial_A(\psi)$$  \hspace{1cm} (13.52)

These are the downward gradient flow equations of the function $C/2 : \tilde{\mathcal{B}}(L) \to \mathbb{R}$

$$C(A, \psi) = \int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \langle \psi, \partial_A(\psi) \rangle$$
To see this take a variation $A = A_0 + a(t) \Rightarrow \dot{A} = \dot{a}$ and use $F_A = F_{A_0} + da$, $F_{\dot{A}} = d\dot{a}$, the Stokes theorem, and the self adjointedness of the Dirac operator:

$$dC_{(A,\psi)}(\dot{A}, \dot{\psi}) = \int_X \dot{a} \wedge (F_A + F_{A_0}) + a \wedge F_{\dot{A}} + \{\dot{\psi}, \partial_A \psi\} + \{\dot{\psi}, \partial_{\dot{A}} \psi\} + \{\psi, \dot{a} \psi\}$$

$$= \int_Y 2[\dot{a} \wedge da + \dot{a} \wedge F_{A_0} + \{\dot{\psi}, \partial_A \psi\}] + \{\dot{\psi}, \dot{a} \psi\}$$

$$= \int_Y 2[\dot{a} \wedge F_A + \{\dot{\psi}, \partial_A \psi\}] + 2i(\dot{a}, \sigma(\psi)) \quad \text{from (13.7)}$$

$$= 2\int_Y [(\dot{a}, *_3 F_A) + (\dot{a}, i\sigma(\psi)) + \{\dot{\psi}, \partial_A \psi\}]$$

$$= 2 <(\dot{a}, \dot{\psi}), (*_3 F_A + i\sigma(\psi), \partial_A \psi)>$$

where $<\ldots, \ldots>$ is the metric on $\mathcal{B}(L)$. In particular at the gradient flow solutions we have

$$dC(\dot{a}, \dot{\psi}) = 2 <(\dot{a}, \dot{\psi}), (-\dot{a}, -\dot{\psi})> = -2(|\dot{a}|^2 + |\dot{\psi}|^2) \quad (13.53)$$

**Remark 13.16.** A gauge transformation $s : X \to S^1$ (13.15) alters $C$ by

$$\int_Y s^{-1} ds \wedge da = \int_Y s^{-1} ds \wedge F_A = (8\pi^2 c_1(L) \cup h, i_*[Y])$$

where $s^{-1} ds = s^*(d\theta) \in \Omega^1(Y)$ is a closed 1-form representing a class $h \in H^1(Y; \mathbb{Z})$, and $i : Y \hookrightarrow X$ is the inclusion.

### 13.8 Adjunction inequality

**Theorem 13.17.** ([KM], [MST]) If $X$ is a closed smooth 4-manifold with $b_2^+(X) > 1$ with a nonzero Seiberg-Witten invariant corresponding to a line bundle $L \to X$, and $\Sigma \subset X$ be a closed genus $g = g(\Sigma) > 1$ oriented surface with $\Sigma . \Sigma \geq 0$. Then we have

$$2g(\Sigma) - 2 \geq \Sigma . \Sigma + |\langle c_1(L), \Sigma \rangle| \quad (13.54)$$

**Proof.** ([KM]) We prove this when $\Sigma . \Sigma = 0$ and. Let $Y = S^1 \times \Sigma \subset \partial(D^2 \times \Sigma) \subset X$. We first stretch $X$ near $Y$ metrically, i.e. we pick a metric $ds_R^2$ on $X$ which is the product metric $dt^2 + ds_R^2$. on $[-R, R] \times S^1 \times \Sigma \subset X$. Call this metrically stretched Riemannian manifold $X_R$. Over $[-R, R] \times Y$ we identify $Spin^c(4)$ bundles $W^* \to [-R, R] \times Y$ with the pull back of the $Spin^c(3)$ bundle $W \to Y$. Choose a base connection $A^0_R$ on $X_R$ such
that $A^0_R$ is fixed in the complement $X_*$ of the product region, independent of $R$. Also since the homology class $i_*[Y] = 0$ by Remark 13.16 the function $C$ is defined on the quotient $\mathcal{B}(L)$ not just on $\mathcal{B}(L)$.

$$C : \mathcal{B}(L) \to \mathbb{R}$$

![Figure 13.1: Deforming the metric of $X$ to get $X_R$](image)

Now we pick a solution $(A_R, \psi_R) = (A^0_R + a_R, \psi_R)$ on $X_R$ for $R$ sufficiently large, such that $A_R$ is in temporal on the product region. Then the change in the quantity $\Delta C(R) = C(a_R, \psi) - C(a_{-R}, \psi)$ is independent of $R$, call it $C_0$. So there is an interval $[-1, 1] \times Y$ on which the change $\Delta C(1)$ is at most $C_0/\text{Volume}(Y)$. By Proposition 13.9 the solution space is compact, so by taking the limit of the solutions as $R \to \infty$ we get $\Delta C(1) = 0$.

$$0 = \Delta C(1) = \int_0^1 \frac{dC}{dt} dt = \int_0^1 dC(\dot{a}, \dot{\psi}) dt = -2 \int_0^1 (|\dot{a}|^2 + |\dot{\psi}|^2) dt$$

Hence $(A(t), \psi(t))$ is constant, giving a translation invariant solution $(A, \psi)$ on $\mathbb{R} \times Y$ in temporal gauge. Here $\Sigma$ is a 2-manifold with constant scalar curvature $s$, we chose the metric so that $\Sigma$ has the unit area so that its scalar curvature (twice the Gaussian curvature) is $-2\pi(2g - 2)$. Also by 13.14, 13.6, 13.7 and 13.16 we calculate

$$\sqrt{2} |F_A^+| = \sqrt{2} |\sigma(\psi)| = |\rho(\sigma(\psi))| = |\sigma(\psi)| = \frac{1}{2} |\psi|^2 \leq 2\pi(2g - 2) \Rightarrow l F_A^+ \leq \sqrt{2} \pi(2g - 2)$$

Because $A$ is translation-invariant in a temporal gauge, the curvature $F_A$ is a 2-form pulled back from $Y$, hence $F_A^+$ and $F_A^-$ have the same magnitude, therefore we have

$$\langle c_1(L), [\Sigma] \rangle = \frac{i}{2\pi} \int_{\Sigma} F_A \leq \frac{1}{2\pi} (\sup |F_A|) \text{Area}(\Sigma) \leq 2g - 2$$

**Exercise 13.2.** (Blowup formula) If $X$ is a closed smooth 4-manifold with $b_2^+(X) > 1$ with a nonzero Seiberg-Witten invariant corresponding to a line bundle $L \to X$, then then the blown up manifold $\bar{X} = X \# k\mathbb{C}P^2 = X \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2$ has nonzero Seiberg-Witten invariant corresponding to the line bundle $\bar{L} = L + \mathbb{C}P^1 + \ldots + \mathbb{C}P^j \ [FS3]$.  

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Exercise 13.3. Prove Theorem 13.17 under the condition $\Sigma.\Sigma = k > 0$ (Hint blow up $X \mapsto X \# k\mathbb{CP}^2$ so that the condition $\Sigma.\Sigma = 0$ holds, then by using Exercise 13.2 apply the proof of the Theorem).

Exercise 13.4. If $X$ is a closed smooth 4-manifold with $b_2^+(X) > 1$ with nonzero Seiberg-Witten invariant, show that it can not contain an imbedded 2-Sphere $S \subset X$ with $[S].[S] \geq 1$ (Hint use $\mathbb{CP}^2$ has positive scalar curvature and 13.16).

Exercise 13.5. If $X$ is a closed smooth 4-manifold with $b_2^+(X) > 1$ with nonzero Seiberg-Witten invariant, show that $X$ can not decompose as $X_1 \# X_2$ with $b_2^+(X_j) > 0, j = 1, 2$ (Hint use the dimension formula Proposition 13.11, if still stuck consult [Sal1]).
Chapter 14

Some applications

14.1 10/8 Theorem

Let $X$ be a simply connected closed smooth 4-manifold. By J.H.C. Whitehead the intersection form

$$q_X : H_2(X; \mathbb{Z}) \otimes H_2(X; \mathbb{Z}) \to \mathbb{Z}$$

determines the homotopy type of $X$. By C.T.C Wall in fact $q_X$ determines the $h$-cobordism class of $X$. Donaldson (c.f. [DK]) showed that if $q_X$ is definite then it is diagonalizable, i.e.

$$q_X = <1> \oplus <1> \oplus ... <1>$$

We call $q_X$ is even if $q(a,a)$ is even for all $a$, otherwise we call $q_X$ odd. Since integral liftings $c$ of the second Steifel Whitney class $w_2$ of $X$ are characterized by $c.a = a.a$ for all $a \in H_2(X; \mathbb{Z})$, the condition of $q_X$ being even is equivalent to $X$ being spin. From classification of unimodular even integral quadratic forms and the Rohlin theorem it follows that the intersection form of a closed smooth spin manifold is in the form:

$$q_X = 2kE_8 \oplus lH \quad (14.1)$$

where $E_8$ is the $8 \times 8$ intersection matrix given by the Dynkin diagram Figure 14.1

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14_1.pdf}
\caption{Figure 14.1}
\end{figure}
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and $H$ is the form $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The intersection form of the manifold $S^2 \times S^2$ realizes the form $H$, and the K3 surface (quadric in $\mathbb{CP}^3$) realizes $2E_8 \oplus 3H$. Donaldson had shown that if $k = 1$, then $l \geq 3$ ([D2]). Clearly connected sums of K3 surface realizes $2kE_8 \oplus 3kH$. In general it is a conjecture that in 14.1 we must necessarily have $l \geq 3k$ (sometimes this is called 11/8 conjecture). By using Seiberg-Witten theory M. Furuta has shown that

**Theorem 14.1.** ([Fu]) Let $X$ be a simply connected closed smooth spin 4-manifold with the intersection form $q_X = 2kE_8 \oplus lH$, then $l \geq 2k + 1$

Proof: We will only sketch the proof of $l \geq 2k$. We pick $L \rightarrow X$ to be the trivial bundle (it is characteristic since $X$ is spin). Notice that the spinor bundles $V^\pm = P \times_{\rho_{\pm}} \mathbb{C}^2 \rightarrow X$

$\rho_{\pm}: x \mapsto q_{\pm}x$, are quaternionic vector bundles. That is, there is an action $j: V^\pm \rightarrow V^\pm$ defined by $[p, x] \mapsto [p, xj]$, which is clearly well defined. This action commutes with $\partial: \Gamma(V^+) \rightarrow \Gamma(V^-)$

Let $A_0$ be the trivial connection, and write $\pm A = A_0 \pm i \alpha \in A(L)$. Recalling 13.1 that the bundle $V^* = V^+ \otimes L^{-1/2} \rightarrow X$ is obtained from the action action $x \mapsto qx\lambda^{-1}$

$\bar{\phi}_A(\psi j) = \sum \rho(e^k) [\nabla_k + i \alpha] (\psi j) = \sum \rho(e^k) [\nabla_k(\psi j) + (\psi j)(-i\alpha)]$  

$= \sum \rho(e^k) [\nabla_k(\psi j) + \psi(ia)j] = \bar{\phi}_A(\psi)j$

$\mathbb{Z}_4$ action $(A, \psi) \mapsto (-A, \psi j)$ on $\Omega^1(X) \times \Gamma(V^+)$ preserves the compact set

$\mathcal{M}_0 = \tilde{\mathcal{M}} \cap ker(d^*) \oplus \Gamma(V^+)$

where $\tilde{\mathcal{M}} = \{(a, \psi) \in \Omega^1(X) \oplus \Gamma(V^+) | \phi_A(\psi) = 0, F_A^+ = \sigma(\psi)\}$. For example from the local description of $\sigma$ in Section 13.2 we can check that

$\sigma(\psi j) = \sigma(z + jw)j = \sigma(-\bar{w} + j\bar{z}) = -\sigma(\psi) = -F_A^+ = F_{-A}^+$

This $\mathbb{Z}_4$ in fact extends to an action of the subgroup $G$ of $SU(2)$ which is generated by $< S^1, j >$, where $S^1$ acts trivially on $\Omega^*$ and by complex multiplication on $\Gamma(V^+)$, and
$j$ acts by $-1$ on $\Omega^*$, and by quaternionic multiplication on $\Gamma(V^+)$. In particular we get a $G$-equivariant map

$$\varphi : \mathcal{V} = \ker(d^* + \Gamma(V^+)) \longrightarrow \mathcal{W} = \Omega^2 \oplus \Gamma(V^-)$$

$$L = \begin{pmatrix} d^* & 0 \\ 0 & 0 \end{pmatrix}$$

and $\theta(a, \psi) = (\sigma(\psi), a\psi)$

with $\varphi^{-1}(0) = \mathcal{M}_0$ and $\varphi(v) = L(v) + \theta(v)$ with $L$ linear Fredholm and $\theta$ quadratic.

We apply the “usual” Kuranishi technique (cf [La]) to obtain a finite dimensional local model $V \hookrightarrow W$ for $\varphi$ as follows:

We let $\mathcal{V} = \oplus V_\lambda$ and $\mathcal{W} = \oplus W_\lambda$, where $V_\lambda$ and $W_\lambda$ be $\lambda$ eigenspaces of $L^*L : \mathcal{V} \to \mathcal{V}$ and $LL^* : \mathcal{W} \to \mathcal{W}$ respectively. Since $LL^*$ is a multiplication by $\lambda$ on $V_\lambda$, for $\lambda > 0$ we have isomorphisms $L : V_\lambda \cong W_\lambda$. Now pick $\Lambda > 0$ and consider projections:

$$\oplus_{\lambda \leq \Lambda} W_\lambda \overset{p_\lambda}{\leftarrow} \mathcal{W} \overset{1-p_\lambda}{\longrightarrow} \oplus_{\lambda > \Lambda} W_\lambda$$

Consider the local diffeomorphism $f_\Lambda : V \longrightarrow V$ given by:

$$u = f_\Lambda(v) = v + L^{-1}(1 - p_\Lambda)\theta(v) \iff L(u) = L(v) + (1 - p_\Lambda)\theta(v)$$

The condition $\varphi(v) = 0$ is equivalent to $p_\Lambda \varphi(v) = 0$ and $(1 - p_\Lambda) \varphi(v) = 0$, but

$$(1 - p_\Lambda) \varphi(v) = 0 \iff (1 - p_\Lambda) L(v) + (1 - p_\Lambda) \theta(v) = 0 \iff$$

$$(1 - p_\Lambda) L(v) + L(u) - L(v) = 0 \iff L(u) = p_\Lambda L(v) \iff u \in \oplus_{\lambda \leq \Lambda} V_\lambda$$

Hence $\varphi(v) = 0 \iff p_\Lambda \varphi(v) = 0$ and $u \in \oplus_{\lambda \leq \Lambda} V_\lambda$, let

$$\varphi_\Lambda : V = \oplus_{\lambda \leq \Lambda} V_\lambda \longrightarrow W = \oplus_{\lambda \leq \Lambda} W_\lambda$$

where $\varphi_\Lambda(u) = p_\Lambda \varphi f_\Lambda^{-1}(u)$

Hence in the local diffeomorphism $f_\Lambda : \mathcal{O} \overset{\cong}{\longrightarrow} \mathcal{O} \subset \mathcal{V}$ takes the piece of the compact set $f_\Lambda(\mathcal{O} \cap \mathcal{M}_0)$ into the finite dimensional subspace $V \subset \mathcal{V}$, where $\mathcal{O}$ is a neighborhood of $(0, 0)$. As a side fact note that near $(0, 0)$ we have

$$\mathcal{M}(L) \cong \mathcal{M}_0(L)/S^1$$

We claim that for $\lambda \gg 1$, the local diffeomorphism $f_\Lambda : \mathcal{O} \overset{\cong}{\longrightarrow} \mathcal{O} \subset V$ extends to a ball $B_R$ of large radius $R$ containing the compact set $\mathcal{M}_0(L)$, i.e. we can make the zero set $\varphi_\Lambda^{-1}(0)$ a compact set. We see this by applying the Banach contraction principle. For example, for a given $u \in B_R$, showing that there is $v \in V$ with $f_\Lambda(v) = u$ is equivalent to
showing that the map $T_u(v) = u - L^{-1}(1 - p_A)\theta(v)$ has a fixed point. Since $L^{-1}(1 - p_A)$ has eigenvalues $1/\lambda$ on each $W_\lambda$ in appropriate Sobolev norm we can write
\[ \|T_u(v_1) - T_u(v_2)\| \leq \frac{C}{\Lambda} \|\theta(v_1) - \theta(v_2)\| \leq \frac{C}{\Lambda} \|v_1 - v_2\| \]

Vector subspaces $V_\lambda$ and $W_\lambda$ are either quaternionic or real depending on whether they are subspaces of $\Gamma(V^\ast)$ or $\Omega^\ast(X)$. For a generic metric we can make the cokernel of $\delta$ zero hence the dimension of the kernel (as a complex vector space) is $ind(\bar{\delta}) = -\sigma/8 = 2k$, and since $H^1(X) = 0$ the dimension of the cokernel of $d^\ast$ (as a real vector space) is $b^+ = l$. Hence $\varphi_\Lambda$ gives a $G$-equivariant map
\[ \varphi : \mathbb{H}^k \oplus \mathbb{R}^x \rightarrow \mathbb{H}^y \oplus \mathbb{R}^{l+x} \]
with compact zero set. From this Furuta shows that $l \geq 2l + 1$. Here we give an easier argument of D. Freed which gives a slightly weaker result of $l \geq 2k$. Let $E_0$ and $E_1$ be the complexifications of the domain and the range of $\varphi$; consider $E_0$ and $E_1$ as bundles over a point $x_0$ with projections $\pi_i : E_i \rightarrow x_0$, and with 0-sections $s_i : x_0 \rightarrow E_i$, $i = 0, 1$. Recall $K_G(x_0) = R(G)$, and we have Bott isomorphisms $\beta(\rho) = \pi_i^*(\rho) \lambda_{E_i}$, for $i = 0, 1$ where $\lambda_{E_i}$ are the Bott classes. By compactness we get an induced map $\varphi^*$:
\[ K_G(B(E_1), S(E_1)) \xrightarrow{\varphi^*} K_G(B(E_0), S(E_0)) \]
\[ \approx \uparrow \beta \quad \approx \uparrow \beta \]
\[ R(G) \quad R(G) \]
Consider $s_i^*(\lambda_{E_i}) = \sum (-1)^k \Lambda^k(E_i) = \Lambda_{-1}(E_i) \in R(G)$, then by some $\rho$ we have
\[ \Lambda_{-1}(E_i) = s_i^*(\lambda_{E_i}) = s_0^* \varphi^*(\lambda_{E_1}) = s_0^* (\pi_0^*(\rho) \lambda_{E_0}) = \rho \Lambda_{-1}(E_0) \]
So in particular $tr_j(\Lambda_{-1}(E_0))$ divides $tr_j(\Lambda_{-1}(E_1))$. By recalling $j : E_i \rightarrow E_i$
\[ tr_j(\Lambda_{-1}(E_i)) = det(I - j) \quad \text{for } i = 0, 1 \]
Since $(z,w)j = (z+jw)j = -\bar{w} + j\bar{z} = (-\bar{w}, \bar{z})$ $j$ acts on $\mathbb{H} \otimes \mathbb{C}$ by matrix
\[ A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]
so $det(I - A) = 4$, and $j$ acts on $\mathbb{R} \otimes \mathbb{C}$ by $j(x) = -x$ so $det(I - (-I)) = 2$. Hence $4^{k+y} 2^x$ divides $4^y 2^{l+x}$ which implies $l \geq 2k$. \[ \square \]

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14.2 Antiholomorphic quotients

It is an old problem whether the quotient of a simply connected smooth complex surface by an antiholomorphic involution $\sigma : \tilde{X} \to \tilde{X}$ (an involution which anticommutes with the almost complex homomorphism $\sigma_* J = -J \sigma_*$) is a “standard” manifold (i.e. connected sums of $S^2 \times S^2$ and $\pm \mathbb{C}P^2$). A common example of an antiholomorphic involution is the complex conjugation on a complex projective algebraic surface with real coefficients. It is known that the quotient of $\mathbb{C}P^2$ by complex conjugation is $S^4$ (Arnold, Massey, Kuiper); and for every $d$ there is a curve of degree $d$ in $\mathbb{C}P^2$ whose two fold branched cover has a standard quotient ([A17]). This problem makes sense only if the antiholomorphic involution has a fixed point, otherwise the quotient space has fundamental group $\mathbb{Z}_2$ and hence it can not be standard. By “connected sum” theorem, Seiberg-Witten invariants of “standard” manifolds vanish, so it is natural question to ask whether Seiberg-Witten invariants of the quotients vanish. The following is the answer.

Theorem 14.2 ([Wg]). Let $\tilde{X}$ be a minimal Kähler surface of general type, and $\sigma : \tilde{X} \to \tilde{X}$ be a free antiholomorphic involution, then the quotient $X = \tilde{X}/\sigma$ has all Seiberg-Witten invariants zero

Proof. Let $h$ be the Kähler metric on $\tilde{X}$, i.e. $\omega(X,Y) = h(X,JY)$ is the Kähler form. Then $\tilde{g} = h + \sigma^* h$ is an invariant metric on $\tilde{X}$ with the Kähler form $\tilde{\omega} = \omega - \sigma^* \omega$. Let $g$ be the “push-down” metric on $X$. Now we claim that all $\text{SW}_X(L) = 0$ for all $L \to X$, in fact we claim there are no solutions to the Seiberg-Witten equations on $X$: Otherwise if $L \to (X,g)$ is the characteristic line bundle supporting a solution $(A,\psi)$, then the pull-back pair $(\tilde{A},\tilde{\psi})$ is a solution for the pull-back line bundle $\tilde{L} \to \tilde{X}$ with the pull-back $\text{Spin}^c$ structure, hence

$$0 \leq \dim \mathcal{M}_L(\tilde{X}) = \frac{1}{4} c_1^2(\tilde{L}) - \frac{1}{4} (3\sigma(\tilde{X}) + 2\chi(\tilde{X}))$$

But $\tilde{X}$ being a minimal Kähler surface of general type $3\sigma(\tilde{X}) + 2\chi(\tilde{X}) = K^2_\tilde{X} > 0$, hence $c_1^2(\tilde{L}) > 0$. This implies that $(\tilde{A},\tilde{\psi})$ must be an irreducible solution (i.e. $\psi \neq 0$), otherwise $F^+_{\tilde{A}} = 0$ would imply $c_1^2(\tilde{L}) < 0$. By 13.35 the nonzero solution $\psi = \alpha u_0 + \beta$ must have either one of $\alpha$ or $\beta$ is zero (so the other one is nonzero), and since $\tilde{\omega} \wedge \tilde{\omega}$ is the volume element:

$$\tilde{\omega}.c_1(\tilde{L}) = \frac{i}{2\pi} \int \tilde{\omega} \wedge F^+_{\tilde{A}} = \frac{i}{2\pi} \int \tilde{\omega} \wedge (\frac{|\beta|^2 - |\alpha|^2}{2}) \tilde{\omega} \neq 0$$

But since $\sigma^*(\tilde{\omega}) = -\tilde{\omega}$, $\sigma^* c_1(\tilde{L}) = c_1(\tilde{L})$, and $\sigma$ is an orientation preserving map we get a contradiction

$$\tilde{\omega}.c_1(\tilde{L}) = \sigma^*(\tilde{\omega}.c_1(\tilde{L})) = -\tilde{\omega}.c_1(\tilde{L}) \quad \square$$

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14.3 Cappell-Shaneson homotopy spheres

In [CS] Cappell and Shaneson constructed a family of homotopy spheres \( \Sigma_m \), \( m = 0, 1, \ldots \), some of them covering exotic \( \mathbb{R}P^4 \)'s. They asked whether they are diffeomorphic to \( S^4 \). \( \Sigma_m \) is obtained by first taking the mapping torus \( M(f) \) of the punctured 3-torus \( T_0^3 \) with a diffeomorphism \( f : T_0^3 \to T_0^3 \) induced by an integral matrix \( C_m \in SL(3, \mathbb{Z}) \)

\[
C_m = \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & m + 1
\end{pmatrix}
\] (14.2)

and then by gluing it to a \( S^2 \times B^2 \) with the nontrivial diffeomorphism of \( S^2 \times S^1 \) along their common boundaries. By [ARu] \( C_0 \) is conjugate to the matrix \( B \) of 4.2. In [AK1] it was shown that \( \Sigma_0 \) is obtained from \( S^4 \) by Gluck twisting (Section 6.1). Already in Section 4.5 a handlebody of the mapping torus \( M(f) \) was constructed. So Figure 14.2 describes \( \Sigma_0 \), which is just the Figure 4.22 drawn in circle-with-dot notation. The extra 1-framed 2-handle linking 1-handle \( A \) describes the \( S^2 \times B^2 \) part of \( \Sigma_0 = M(f) \sim S^2 \times B^2 \).

Figure 14.2: \( \Sigma_0 \)

It turns out all \( \Sigma_m \) are diffeomorphic to \( S^4 \) (\( m = 0 \) case in [G3], and all other cases in [A5]). Both proofs are based on the basic Proposition 14.3 below, proved by the techniques developed earlier. Let us go through it. First let \( M = \Sigma_0 - B^4 \) be the punctured homotopy sphere \( \Sigma_0 \), clearly proving \( M = B^4 \) is equivalent to proving \( \Sigma_0 = S^4 \).
Proposition 14.3. ([AK4]) The Figure 14.3 which consists of two 1-handles and two 2-handles describes a handlebody of the homotopy ball $M$. Furthermore, $M$ smoothly imbeds into $S^4$ (Hence by Schoenflies theorem [Bn], [Maz] it is homeomorphic to $B^4$).
Having identified the framings, we go back to Figure 14.2 and obtain Figure 14.6 by first sliding $b_1$ over $a_1$, $b_3$ over $a_3$ (thereby changing their framings) and then drawing the resulting figure in the circle-with-dot notation.

Then by canceling the 2-handle $b_2$ with the middle 1-handle in Figure 14.6 we get Figure 14.7, and in that figure by canceling the 2-handle $b_3$ with the left 1-handle we get Figure 14.8, and then by an isotopy we get Figure 14.9. Then finally in Figure 14.9 by canceling 2-handle $b_1$ with the 1-handle next to it we obtain Figure 14.10. Then by further isotopies we arrive to the first picture of Figure 14.11. The dotted lines in the Figures 14.10 and 14.11 signify the “ribbon move” of the ribbon 1-handle.
14.3 Cappell-Shaneson homotopy spheres

Figure 14.7

Figure 14.8

Figure 14.9
Similar to the Figure 1.20 (and Exercises 1.8) the Figure 14.11 describes an imbedding $S^2 \rightarrow S^4$ consisting of the union of two ribbon disks (one is in the upper hemisphere the other is in the lower hemisphere) meeting along the common boundary ribbon knot $K$. The small 1-framed linking circle in the figures shows that this 2-sphere is being Gluck-twisted (Section 6.1). Now we need to Gluck twist $S^4$ along this imbedding $S^2 \rightarrow S^4$. 

Figure 14.10

Figure 14.11
The first two pictures of Figure 14.12 is the Gluck twisting of $S^4$ along this 2-sphere. In the third picture we are introducing a canceling $2/3$ handle pair, for a future convenience (check that the 2-handle of this pair is attached to the unknot).

![Figure 14.12](image1)

By sliding over the new 2-handle, we can unlink the clasp of one of the 1-framed 2-handles from the rest (this changes its framing to $-1$), as shown in the first picture of Figure 14.13. Now clearly the $-1$ framed 2-handle is free to cancel the 1-handle. So the first picture of Figure 14.13 can be viewed as a handlebody consisting of a pair of 2-handles and a pair of 3-handles. Now we immediately turn this handlebody upside down (by using the method of Section 3.1) and obtain a handlebody of $M$ consisting of a pair of 1-handles and a pair of 2-handles.

![Figure 14.13](image2)
For this, we take the dual circles of the 2-handles (as indicated in the first picture of Figure 14.13) and find any diffeomorphism from the boundary to the boundary of \((S^1 \times B^3) \# (S^1 \times B^3)\) and attach 2-handles to \((S^1 \times B^3) \# (S^1 \times B^3)\) along the images of these dual circles. The second and the third pictures of Figure 14.13 are obtained by first blowing up-sliding-blowing down operations, and then by a handle slide. By the similar steps we obtain the first second and third pictures of Figure 14.14. Then finally in the last picture, we blow down ±1 framed circles to arrive Figure 14.3, which was our goal. The proof of the last part of this proposition is in the following exercise.

\[\begin{array}{c}
\includegraphics{figure14.14.png}
\end{array}\]

**Exercise 14.1.** (a) Show that \(M\) imbeds into \(S^4\) (Hint attached two 2-handles to \(M\) as indicated in Figure 14.15 to show that the handlebody collapses to become \(S^4\))

(b) Show as 2-complex \(M\) has the fundamental group presentation

\[\pi_1(M) = \{x, y \mid xyx = yxy, x^4 = y^5\}\] (a presentation of the trivial group)

\[\begin{array}{c}
\includegraphics{figure14.15.png}
\end{array}\]

**Remark 14.4.** By using Proposition 14.3 one can prove \(M\) is in fact diffeomorphic to \(B^4\) not just homomorphic to it (see [G3] for the first proof, and [A21] for an easier recent proof).

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In the general case of \( m > 0 \), imitating the above steps gives Figure 14.16, which is the handlebody of \( M_m = \Sigma_m - B^4 \), where \( \Sigma_m = M(f) \cup S^2 \times B^2 \) and \( f \) is the diffeomorphism corresponding to the matrix \( C_m \) (see [G1]).

\[
\begin{align*}
M_m &= \Sigma_m - B^4
\end{align*}
\]

\[\text{Figure 14.16: } M_m = \Sigma_m - B^4\]

**Theorem 14.5.** \( \Sigma_m \) is diffeomorphic to \( S^4 \) ([A5])

**Proof.** We first find a specific boundary diffeomorphism \( \partial M_m \cong S^3 \).

\[\text{Figure 14.17}\]

This diffeomorphism shows that the \( \alpha \) and the \( \beta \) circles on the boundary of the Figure 14.18 are isotopic to each other, in fact each are 1-framed unknots in \( S^3 \).
Hence by attaching a $-1$ framed 2-handle to either $\alpha$, or to $\beta$ gives $S^1 \times S^2$, which we can cancel it immediately with a 3-handle, i.e. we get diffeomorphisms of the handlebodies (the second and the third handlebodies have 3-handles):

$$\Sigma_m \approx \Sigma_m + \alpha^{-1} \approx \Sigma_m + \beta^{-1}$$

Now observe that by attaching a $-1$ framed 2-handle to $\Sigma_m$ along $\beta$, and then by sliding the other 2-handle going through it, as shown in Figure 14.19 we get $\Sigma_{m-1}$ with a 2-handle attached to $\alpha$ with $-1$ framing, so we have:

$$\Sigma_m + \beta^{-1} \approx \Sigma_{m-1} + \alpha^{-1}$$

Hence we have diffeomorphisms $\Sigma_m \approx \Sigma_{m-1} \ldots \approx \Sigma_0 \approx S^4$
14.4 Some small closed exotic manifolds

By techniques developed in the earlier chapters, here we will construct handlebodies of some closed symplectic 4-manifolds built by A.Akhmedov and D.Park in [AP1], [AP2]. They are closed smooth 4-manifolds $M$ and $N$, which are exotic copies of the standard manifolds $\mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}^2}$, respectively. We will construct their handlebodies, from which it will follow that they are homotopy equivalent to (and hence by [F] homemorphic to) the standard manifolds (cf [A9], [A10]). They are symplectic because they are obtained by symplectic gluing process from basic symplectic pieces, and they are exotic because their Kodaira dimensions (a symplectic invariant defined in [L]) are different from that of the standard manifolds [HL], [LZ].

14.4.1 An exotic $\mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2}$

$M$ is obtained from two codimension zero pieces, glued along their common bondaries:

$$M = \tilde{E}_0 \sim_\partial \tilde{E}_2$$

To construct the pieces, we start with the product of a genus 2 surface and the torus $E = \Sigma_2 \times T^2$, and let $E_0 = \Sigma_2 \times T^2$ be the punctured $E$. Figure 4.10 shows $E_0$ and the adiffeomorphism $f: \partial E_0 \to \Sigma_2 \times S^1$. As explained in Section 3.2, to describe the diffeomorphism $f$ we are indicating where the “basic” arcs of the figure are thrown by $f$. Now let $<a_1, b_1, a_2, b_2>$ and $<C, D>$ be the standard circles generating the first homology of $\Sigma_2$ and $T^2$ respectively, i.e. the cores of the 1-handles (as indicated at the top of Figure 14.20). Let $\tilde{E}_0$ be the manifold obtained from $E_0$ by doing 1-log transforms to the four subtori $(a_1 \times C, a_1), (b_1 \times C, b_1), (a_2 \times C, C), (a_2 \times D, D)$ (see Section 6.3).

**Lemma 14.6.** The handlebody of $\tilde{E}_0$ and the diffeomorphism $f: \partial \tilde{E}_0 \to \Sigma_2 \times S^1$ is as shown in Figure 14.20.

**Proof.** By performing the indicated handle slide to $E_0$ (indicated by dotted arrow) in Figure 14.21, we obtain a second equivalent picture of $E_0$. By performing 1-log transforms to Figure 14.21 along $(a_1 \times C, a_1), (b_1 \times C, b_1)$ (as described in Section 6.3) we obtain the first picture of Figure 14.22, and then by handle slides obtain the second picture. First by an isotopy then a handle slide to Figure 14.22 we obtain the first and second pictures in Figure 14.23. By a further isotopy we obtain the first picture of Figure 14.24, and then by 1-log transforms to $(a_2 \times C, C), (a_2 \times D, D)$ we obtain the second picture in Figure 14.24. By introducing canceling handle pairs we express this last picture by a simpler looking first picture of Figure 14.25. Then by indicated handle slides we obtain the second picture of Figure 14.25, which is $\tilde{E}_0$ as drawn in Figure 14.20. \qed
Figure 14.20: $f : \partial E_0 \rightarrow \Sigma_2 \times S^1$ and $f : \partial \tilde{E}_0 \rightarrow \Sigma_2 \times S^1$
Figure 14.21: Handle slide (the pair of thick arrows in the second picture indicate where we will perform Log transforms next)
Figure 14.22: First we performed Log transforms along \((a_1 \times C, a_1), (b_1 \times C, b_1)\), then handle slides (thick arrows indicate where we will perform Luttinger surgeries next)
Figure 14.23: Isotopy and a handle slides
Figure 14.24: Log transforms along \((a_2 \times C, C), (a_2 \times D, D)\)
Figure 14.25: More isotopies and handle slides and getting $\tilde{E}_0$
To built the other piece, let $K \subset S^3$ be the trefoil knot, and $S^3_0(K)$ be the 3-manifold obtained doing 0-surgery to $S^3$ along $K$. Being a fibered knot, $K$ induces a fibration $T^2 \to S^3_0(K) \to S^1$ and the fibration

$$T^2 \to S^3_0(K) \times S^1 \to T^2$$

Let $T^2_1$ and $T^2_2$ be the vertical (fiber) and the horizontal (section) tori of this fibration, intersecting at one point $p$. Smoothing $T^2_1 \cup T^2_2$ at $p$ gives an imbedded genus 2 surface with self intersection 2, so by blowing up the total space twice (at points on this surface) we get a genus 2 surface $\Sigma_2 \subset \left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2$ with trivial normal bundle. Define:

$$\tilde{E}_2 = \left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2 - \Sigma_2 \times D^2$$

Now to see $\tilde{E}_2$ we construct a handlebody of $\left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2$ so that $\Sigma_2 \times D^2$ is clearly visible inside. This process is explained in the steps of Figure 14.26:

1. The first picture is $S^3_0(K) \times S^1$ (Section 6.4). In this picture the horizontal $T_2 \times D^2$ is visible, but not the vertical torus $T^2_1$ (which consist of the Seifet surface of $K$ capped off by the 2-handle given by the zero framed trefoil). We redraw the first picture so that both vertical and horizontal tori are visible (reader can check this by canceling 1- and 2- handle pairs from the second picture to obtain the first picture).

2. In $\left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2$ by sliding the 2-handle of $T_1$ over the 2-handle of $T_2$ (and by sliding over the two $\mathbb{CP}^1$'s of the $\mathbb{CP}^2$'s) we obtained a picture of the imbedding $\Sigma_2 \times D^2 \subset \left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2$.

3. The rest of the steps (4) and (5) are isotopies and handle slides (indicated by dotted arrows) to obtain the last picture $\left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2$, where $\Sigma_2 \times D^2$ is clearly visible inside.

Figure 14.27 is the combination of the last picture of Figure 14.26 and Figure 14.20, except in the figure by using Exercise 1.8 we replaced the ribbon complement by a pair of 1-handles and a 2-handle. We now want to remove this $\Sigma_2 \times D^2$ from the handlebody of $\left(S^3_0(K) \times S^1\right) \# 2\mathbb{CP}^2$ and replace it with $\tilde{E}_0$ by the boundary identification. The arcs in Figure 14.20 and Figure 14.27 describe the diffeomorphism $f$, which guides us how to do the boundary identification, resulting with the handlebody of $M$ in Figure 14.28.
14.4 Some small closed exotic manifolds

Figure 14.26: \((S^3_0(K) \times S^1) \# 2\mathbb{CP}^2\)
Figure 14.27: Getting ready to replace $\Sigma_2 \times D^2$ with $\tilde{E}_0$
Figure 14.28: $M$
14.4.2 An exotic $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$

Akhmedov-Park’s other manifold $N$ consists of two codimension zero pieces glued along the common boundaries: $\tilde{E}_0$ (which is defined in the previous section) and $\tilde{E}_1$.

$$N = \tilde{E}_0 \cup \tilde{E}_1$$

$\tilde{E}_1$ defined as follows: As before let $S^3_0(K)$ be the 3-manifold obtained by 0-surgering $S^3$ along trefoil knot $K$. Consider the induced fibration $T^2 \to S^3_0(K) \times S^1 \to T^2$ of 14.3. Let $T^2_1$ and $T^2_2$ be the vertical (fiber) and the horizontal (section) tori of this fibration, transversally intersecting one point $p$. The degree 2 map $S^1 \to S^1$ induces an imbedding $S^1 \times D^3 \to S^1 \times D^3$, and by using one of the circle factors of $T^2$ this induces an imbedding $\rho : T^2 \times D^2 \to T^2 \times D^2$. The torus $T^2_3 := \rho(T^2 \times 0) \subset T^2_1 \times D^2$ intersects a normal disk $D^2$ of $T^2_1$ transversally at two points $\{p_1, p_2\}$, hence it intersects $T^2_2$ transversally at $\{p_1, p_2\}$. Therefore by blowing up the total space at $p_1$, we get a pair of imbedded pair tori $T^2_3 \cup T^2_2 \subset (S^3_0(K) \times S^1) \# \mathbb{C}P^2$, each with self intersection $-1$, and transversally intersecting each other at one point $p_2$. Smoothing $T^2_2 \cup T^2_3$ at $p_2$ gives a genus 2 surface $\Sigma_2 \subset (S^3_0(K) \times S^1) \# \mathbb{C}P^2$ with trivial normal bundle $\Sigma_2 \times D^2$, then let:

$$\tilde{E}_1 = (S^3_0(K) \times S^1) \# \mathbb{C}P^2 - \Sigma_2 \times D^2$$

![Diagram](Figure 14.29)

The first picture of Figure 14.30 is $S^3_0(K) \times S^1$ (Figure 6.12), where in this picture the horizontal $T_2 \times D^2$ is visible, but not the vertical torus $T^2_1$ (which consists of the Seifert surface of $K$ capped off by the 2-handle bounded by the trefoil knot). In the second picture of Figure 14.30 as before, we redraw this handlebody so that both the vertical and the horizontal tori are visible. By an isotopy and creating a canceling 1- and 2- handle pair we obtain the third and the forth pictures of Figure 14.30.
14.4 Some small closed exotic manifolds

Now we want to find an imbedded copy of $T^2 \times D^2$ inside of the neighborhood $T^2_1 \times D^2$ of the vertical torus (indicated by dotted rectangle) imbedded by the degree two map, i.e. we want to locate $T^2_3$. For this, we first start with $S^1 \times D^2$ inside of another $S^1 \times D^2$ imbedded by a degree two map. This the first picture of Figure 14.31 (Heegaard picture). Here the 1-handle of the sub $S^1 \times D^2$ is $B$ and the 1-handle of the ambient $S^1 \times D^2$ is $A$. Crossing this picture $S^1$ we get a handle description of $T^2 \times D^2$ inside of another $T^2 \times D^2$ imbedded by the degree two map, which is the second picture of Figure 14.31. By changing 1-handle notations to circle with dot notation, we get the last picture of Figure 14.31, where the degree two imbedding of $T^2 \times D^2$ inside of $T^2 \times D^2$ is clearly visible. This picture is just $T^2 \times D^2$, drawn in a nonstandard way so that the degree two imbedded copy of the sub $T^2 \times D^2$ is clearly visible inside. Next we want to install this picture inside of the dotted rectangle (i.e. inside of $T^2_1 \times D^2$) in Figure 14.30.
Figure 14.31: Degree 2 imbedding $S^1 \times (S^1 \times D^2) \hookrightarrow S^1 \times (S^1 \times D^2)$

Figure 14.32, describes a concrete diffeomorphism $h$ (in terms of handle slides) from the nonstandard copy of $T^2 \times D^2$ in Figure 14.31 to the standard copy of $T^2 \times D^2$. Then we apply the inverse $h^{-1}$ of this diffeomorphism to the region enclosed by the dotted rectangle in Figure 14.30. This gives the second picture of Figure 14.33. Think of Figure 14.33 as a piece of $S^3_0(K) \times S^1$ (i.e. the region enclosed by dotted rectangle in Figure 14.30). Now we want to work only in this region (i.e. manipulate it by diffeomorphisms), then at the end install it back in Figure 14.30. This will help to simplify our pictures, otherwise we have to carry the rest of whole Figure 14.30 every step of the way, even though we are only working inside of $T^2_1 \times D^2$.

After blowing up the second picture of Figure 14.33 at the indicated point we get the first picture of 14.34, i.e. we performed the operation

$S^3_0(K) \times S^1 \mapsto (S^3_0(K) \times S^1) \# \overline{CP}^2$

(actually the figure shows only piece of it). The thick arrow in Figure 14.33 indicates where we performed the blowing up operation. Then by isotopies, handle slides (indicated by dotted arrows), and handle cancelations we go from first to the last picture of Figure 14.34. By installing this last picture inside of the dotted square in Figure 14.30 we obtain the whole picture of $(S^3_0(K) \times S^1) \# \overline{CP}^2$, which is Figure 14.35 (two pictures are the same, in the second one we draw the slice 1-handle as a pair standard 1-handles and a 2-handle as discussed in Exercises 1.8), where we see clearly the imbedded copy $\Sigma_2 \times D^2 \subset (S^3_0(K) \times S^1) \# \overline{CP}^2$ discussed in the introduction. To obtain $N$ we need to remove this copy of $\Sigma_2 \times D^2$ from inside of $(S^3_0(K) \times S^1) \# \overline{CP}^2$ and replace it with $\tilde{E}_0$. The arcs in Figure 14.36 (describing the diffeomorphism $f$) show us how to do this, resulting with the handlebody picture of $M$ in Figure 14.37. Here we have used the convention that, when not indicated the framings of the 2- handles are zero.
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Figure 14.32: Checking the boundary diffeomorphism

Blow up here

Figure 14.33: $T^2 \times D^2$ containing a degree 2 imbedded copy of itself, with some reference arcs drawn to describe the imbedding $T^2 \times D^2 \subset S_0^3(K) \times S^1$
Figure 14.34: Blow up \( (T^2 \times D^2) \rightarrow (T^2 \times D^2) \# CP^2 \), then simplify.
Figure 14.35: Installing \((T^2 \times D^2) \# \mathbb{C}P^2\) (Figure 14.34) into Figure 14.30
Figure 14.36: Getting ready to replace $\Sigma_2 \times D^2$ with $\tilde{E}_1$
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Figure 14.37: $N$
14 Some applications
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