Progress on the Curvature Problem

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April 7, 2002

Abstract

This writeup presents the known results about the curvature problem: the proof for the $n = 2$ case, and the argument generalizing the R^2 result to R^n when $w \geq 1$.

1 The Conjecture

The conjecture we are studying comes out of the Hutchings theory of component bounds on double bubbles in general dimensions. The basic setup is illustrated in Figure 1.

Note: All the figures and other related materials are available on the Web at

http://www.math.msu.edu/˜dfuter/research/curvature/

Conjecture 1.1. *In R^n (with $n \geq 2$), let H_0, H_1, and H_2, respectively, denote the mean curvature of a sphere of volume w, a sphere of volume $w + 1$, and the exterior of the second region of the standard double bubble of volumes $1, w$. Then*

$$H_2 > \frac{H_0 + H_1}{2} .$$

In studying this problem, it is convenient to divide the double bubble of volumes 1, w into four regions separated by pieces of the spheres. We call these regions R_0, R_1, R_2, and R_3. Thus $R_0 \cup R_1$ is the bubble of volume w and $R_2 \cup R_3$ is the bubble of volume 1. Regions R_2 and R_3 are of particular interest to us, so we call their volumes A and B, respectively. We usually think think of volumes A and $B = 1 - A$ as functions $A_n(w)$ and $B_n(w)$, where n indicates dimension.

It turns out to be convenent to rephrase the problem in terms of w and $A_n(w)$ rather than in terms of curvature. To that end, let $V(n)$ be the volume of the n-dimensional unit ball. (This can be calculated in closed form using the Gamma function. It is interesting to note, and occasionally relevant to our estimates, that $V(n)$ reaches a maximum at $n = 5$, and decreases steadily thereafter.) Because mean curvature is just the reciprocal of the radius, we have

$$H_0 = (V(n)/w)^{1/n} \qquad H_1 = (V(n)/(w+1))^{1/n} \qquad H_2 = (V(n)/(w+A_n))^{1/n}$$

Thus, substituting this into the conjectured inequality and solving for A_n, the conjecture is equivalent to the statement that

$$A_n(w) < M_n(w) \equiv 2^n \frac{w(w+1)}{(w^{1/n} + (w+1)^{1/n})^n} - w \, .$$

Thus we have an explicitly defined function $M_n(w)$ expressing the bound that $A_n(w)$ must satisfy for the conjecture to hold. Before we proceed with the task of comparing the two functions, the following lemmas provide an appreciation of how similar they are.

Lemma 1.2. *For each n, the function $A_n(w)$ is strictly increasing in w. As $w \to 0$, it shrinks to 0; as $w \to \infty$, it is asymptotic to $1/2$.*

Proof: This proof is similar in spirit to Frank Morgan's proof that only one standard double bubble exists for every pair of volumes. Let $r_n(w)$ be the ratio between the radii of the spherical caps outside regions R_0 and R_3, respectively. Now, $r_n(w)$ must increase monotonically with w – otherwise there will be two double bubbles with radii in the same proportion ($r_n(w) = r_n(w')$) but disproportional volumes ($w/1 \neq w'/1$). This would violate the scaling symmetry of R^n.

Now, let us focus our attention on the the sphere S containing regions R_1, R_2, and R_3, and on one particular point where the spherical caps meet at 120° angles. Consider what happens to S as we vary the other sphere but keep the intersection tied to that point. As w increases, the angle is conserved but the ratio $r_n(w)$ goes up – so regions R_1 and R_2 take up a bigger and bigger portion of this sphere. (See Figure 2.) So $B_n(w)$, the volume of R_3, is decreasing in w, and thus $A_n(w) = 1 - B_n(w)$ is increasing.

As $w \to 0$, the bubble of volume w looks like a small lens on a vastly larger sphere. (See Figure 3.) Thus the sphere containing this bubble in addition to region R_2 grows smaller and smaller – forcing $A_n(w)$, the volume of R_2 to approach 0. (In fact, the ratio $A_n(w)/w$ approaches a constant – more on this later.) As $w \to \infty$, the picture is reversed: now, the bubble of volume 1 is a lens on a much larger sphere. Regions R_2 and R_3 are symmetric with respect to inversion in this sphere; thus, in the limit, they are reflections of one another in a flat plane and have equal volume. So $A_n(w) \to B_n(w) = 1 - A_n(w)$, and thus $A_n(w) \to 1/2$. □

Lemma 1.3. *Let $m < n$. Then $M_m(w) < M_n(w)$ for all w.*

Proof: For this proof, it is convenient to rewrite

$$M_n(w) = \frac{w(w+1)}{\left(\frac{w^{1/n}+(w+1)^{1/n}}{2}\right)^n} - w \, .$$

Thus the only difference as we vary n is in the averages in the denominator. Now, since $m < n$, $f(x) = x^{n/m}$ is a strictly convex function on $[0, \infty)$. So, for all distinct $x, y \geq 0$,

$$\left(\frac{x+y}{2}\right)^{n/m} < \frac{x^{n/m} + y^{n/m}}{2}.$$

Let $x = w^{1/n}$ and $y = (w+1)^{1/n}$. Then

$$\begin{aligned}
\left(\frac{w^{1/n} + (w+1)^{1/n}}{2}\right)^{n/m} &< \frac{w^{1/m} + (w+1)^{1/m}}{2}\,, \\
\left(\frac{w^{1/n} + (w+1)^{1/n}}{2}\right)^{n} &< \left(\frac{w^{1/m} + (w+1)^{1/m}}{2}\right)^{m}\,,
\end{aligned}$$

and thus $M_m(w) < M_n(w)$. $\square$

Lemma 1.4. *$M_n(w)$ has the same limits as $A_n(w)$: it approaches 0 as $w \to 0$ and $1/2$ as $w \to \infty$.*

Proof: Observe that

$$M_n(w) = w\left(2^n \frac{w+1}{(w^{1/n} + (w+1)^{1/n})^n} - 1\right)\,,$$

and thus clearly vanishes as $w \to 0$. Now, to compute the limit of $M_n(w)$ as $w \to \infty$, we will need to squeeze it between two other functions. We know, by Lemma 1.3, that $M_1(w) < M_n(w)$ for $n \geq 2$. For an upper bound on $M_n(w)$, recall that the geometric mean is always lower than the arithmetic. Thus

$$\sqrt{w^{1/n}(w+1)^{1/n}} < \frac{w^{1/n} + (w+1)^{1/n}}{2}\,, \text{ so } \sqrt{w(w+1)} < \left(\frac{w^{1/n} + (w+1)^{1/n}}{2}\right)^n.$$

Therefore we have

$$\begin{aligned}
M_1(w) = \frac{w(w+1)}{\frac{w+(w+1)}{2}} - w \;&<\; M_n(w) \;<\; \frac{w(w+1)}{\sqrt{w(w+1)}} - w \\
\frac{2w(w+1)}{2w+1} - w \;&<\; M_n(w) \;<\; \sqrt{w^2+w} - w \\
\frac{2w^2 + 2w - 2w^2 - w}{2w+1} \;&<\; M_n(w) \;<\; \sqrt{w^2 + w + \frac{1}{4}} - w \\
\frac{w}{2w+1} \;&<\; M_n(w) \;<\; w + \frac{1}{2} - w \\
\frac{w}{2w+1} \;&<\; M_n(w) \;<\; \frac{1}{2}\,.
\end{aligned}$$

Thus $\lim_{w \to \infty} M_n(w) = 1/2$. $\square$

Lemma 1.5. *For each n, the function $M_n(w)$ is strictly increasing and concave down.*

Proof: We need to check that $M_n'(w) > 0$ and $M_n''(w) < 0$.

$$\begin{aligned}
M_n'(w) &= 2^n \frac{(w^{1/n} + (w{+}1)^{1/n})^n (2w{+}1)}{(w^{1/n} + (w{+}1)^{1/n})^{2n}} - 1 \\
&\quad - 2^n \frac{w(w{+}1) \cdot n(w^{1/n} + (w{+}1)^{1/n})^{n-1} \cdot \frac{1}{n}(w^{1/n-1} + (w{+}1)^{1/n-1})}{(w^{1/n} + (w{+}1)^{1/n})^{2n}} \\
&= 2^n \frac{(w^{1/n} + (w+1)^{1/n})(w + (w+1))}{(w^{1/n} + (w+1)^{1/n})^{n+1}} \\
&\qquad - 2^n \frac{w(w+1)(w^{1/n-1} + (w+1)^{1/n-1})}{(w^{1/n} + (w+1)^{1/n})^{n+1}} - 1 \\
&= 2^n \frac{w^{1+1/n} + w(w+1)^{1/n} + (w+1)w^{1/n} + (w+1)^{1+1/n}}{(w^{1/n} + (w+1)^{1/n})^{n+1}} \\
&\qquad - 2^n \frac{(w+1)w^{1/n} - w(w+1)^{1/n}}{(w^{1/n} + (w+1)^{1/n})^{n+1}} - 1 \\
&= 2^n \frac{w^{1+1/n} + (w+1)^{1+1/n}}{(w^{1/n} + (w+1)^{1/n})^{n+1}} - 1\,.
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{w\to\infty} M_n'(w) &= \lim_{w\to\infty} 2^n \frac{(\frac{w}{w+1})^{1+1/n} + 1}{\left((\frac{w}{w+1})^{1/n} + 1\right)^{n+1}} - 1 \\
&= 2^n \frac{2}{2^{n+1}} - 1 \\
&= 0\,.
\end{aligned}$$

Thus, once we establish that $M'(w)$ is decreasing, it will follow that it's everywhere positive.

$$
\begin{aligned}
M_n''(w) &= 2^n \frac{(w^{1/n} + (w{+}1)^{1/n})^{n+1}(1+\frac{1}{n})(w^{1/n} + (w{+}1)^{1/n})}{(w^{1/n} + (w{+}1)^{1/n})^{2n+2}} \\
&\quad - 2^n \frac{(w^{1+1/n}{+}(w{+}1)^{1+1/n}) \cdot (n{+}1)(w^{1/n}{+}(w{+}1)^{1/n})^n \cdot \frac{1}{n}(w^{1/n-1}{+}(w{+}1)^{1/n-1})}{(w^{1/n} + (w{+}1)^{1/n})^{2n+2}} \\
&= 2^n \left(1+\frac{1}{n}\right) \frac{(w^{1/n} + (w{+}1)^{1/n})^2}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&\qquad - 2^n \left(1+\frac{1}{n}\right) \frac{(w^{1+1/n} + (w{+}1)^{1+1/n})(w^{1/n-1} + (w{+}1)^{1/n-1})}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&= 2^n \left(1+\frac{1}{n}\right) \frac{(w^{1/n} + (w{+}1)^{1/n})^2}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&\qquad - 2^n \left(1+\frac{1}{n}\right) \frac{(\frac{w^{1/n}}{w+1} + \frac{(w+1)^{1/n}}{w})((w{+}1)w^{1/n} + w(w{+}1)^{1/n})}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&= 2^n \left(1+\frac{1}{n}\right) \frac{w^{2/n} + 2w^{1/n}(w{+}1)^{1/n} + (w{+}1)^{2/n}}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&\qquad - 2^n \left(1+\frac{1}{n}\right) \frac{w^{2/n} + \frac{w}{w+1}w^{1/n}(w{+}1)^{1/n} + \frac{w+1}{w}w^{1/n}(w{+}1)^{1/n} + (w{+}1)^{2/n}}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&= 2^n \left(1+\frac{1}{n}\right) \frac{2w^{1/n}(w{+}1)^{1/n} - \frac{w}{w+1}w^{1/n}(w{+}1)^{1/n} - \frac{w+1}{w}w^{1/n}(w{+}1)^{1/n}}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&= 2^n \left(1+\frac{1}{n}\right) \frac{w^{1/n}(w{+}1)^{1/n}(2 - \frac{w}{w+1} - \frac{w+1}{w})}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&= 2^n \left(1+\frac{1}{n}\right) \frac{w^{1/n}(w{+}1)^{1/n}\frac{-1}{w(w+1)}}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&= -2^n \left(1+\frac{1}{n}\right) \frac{w^{1/n-1}(w{+}1)^{1/n-1}}{(w^{1/n} + (w{+}1)^{1/n})^{n+2}} \\
&< 0\,.
\end{aligned}
$$

□

2 Asymptotic analysis for small w

Because the functions $A_n(w)$ and $M_n(w)$ are so similar in their asymptotic behavior, we need to analyze carefully the intervals when w is very small and very large. In this section, we present the proof that $A_n(w) < M_n(w)$ for sufficiently small w; the proof for large w in the case $n = 2$ will come in the next section.

Lemma 2.1.

$$M_n(w) \geq (2^n - 1)w - n2^{n-1}w^{1+1/n}\,.$$

Proof: Taylor's Theorem implies that

$$M_n(w) = M_n(0) + M_n'(0)w + \frac{1}{2}\int_0^w M_n''(t)(w-t)\,dt\,.$$

From Lemmas 1.4 and 1.5, we have $M_n(0) = 0$ and $M_n'(0) = 2^n - 1$. Also,

$$\begin{aligned} M_n''(w) &= -2^n\left(1+\frac{1}{n}\right)\frac{w^{1/n-1}}{(w^{1/n}+(w{+}1)^{1/n})^{n+2}(w{+}1)^{1-1/n}} \\ &\geq -2^n\left(1+\frac{1}{n}\right)w^{1/n-1}\,, \end{aligned}$$

since $(w^{1/n}+(w{+}1)^{1/n})^{n+2}(w{+}1)^{1-1/n} \geq 1$ for all w. Thus

$$\begin{aligned} \frac{1}{2}\int_0^w M_n''(t)(w-t)\,dt &\geq \frac{1}{2}\int_0^w -2^n\left(1+\frac{1}{n}\right)t^{1/n-1}(w-t)\,dt \\ &= -2^{n-1}\left(1+\frac{1}{n}\right)\int_0^w t^{1/n-1}w - t^{1/n}\,dt \\ &= -2^{n-1}\frac{n+1}{n}\left[nwt^{1/n} - \frac{n}{n+1}t^{1+1/n}\right]_0^w \\ &= -2^{n-1}\left((n+1)w^{1+1/n} - w^{1+1/n}\right) \\ &= -n\,2^{n-1}w^{1+1/n}\,, \end{aligned}$$

completing the proof. □

Now that we have a lower bound on $M_n(w)$, it would help to have an upper bound on $A_n(w)$. This is obtained with the help of some geometry. When w is very small, the bubble enclosing volume w looks like a lens on a nearly flat surface. (See Figure 3.) Because the boundary surfaces of a double bubble meet at 120° angles, each of the regions R_0 and R_1 looks like a truncated portion of an n-ball; specifically, the part of the ball when (for unit radius) $x_n \geq \frac{1}{2}$. Region R_2, of volume A, fills out the remainder of this n-ball. This picture provides an upper bound on the ratio A/w: as w grows, the bubble of volume w fills out a greater and greater portion of the ball.

Thus, to get an upper bound on the ratio A/w, we need to calculate the volume of this lens in an n-ball and compare it to the volume of the remainder of the ball.

Lemma 2.2.

$$A_n(w) < n\left(\frac{2}{\sqrt{3}}\right)^{n-1} w\,.$$

Proof: Because we are calculating the ratio between the volumes of two different portions of a ball, we may suppose that the ball has unit radius. We can compute the n-dimensional volume $L(n)$ of the lens by integrating by cylindrical shells. (See Figure 4.) Each of these cylindrical shells is an interval times an $(n-1)$-sphere, whose "surface area" ($(n-2)$-dimensional measure) for radius r is $(n-1)V(n-1)r^{n-2}$. Then

$$\begin{aligned}
L(n) &= 2\int_0^{\sqrt{3}/2}(n-1)V(n-1)r^{n-2}\left(\sqrt{1-r^2}-\frac{1}{2}\right)dr \\
&\geq 2(n-1)V(n-1)\int_0^{\sqrt{3}/2} r^{n-2}\left(\frac{1}{2}-\frac{1}{\sqrt{3}}r\right)dr \\
&= (n-1)V(n-1)\int_0^{\sqrt{3}/2} r^{n-2}-\frac{2}{\sqrt{3}}r^{n-1}dr \\
&= (n-1)V(n-1)\left[\frac{r^{n-1}}{n-1}-\frac{2}{\sqrt{3}}\frac{r^n}{n}\right]_0^{\sqrt{3}/2} \\
&= V(n-1)\left[r^{n-1}-\frac{n-1}{n}\frac{2}{\sqrt{3}}r^n\right]_0^{\sqrt{3}/2} \\
&= V(n-1)\left(\left(\frac{\sqrt{3}}{2}\right)^{n-1}-\frac{n-1}{n}\left(\frac{\sqrt{3}}{2}\right)^{n-1}\right) \\
&= \frac{V(n-1)}{n}\left(\frac{\sqrt{3}}{2}\right)^{n-1}.
\end{aligned}$$

$$\begin{aligned}
\frac{A_n(w)}{w} &\leq \frac{V(n)-L(n)}{L(n)} \\
&< \frac{V(n)}{L(n)} \\
&\leq n\frac{V(n)}{V(n-1)}\left(\frac{2}{\sqrt{3}}\right)^{n-1} \\
&< n\left(\frac{2}{\sqrt{3}}\right)^{n-1} \quad \text{when } n \geq 6.
\end{aligned}$$

When $n \leq 5$, we can compute the constant $c(n) = (V(n)-L(n))/L(n)$ explicitly from the integral expression for $L(n)$:

n	$c(n)$
2	$\frac{2\pi+3\sqrt{3}}{4\pi-3\sqrt{3}} \approx 1.5575$
3	$\frac{11}{5} = 2.2$
4	$\frac{4\pi+9\sqrt{3}}{8\pi-9\sqrt{3}} \approx 2.9499$
5	$\frac{203}{53} \approx 3.8302$

Thus $A_n(w) < n(2/\sqrt{3})^{n-1} w$ for all n. $\square$

Theorem 2.3.

$$A_n(w) < M_n(w) \text{ whenever } w \le \left(\frac{2}{n} - \frac{1}{n\,2^{n-1}} - \left(\frac{1}{\sqrt{3}}\right)^{n-1} \right)^n .$$

Proof: The proof is just a computation based on the past two lemmas.

$$\begin{aligned}
w &\le \left(\frac{2}{n} - \frac{1}{n\,2^{n-1}} - \left(\frac{1}{\sqrt{3}}\right)^{n-1} \right)^n \\
w^{1/n} &\le \frac{2}{n} - \frac{1}{n\,2^{n-1}} - \left(\frac{1}{\sqrt{3}}\right)^{n-1} \\
n\,2^{n-1} w^{1/n} &\le 2^n - 1 - n\left(\frac{2}{\sqrt{3}}\right)^{n-1} \\
n\left(\frac{2}{\sqrt{3}}\right)^{n-1} &\le (2^n - 1) - n\,2^{n-1} w^{1/n} \\
n\left(\frac{2}{\sqrt{3}}\right)^{n-1} w &\le (2^n - 1)\,w - n\,2^{n-1} w^{1+1/n} \\
A_n(w) &< M_n(w) .
\end{aligned}$$

$\square$

3 The two-dimensional case

Our proof of Conjecture 1.1 for $n = 2$ comes in three pieces. First, we know from Theorem 2.3 that the conjecture holds for sufficiently small volumes. Second, asymptotic analysis at the other end will prove that the conjecture holds for sufficiently large volumes. And finally, the compact interval in between can be checked numerically, relying on the result (Lemmas 1.2 and 1.5) that both M_n and A_n are increasing functions.

3.1 Different parameters and explicit formulae

Since infinity is such an unwieldy notion, it would be preferable to replace w with some other parameter that stays finite as $w \to \infty$. As it happens, not one but two alternate parameters are useful to our purposes here. First, we can rescale the double bubble of volumes w and 1 so that instead it has volumes v and $1 - v$. Now, as w varies from 0 to ∞, v varies from 0 to 1. They are related by the equations

$$v = \frac{w}{w+1} \text{ and } w = \frac{v}{1-v} .$$

Now, we can express the bound $M_2(w)$ in terms of v:

$$\begin{aligned}
M_2 \circ w(v) &= M_2(w(v)) \\
&= w\left(4\frac{w+1}{(\sqrt{w}+\sqrt{w+1})^2}-1\right) \\
&= \frac{v}{1-v}\left(4\frac{\frac{1}{1-v}}{\left(\sqrt{\frac{v}{1-v}}+\sqrt{\frac{1}{1-v}}\right)^2}-1\right) \\
&= \frac{v}{1-v}\left(\frac{4}{(\sqrt{v}+1)^2}-1\right).
\end{aligned}$$

The other parameter, even more useful for this section, is the (oriented) angle θ between the chord and the separating cap in Figure 1. When $w < 1$ and the separating cap bulges into the bubble of volume 1, we say that $\theta < 0$; when $w > 1$ and the separating cap bulges the other way, we say $\theta > 0$. Thus, as w varies from 0 to ∞ (and v varies from 0 to 1), θ varies from $-\pi/3$ to $\pi/3$. (Because w, v and θ are all strictly increasing functions of one another, the monotonicity results of Lemmas 1.2 and 1.5 apply with any parameter.)

The advantage of θ as a parameter is that both A_2 and v can be written explicitly in terms of it, allowing us to compare A_2 and M_2 directly. The computations hinge on the geometrical formula in the following lemma.

Definition 3.1. *For an angle θ, define a function $\varphi(\theta)$ by*

$$\varphi(\theta) = \frac{\theta - \sin\theta\cos\theta}{\sin^2\theta}.$$

Lemma 3.2. *Consider the sector S of a circle subtended by a chord of length $2c$, where the chord meets the circle at internal angle θ. (See Figure 5.) Then the area of S is*

$$a(\theta, c) = c^2\varphi(\theta) = c^2\frac{\theta - \sin\theta\cos\theta}{\sin^2\theta}.$$

Proof: Connect the intersection points of the chord and the circle to the center by a pair of radii; then the length of the radius is $r = c/(\sin\theta)$. (See Figure 5.) Now, S can be described as the wedge between the two radii, minus the triangle of the chord and the two radii. In terms of r, the area of the wedge is $r^2\theta$ and the area of the triangle is $r^2 \sin\theta\cos\theta$. The area of S is the difference. $\square$

Lemma 3.3. *Consider a planar double bubble of areas $w, 1$, where the separating cap meets the chord at oriented angle θ, as above. Then*

$$A_2(\theta) = \frac{\varphi(\theta)+\varphi(\frac{\pi}{3}-\theta)}{\varphi(\theta)+\varphi(\frac{2\pi}{3}-\theta)} \quad \textit{and} \quad v(\theta) = \frac{\varphi(\frac{2\pi}{3}+\theta)-\varphi(\theta)}{\varphi(\frac{2\pi}{3}+\theta)+\varphi(\frac{2\pi}{3}-\theta)}.$$

Proof: We compute the areas using Figure 6. It is evident from the picture that, when $\theta > 0$,

$$A = a(\theta, c) + a(\frac{\pi}{3} - \theta, c) \text{ and } w = a(\frac{2\pi}{3} + \theta, c) - a(\theta, c).$$

When $\theta < 0$, extending the formula of Lemma 3.2 gives a negative expression for $a(\theta, c)$, so in fact the same formulae for A and w still apply. To expand them completely, we need an expression for c in terms of θ. To that end, observe that

$$1 \ = \ A + B \ = \ a(\theta, c) + a(\frac{2\pi}{3} - \theta, c) \ = \ c^2 \left(\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta) \right) \ .$$

Thus

$$c^2 = \frac{1}{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)} \ ,$$

so

$$A = \frac{\varphi(\theta) + \varphi(\frac{\pi}{3} - \theta)}{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)} \text{ and } w = \frac{\varphi(\frac{2\pi}{3} + \theta) - \varphi(\theta)}{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)} \ .$$

From this, we can compute

$$w + 1 \ = \ \frac{\varphi(\frac{2\pi}{3} + \theta) - \varphi(\theta)}{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)} + \frac{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)}{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)} \ = \ \frac{\varphi(\frac{2\pi}{3} + \theta) + \varphi(\frac{2\pi}{3} - \theta)}{\varphi(\theta) + \varphi(\frac{2\pi}{3} - \theta)} \ ,$$

and thus

$$v = \frac{w}{w+1} = \frac{\varphi(\frac{2\pi}{3} + \theta) - \varphi(\theta)}{\varphi(\frac{2\pi}{3} + \theta) + \varphi(\frac{2\pi}{3} - \theta)} \ .$$

□

3.2 Asymptotic analysis for large w

Our plan is to prove that $A_2 < M_2$ when θ is close to $\frac{\pi}{3}$ by bounding their derivatives $\frac{dA}{d\theta}$ and $\frac{dM}{d\theta}$. In order to do that, we need to know more about the building-block function $\varphi(\theta)$.

Lemma 3.4. *$\varphi(\theta)$ is positive, increasing, and concave up on $(0, \pi)$. Specifically,*

$$\varphi'(\theta) = 2\,\frac{\sin\theta - \theta\cos\theta}{\sin^3\theta} \ .$$

Proof: Lemma 3.2 tells us that $\varphi(\theta)$ is the area of a sector $S(\theta)$ of a circle of radius $1/\sin\theta$ cut by a chord of length 2. (See Figure 7.) So $\varphi(\theta)$ is clearly positive. The chord length is fixed, so as θ increases, the sector $S(\theta)$ will grow larger. Thus $\varphi(\theta)$ is increasing.

Now consider the change in area between $S(\theta)$ and $S(\theta+\epsilon)$ for some small ϵ. The difference between the two sectors is a narrow strip along the circumference. Now, keep ϵ fixed and vary θ. For larger θ, the strip is both longer and wider, and thus has larger area. So $\varphi'(\theta)$ is increasing and $\varphi(\theta)$ is concave up.

$$
\begin{aligned}
\varphi'(\theta) &= \frac{\sin^2\theta(1+\sin^2\theta-\cos^2\theta)-(\theta-\sin\theta\cos\theta)(2\sin\theta\cos\theta)}{\sin^4\theta} \\
&= \frac{\sin^2\theta(2\sin^2\theta)-2\theta\sin\theta\cos\theta+2\sin^2\theta\cos^2\theta}{\sin^4\theta} \\
&= \frac{2\sin^2\theta-2\theta\sin\theta\cos\theta}{\sin^4\theta} \\
&= 2\,\frac{\sin\theta-\theta\cos\theta}{\sin^3\theta}\,.
\end{aligned}
$$

□

For computing derivatives of A_2 and M_2 we introduce the following notation: $\alpha=\frac{\pi}{3}-\theta$, $\beta=\frac{2\pi}{3}-\theta$, and $\gamma=\frac{2\pi}{3}+\theta$. With this notaton, we have

$$A_2(\theta)=\frac{\varphi(\theta)+\varphi(\alpha)}{\varphi(\theta)+\varphi(\beta)} \quad\text{and}\quad v(\theta)=\frac{\varphi(\gamma)-\varphi(\theta)}{\varphi(\gamma)+\varphi(\beta)}\,.$$

Lemma 3.5. *When* $\theta\in(1,\frac{\pi}{3})$, $\frac{dA}{d\theta}>\frac{1}{6}$.

Proof: Let $\theta\in(1,\frac{\pi}{3})$. Then

$$
\begin{aligned}
\frac{dA}{d\theta} &= \frac{(\varphi(\theta)+\varphi(\beta))(\varphi'(\theta)-\varphi'(\alpha))-(\varphi(\theta)+\varphi(\alpha))(\varphi'(\theta)-\varphi'(\beta))}{(\varphi(\theta)+\varphi(\beta))^2} \\
&= \frac{(\varphi(\theta)+\varphi(\beta))(\varphi'(\theta)-\varphi'(\alpha))+(\varphi(\theta)+\varphi(\alpha))(\varphi'(\beta)-\varphi'(\theta))}{(\varphi(\theta)+\varphi(\beta))^2} \\
&> \frac{(\varphi(\theta)+\varphi(\beta))(\varphi'(\theta)-\varphi'(\alpha))}{(\varphi(\theta)+\varphi(\beta))^2}\,, \qquad \text{since } \varphi'(\beta)>\varphi'(\theta) \\
&> \frac{2\varphi(\theta)(\varphi'(\theta)-\varphi'(\alpha))}{(2\varphi(\beta))^2}\,, \qquad \text{since } \varphi(\beta)>\varphi(\theta) \\
&> \frac{\varphi(1)(\varphi'(1)-\varphi'(\frac{\pi}{3}-1))}{2\varphi(\frac{2\pi}{3}-1)^2} \\
&\approx 0.1749\,.
\end{aligned}
$$

□

Lemma 3.6. *When* $\theta\in(1,\frac{\pi}{3})$, $0<\frac{dv}{d\theta}<\frac{1}{30}$.

Proof: Let $\theta \in (1, \frac{\pi}{3})$. Since v is increasing, $\frac{dv}{d\theta} > 0$. Also,

$$
\begin{aligned}
\frac{dv}{d\theta} &= \frac{(\varphi(\gamma)+\varphi(\beta))(\varphi'(\gamma)-\varphi'(\theta)) - (\varphi(\gamma)+\varphi(\theta))(\varphi'(\gamma)-\varphi'(\beta))}{(\varphi(\gamma)+\varphi(\beta))^2} \\
&= \frac{\varphi(\gamma)\varphi'(\gamma) - \varphi(\gamma)\varphi'(\theta) + \varphi(\beta)\varphi'(\gamma) - \varphi(\beta)\varphi'(\theta)}{(\varphi(\gamma)+\varphi(\beta))^2} \\
&\qquad + \frac{-\varphi(\gamma)\varphi'(\gamma) + \varphi(\gamma)\varphi'(\beta) + \varphi(\theta)\varphi'(\gamma) - \varphi(\theta)\varphi'(\beta)}{(\varphi(\gamma)+\varphi(\beta))^2} \\
&= \frac{-\varphi(\gamma)\varphi'(\theta) + \varphi(\beta)\varphi'(\gamma) - \varphi(\beta)\varphi'(\theta) + \varphi(\gamma)\varphi'(\beta) + \varphi(\theta)\varphi'(\gamma) - \varphi(\theta)\varphi'(\beta)}{(\varphi(\gamma)+\varphi(\beta))^2} \\
&< \frac{\varphi(\beta)\varphi'(\gamma) + \varphi(\gamma)\varphi'(\beta) + \varphi(\theta)\varphi'(\gamma)}{(\varphi(\gamma)+\varphi(\beta))^2} \\
&< \frac{2\varphi(\beta)\varphi'(\gamma) + \varphi(\gamma)\varphi'(\beta)}{\varphi(\gamma)^2} \\
&= 2\,\frac{\varphi'(\gamma)}{\varphi(\gamma)^2}\,\varphi(\beta) + \frac{\varphi'(\beta)}{\varphi(\gamma)} \\
&< 2\,\frac{\varphi'(\gamma)}{\varphi(\gamma)^2}\,\varphi\!\left(\frac{2\pi}{3}-1\right) + \frac{\varphi'(\frac{2\pi}{3}-1)}{\varphi(\gamma)}\,.
\end{aligned}
$$

We now substitute $\varphi(\frac{2\pi}{3}-1) \approx 0.8698$ and $\varphi'(\frac{2\pi}{3}-1) \approx 0.6673$.

$$
\begin{aligned}
\frac{dv}{d\theta} &< 1.8\,\frac{\varphi'(\gamma)}{\varphi(\gamma)^2} + \frac{1}{\varphi(\gamma)} \\
&= 1.8\,\frac{\sin\gamma - \gamma\cos\gamma}{\sin^3\gamma}\cdot\frac{\sin^4\gamma}{(\gamma-\sin\gamma\cos\gamma)^2} + \frac{\sin^2\gamma}{\gamma-\sin\gamma\cos\gamma} \\
&= 1.8\,\frac{\sin\gamma(\sin\gamma-\gamma\cos\gamma)}{(\gamma-\sin\gamma\cos\gamma)^2} + \frac{\sin^2\gamma}{\gamma-\sin\gamma\cos\gamma} \\
&< \frac{1}{5}\,(\sin\gamma)(\sin\gamma-\gamma\cos\gamma) + \frac{1}{3}\,\sin^2\gamma\,, \qquad \text{since } \gamma-\sin\gamma\cos\gamma > 3 \\
&< \frac{\pi}{5}\,\sin\gamma + \frac{1}{3}\,\sin^2\gamma\,, \qquad \text{since } \sin\gamma-\gamma\cos\gamma < \pi \\
&< \frac{\pi}{5}\,\sin\!\left(\frac{2\pi}{3}+1\right) + \frac{1}{3}\,\sin^2\!\left(\frac{2\pi}{3}+1\right) \\
&\approx 0.0304\,.
\end{aligned}
$$

$\square$

Lemma 3.7. *When $\theta \in (1, \frac{\pi}{3})$, $0 < \frac{dM}{dv} < \frac{3}{8}$.*

Proof: Recall that $M_2(w) = \frac{v}{1-v}(4(1+\sqrt{v})^{-2}-1)$. Thus

$$
\begin{aligned}
\frac{dM}{dv} &= \frac{(1-v)-v(-1)}{(1-v)^2}\left(\frac{4}{(1+\sqrt{v})^2}-1\right)+\frac{v}{1-v}\left(\frac{-8}{(1+\sqrt{v})^3}\cdot\frac{1}{2\sqrt{v}}\right) \\
&= \frac{1}{(1-v)^2}\cdot\frac{4-(1+\sqrt{v})^2}{(1+\sqrt{v})^2} - 4\,\frac{\sqrt{v}}{1-v}\cdot\frac{1}{(1+\sqrt{v})^3} \\
&= \frac{(2+(1+\sqrt{v}))(2-(1+\sqrt{v}))}{(1-\sqrt{v})^2(1+\sqrt{v})^4} - \frac{4\sqrt{v}}{(1-\sqrt{v})(1+\sqrt{v})^4} \\
&= \frac{3+\sqrt{v}}{(1-\sqrt{v})(1+\sqrt{v})^4} - \frac{4\sqrt{v}}{(1-\sqrt{v})(1+\sqrt{v})^4} \\
&= \frac{3-3\sqrt{v}}{(1-\sqrt{v})(1+\sqrt{v})^4} \\
&= \frac{3}{(1+\sqrt{v})^4}\,.
\end{aligned}
$$

Now, when $\theta > 1$, certainly $w > 1$, and thus $v > 1/2$. Therefore

$$\frac{dM}{dv} < \frac{3}{\left(1+\frac{1}{\sqrt{2}}\right)^4} \approx 0.3532\,.$$

□

Theorem 3.8. *When $\theta > 1$, $A_2 < M_2$.*

Proof: By the Fundamental Theorem of Calculus,

$$
\begin{aligned}
A_2(\theta) &= A_2\left(\frac{\pi}{3}\right) - \int_\theta^{\frac{\pi}{3}} \frac{dA}{d\lambda}\,d\lambda\,, \\
M_2\circ w(\theta) &= M_2\circ w\left(\frac{\pi}{3}\right) - \int_\theta^{\frac{\pi}{3}} \frac{dM}{d\lambda}\,d\lambda\,.
\end{aligned}
$$

But by Lemmas 1.2 and 1.4, $A_2(\frac{\pi}{3}) = M_2\circ w(\frac{\pi}{3}) = \frac{1}{2}$. Also, by the lemmas in this section,

$$\frac{dM}{d\lambda} = \frac{dM}{dv}\cdot\frac{dv}{d\lambda} < \frac{3}{8}\cdot\frac{1}{30} < \frac{1}{6} < \frac{dA}{d\lambda}$$

when $\theta \in (1, \frac{\pi}{3})$. Thus $A_2 < M_2$ on this interval. □

3.3 Proof of the conjecture

Theorem 3.9. *Conjecture 1.1 is true when $n = 2$.*

Proof: By Theorem 2.3, $A_2(w) < M_2(w)$ when

$$w \le \left(\frac{2}{2} - \frac{1}{2 \cdot 2^1} - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{3}{4} - \frac{1}{\sqrt{3}}\right)^2 \approx 0.02981\,.$$

Using the formula for $w(\theta)$ from Lemma 3.3, we can check that

$$w < 0.02 \text{ when } \theta < -0.9\,.$$

Also, Theorem 3.8 tells us that $A_2 < M_2$ when $\theta > 1$. Thus the only remaining task is to check the conjecture when $\theta \in [-0.9, 1]$. This is quite easy to do numerically. Lemmas 1.2 and 1.5 tell us that both functions are increasing, so the graph in Figure 8 is in fact rigorous. Alternately, one can find a collection of angles $-0.9 = \theta_0 < \theta_1 < \ldots < \theta_k = 1$ such that $A_2(\theta_i) < M_2 \circ w(\theta_{i-1})$. One such collection is given below.

θ_i	$A(\theta_i)$	$M \circ w(\theta_i)$
$\theta_0 = -0.9$	0.0136	0.0259
$\theta_1 = -0.84$	0.0245	0.0463
$\theta_2 = -0.75$	0.0441	0.0824
$\theta_3 = -0.6$	0.0817	0.1484
$\theta_4 = -0.4$	0.1356	0.2349
$\theta_5 = -0.05$	0.2299	0.3584
$\theta_6 = 0.4$	0.3446	0.4556
$\theta_7 = 0.8$	0.4414	0.4939
$\theta_8 = 1$	0.4888	0.4998

□

4 Centers of mass and inversion in spheres

4.1 Generalized centers of mass.

Now that we have proved Conjecture 1.1 for $n = 2$, we must turn our attention to higher dimensions. Instead of approaching the general problem from scratch, it would be much easier to somehow make use of what we already know. In order to pass from the planar picture to the higher-dimensional one, it becomes important to know the relationship between the volumes of corresponding regions in different dimensions.

Let $G_n \subset R^n$ ($n \ge 3$) be a measurable set invariant under any rotation preserving the x_1-axis; that is, a region of revolution. We can think of G_n as being generated by a subset G_2 of the upper half-plane: G_n is what we get when we "revolve" G_2 about the x-axis. More precisely, each point $(x, y) \in G_2$ corresponds to an $(n-2)$-sphere of radius y in the $(n-1)$-dimensional cross-section of G_n whose first coordinate is x. (See Figure 9.) The surface area ($(n-2)$-dimensional measure) of such a sphere is $(n-1)V(n-1)y^{n-2}$, in terms of the already familiar constant $V(n-1)$ describing the volume of the unit $n-1$ ball. Now, this setup allows us to compute the volume of G_n:

$$\mathrm{Vol}(G_n) = \int_{G_2} (n-1)\, V(n-1)\, y^{n-2}\, dA\,.$$

When $n = 3$, the above setup brings to mind Pappus' Theorem:

$$\mathrm{Vol}(G_3) = 2\pi\, \overline{y}\, \mathrm{Area}(G_2)\,,$$

where $\overline{y}$ is the average distance of G_2 from the x-axis, *i.e.* the y-coordinate of its center of mass. One way to compute the center of mass of a region is with the same integral that we have above:

$$\overline{y} = \frac{\int_{G_2} y\, dA}{\mathrm{Area}(G_2)}\,.$$

This suggests a way to generalize the notion of center of mass to correspond to volumes of higher-dimensional regions of revolution.

Definition 4.1. *Let G be a bounded, measurable subset of the upper half-plane, and let $n \geq 3$. We define the* n**-dimensional center of mass** $c_n(G)$ *by*

$$c_n(G) = \frac{\int_G y^{n-2}\, dA}{Area(G)}\,.$$

Our computations above imply

Lemma 4.2. *Let $G_n \subset R^n$ be a bounded, measurable set invariant under rotations about the x_1-axis. Let G_2 be its generating region in the upper half-plane. Then*

$$Vol(G_n) = (n-1)\, V(n-1)\, c_n(G_2)\, Area(G_2)\,.$$

The following two lemmas are also immediate consequences of the definition of $c_n(G)$.

Lemma 4.3. *Let G and H be disjoint, bounded, measurable subsets of the upper half-plane. Then the n-dimensional center of mass of their union is a weighted average of their centers of mass, the weights being the respective areas. In other words,*

$$c_n(G \cup H) = \frac{Area(G)\, c_n(G) + Area(H)\, c_n(H)}{Area(G \cup H)}\,.$$

Proof: Obvious from the definition. $\square$

Lemma 4.4. *Let G be a bounded, measurable subset of the upper half-plane, and let λG be the image of G under scaling by some factor $\lambda > 0$. Then*

$$c_n(\lambda G) = \lambda^{n-2}\, c_n(G)\,.$$

Proof: Let $G_n \subset R^n$ be the region obtained by revolving G around the x-axis. Scaling by λ increases the area of G by a factor of λ^2 and the volume of G_n by a factor of λ^n. The result now follows from Lemma 4.2. $\square$

4.2 Centers of mass of circular sectors

In our problem, we are specifically concerned with regions bounded between circles in R^2 or spheres in R^n. The following regions turn out to be fundamental building blocks of double bubbles.

Definition 4.5. *Consider a circle whose center lies on the x-axis, and a vertical chord of length 2 through the circle. Let θ be the internal angle between the chord and the circular arc to the right of the chord. (See Figure 10.) Let $G(\theta)$ be the region contained to the right of the chord, inside the circle, and above the x-axis. (For any $\theta \in (0, \pi)$, this construction defines $G(\theta)$ uniquely up to horizontal translation.) For each $n \geq 3$, define a function f_n on $(0, \pi)$ by*

$$f_n(\theta) = c_n(G(\theta)) .$$

Lemma 4.6. *For each $n \geq 3$, $f_n(\theta)$ is strictly increasing in θ.*

Proof: In order to make it easier to compute an explicit formula for $f_n(\theta)$, let us rescale the picture so that $G(\theta)$ is a sector of the unit circle. Since the radius of the original circle is $1/\sin(\theta)$, we need to scale by a factor $\lambda = \sin(\theta)$. After scaling, $\lambda G(\theta)$ is the subset of the unit circle above the x-axis and to the right of the line $x = \cos(\theta)$. By Lemma 3.2, the area of this region is

$$\text{Area}(\lambda G(\theta)) = \frac{1}{2} \sin^2 \theta \, \varphi(\theta) = \frac{\theta - \sin\theta\cos\theta}{2} .$$

Now,

$$\begin{aligned}
f_n(\theta) &= \frac{1}{\lambda^{n-2}} \frac{\int_{\lambda G(\theta)} y^{n-2}\, dA}{\text{Area}(\lambda G(\theta))} \\
&= \frac{2\int_{\cos\theta}^{1} \int_0^{\sqrt{1-x^2}} y^{n-2}\, dy\, dx}{\sin^{n-2}\theta(\theta - \sin\theta\cos\theta)} \\
&= \frac{2\int_{\cos\theta}^{1} \frac{1}{n-1} \sqrt{1-x^2}^{\,n-1}\, dx}{\sin^{n-2}\theta(\theta - \sin\theta\cos\theta)} \\
&= \frac{2}{n-1} \cdot \frac{\int_\theta^0 \sin^{n-1} u(-\sin u)\, du}{\sin^{n-2}\theta(\theta - \sin\theta\cos\theta)} \\
&= \frac{2}{n-1} \cdot \frac{\int_0^\theta \sin^n u\, du}{\sin^{n-2}\theta(\theta - \sin\theta\cos\theta)} .
\end{aligned}$$

To write the derivative of $f_n(\theta)$ explicitly, without any integrals, one needs to use the messy sum formula for $\int_0^\theta \sin^n u\, du$. Instead of doing this, we resort to a trick. Define a function $g_n(\theta)$ by

$$g_n(\theta) = \frac{(\sin^{n+1}\theta)(\theta - \sin\theta\cos\theta)}{\theta(n-2)\cos\theta + n\sin^3\theta - (n-2)\sin\theta} - \int_0^\theta \sin^n(u)\,du$$

It is easy to check that $g_n(\theta)$ is related to $f_n'(\theta)$ by

$$f_n'(\theta) = g_n(\theta) \cdot \frac{2}{n-1} \cdot \frac{\theta(n-2)\cos\theta + n\sin^3\theta - (n-2)\sin\theta}{(\sin^{n-1}\theta)(\theta - \sin\theta\cos\theta)^2}\,.$$

We shall prove that $f_n'(\theta) > 0$ by considering the sign of $g_n(\theta)$ and of the expression multiplying it. In the expression above, the denominator is always positive. However, the numerator

$$\theta\,(n-2)\cos\theta + n\,\sin^3\theta - (n-2)\,\sin\theta$$

changes sign once on $(0,\pi)$: it is positive on $(0, b_n)$ and negative on (b_n, π) for some bound $b_n > \pi/2$. On (b_n,π), $g_n(\theta)$ is negative and the factor multiplying it is also negative, making $f_n'(\theta)$ clearly positive. At $\theta = b_n$, when this expression is 0, $f_n'(b_n)$ is easily seen to be positive by computing it directly. (Thus the fact that $g_n(\theta)$ has a discontinuity at b_n does not concern us.)

On $(0, b_n)$, we need to prove that $g_n(\theta) > 0$; this takes some work. For starters,

$$\begin{aligned}
\lim_{\theta\to 0} g_n(\theta) &= \lim_{\theta\to 0}\left(\frac{(\sin^{n+1}\theta)(\theta - \sin\theta\cos\theta)}{\theta\,(n-2)\cos\theta + n\,\sin^3\theta - (n-2)\,\sin\theta} - \int_0^\theta \sin^n(u)\,du\right)\\
&= \lim_{\theta\to 0}\frac{(\sin^{n+1}\theta)(\theta - \sin\theta\cos\theta)}{\theta\,(n-2)\cos\theta + n\,\sin^3\theta - (n-2)\,\sin\theta}\\
&= 0\,,
\end{aligned}$$

because the numerator has a zero of order at least $n+2 \geq 5$ at 0, and the denominator has a zero of order 3. To complete the proof, we show that $g_n(\theta)$ is strictly increasing for as long as it's continuous; then it will have to be positive for $\theta \in (0, b_n)$.

$$\begin{aligned}
g_n'(\theta) &= \frac{(\theta(n-2)\cos\theta + n\sin^3\theta - (n-2)\sin\theta)(\theta(n+1)\sin^n\theta\cos\theta + (n+3)\sin^{n+3}\theta - (n+1)\sin^{n+1}\theta)}{(\theta\,(n-2)\cos\theta + n\,\sin^3\theta - (n-2)\,\sin\theta)^2}\\
&\quad - \frac{(\sin^{n+1}\theta)(\theta - \sin\theta\cos\theta)(3n\,\sin^2\theta\cos\theta - \theta\,(n-2)\sin\theta)}{(\theta\,(n-2)\cos\theta + n\,\sin^3\theta - (n-2)\,\sin\theta)^2} - \sin^n\theta\,.
\end{aligned}$$

Since we are only concerned with the sign of $g_n'(\theta)$, we may multiply through by the denominators and factor out $\sin^n\theta$. This gives us

$$
\begin{aligned}
h_n(\theta) &= (\theta(n-2)\cos\theta + n\sin^3\theta - (n-2)\sin\theta)(\theta(n+1)\cos\theta + (n+3)\sin^3\theta - (n+1)\sin\theta) \\
&\qquad - (\theta - \sin\theta\cos\theta)(3n\,\sin^3\theta\cos\theta - \theta\,(n-2)\sin^2\theta) \\
&\qquad - (\theta\,(n-2)\cos\theta + n\,\sin^3\theta - (n-2)\,\sin\theta)^2 \\
&= (\theta(n-2)\cos\theta + n\sin^3\theta - (n-2)\sin\theta)(3\theta\cos\theta + 3\sin^3\theta - 3\sin\theta) \\
&\qquad - (\theta - \sin\theta\cos\theta)(3n\,\sin^3\theta\cos\theta - \theta\,(n-2)\sin^2\theta) \\
&= (3n-6)\theta^2\cos^2\theta + 3n\sin^6\theta + (3n-6)\sin^2\theta + (6n-6)\theta\sin^3\theta\cos\theta \\
&\qquad - (6n-12)\theta\sin\theta\cos\theta - (6n-6)\sin^4\theta - 3n\theta\sin^3\theta\cos\theta \\
&\qquad + (n-2)\theta^2\sin^2\theta + 3n\sin^4\theta\cos^2\theta - (n-2)\theta\sin^3\theta\cos\theta \\
&= (3n-6)\theta^2\cos^2\theta + (n-2)\theta^2\sin^2\theta + (2n-4)\theta\sin^3\theta\cos\theta - (6n-12)\theta\sin\theta\cos\theta \\
&\qquad + (3n-6)\sin^2\theta + 3n\sin^6\theta + 3n\sin^4\theta\cos^2\theta - (6n-6)\sin^4\theta \\
&= (n-2)\theta^2 + (2n-4)\theta^2\cos^2\theta + (2n-4)\theta\sin\theta\cos\theta\,(\sin^2\theta - 1) \\
&\qquad - (4n-8)\theta\sin\theta\cos\theta + (3n-6)\sin^2\theta + 3n\sin^4\theta - (6n-6)\sin^4\theta \\
&= (n-2)\theta^2 - (n-2)\theta\sin\theta\cos\theta + (2n-4)\theta^2\cos^2\theta + (2n-4)\theta\sin\theta\cos^3\theta \\
&\qquad - (3n-6)\theta\sin\theta\cos\theta + (3n-6)\sin^2\theta - (3n-6)\sin^4\theta \\
&= (n-2)(\theta^2 - \theta\sin\theta\cos\theta) + (n-2)(2\theta^2\cos^2\theta - 2\theta\sin\theta\cos^3\theta) \\
&\qquad - (n-2)(3\theta\sin\theta\cos\theta - 3\sin^2\theta\cos^2\theta) \\
&= (n-2)(\theta - \sin\theta\cos\theta)(\theta + 2\theta\cos^2\theta - 3\sin\theta\cos\theta)\,.
\end{aligned}
$$

Now, $(n-2)(\theta - \sin\theta\cos\theta) > 0$ whenever $\theta > 0$ and $n \geq 3$, so this term is not a concern. To show that

$$k(\theta) = (\theta + 2\theta\cos^2\theta - 3\sin\theta\cos\theta) > 0\,,$$

observe that this term equals zero at 0. We will show that it increases on $(0, \pi)$, guaranteeing that it will be positive.

$$
\begin{aligned}
k'(\theta) &= 1 + 2\cos^2\theta - 4\theta\sin\theta\cos\theta - 3\cos^2\theta + 3\sin^2\theta \\
&= 4\sin^2\theta - 4\theta\sin\theta\cos\theta \\
&= 4\sin\theta\,(\sin\theta - \theta\cos\theta) \\
&> 0 \text{ when } \theta \in (0, \pi)\,.
\end{aligned}
$$

Thus $h_n(\theta) > 0$ for $n \geq 3$, implying that $g_n(\theta)$ is increasing and therefore positive. This completes the proof. □

4.3 Back to double bubbles: inversion in spheres

Now we have the tools to prove the result that will be useful in generalizing our results about two-dimensional bubbles to higher dimensions. As a motivation for the following theorem, notice that the bubble of volume 1 is the image, under inversion in the separating cap, of the bubble of volume w. Regions R_0 and R_1

are also symmetric with respect to inversion in a sphere, as are regions R_2 and R_3.

Theorem 4.7. *Let $D \subset R^2$ be a closed disk, centered on the positive x-axis, and not containing the origin. Let G be the intersection of the unit disk with the upper half of D. Let H be the image of G under inversion in the unit circle. Then, for each $n \geq 3$,*

$$c_n(G) < c_n(H) .$$

Proof: Suppose first that D is entirely contained in the unit disk. Then G is just a half-disk, and its image H is also a half-disk, congruent to λG for some scaling factor $\lambda > 1$. Then $c_n(G) < c_n(H)$ by Lemma 4.4.

If, on the other hand, D is not contained in the unit disk, we can draw a vertical chord connecting the two points where ∂D intersects the unit circle. This chord divides G into two regions, R and S (See Figure 11.) Let α be the internal angle between the vertical chord and the unit circle, and β the angle between the vertical chord and ∂D. Because inversion is conformal, the internal angle of H at the same vertex is $\alpha + \beta$. The length of the vertical segment between this vertex and the x-axis is $\lambda = \sin(\alpha)$, so we see that R, S, and $S \cup H$ are congruent to $\lambda G(\beta)$, $\lambda G(\alpha)$, and $\lambda G(2\alpha + \beta)$, respectively.

By Lemma 4.6, $f_n(2\alpha+\beta) > f_n(\alpha)$, so $c_n(S \cup H) > c_n(S)$. Since (by Lemma 4.3) $c_n(S \cup H)$ is a weighted average of $c_n(S)$ and $c_n(H)$, this implies that

$$c_n(S) < c_n(S \cup H) < c_n(H) .$$

Applying Lemma 4.6 again, we see that $f_n(\beta) < f_n(2\alpha + \beta)$, and thus

$$c_n(R) < c_n(S \cup H) < c_n(H) .$$

Therefore, since both $c_n(R) < c_n(H)$ and $c_n(S) < c_n(H)$, we can conclude (again by taking weighted averages) that

$$c_n(G) = c_n(R \cup S) < c_n(H) .$$

□

Corollary 4.8. *Let $G_n \subset R^n$ be the intersection of two closed n-balls B_1 and B_2, not containing the center of B_1. Let H_n the the image of G_n under inversion in the boundary of B_1. Let G_2 and H_2 be the "generating regions" for G_n and H_n, respectively, in any half-plane through the centers of B_1 and B_2. (See Figure 12.) Then*

$$\frac{Area(H_2)}{Area(G_2)} < \frac{Vol(H_n)}{Vol(G_n)} .$$

Proof: Translating, rotating, and scaling the whole picture will not change the ratios of areas and volumes. Thus we may assume that B_1 is the unit ball, and that the center of B_2 lies on the positive x_1-axis. Thus $c_n(G_2) < c_n(H_2)$, and the result follows by Lemma 4.2. □

Given these results, it is natural to ask if Theorem 4.7 applies in a more general context than regions bounded between circles. It turns out that if we let G be any measurable subset of the upper half-disk whose closure excludes the origin, and H its image under inversion, then it no longer follows that $c_n(G) < c_n(H)$. Counterexamples exist where G is the union of two half-disks centered on the x-axis. For one specific numerical example, let G consist of the upper halves of $B(0.7, 0.003)$ and $B(0.98, 0.01)$. Then we have

$$c_3(G) \approx 0.0039988 \qquad\qquad c_3(H) \approx 0.0039158\,,$$

violating the desired inequality. Other counterexamples can be constructed that violate the inequality for c_3 and also for c_n with higher values of n. However, in all of the counterexamples known to us, G is either a disjoint union of several pieces, or else a region whose parts are connected by very narrow corridors. These examples lead us to conjecture that the key sufficient condition is convexity.

Conjecture 4.9. *Let G be a closed, convex subset of the upper half of the unit disk, that does not contain the origin. Let H be the image of G under inversion in the unit circle. Then $c_n(G) < c_n(H)$.*

5 Generalizing from R^2 to R^n

To show that Conjecture 1.1 for $n = 2$ implies the conjecure for $n > 2$, we have to consider the cases $w \geq 1$ and $w < 1$ separately. In the former case, we have a proof using Lemma 1.3 and Corollary 4.8; but in the latter case much remains unknown.

5.1 Case 1: $w \geq 1$

Theorem 5.1. *Conjecture 1.1 holds for all $n \geq 2$ when $w \geq 1$.*

Proof: Theorem 3.9 states the result for $n = 2$, and we know (by Lemma 1.3) that $M_n(w) > M_2(w)$ when $n > 2$. Thus if we can prove that $A_n(w) < A_2(w)$ when $n > 2$, we will have the result for general n. This can be shown when $w \geq 1$.

Consider our problem in R^2, and suppose $w > 1$. Then the bubble of area 1 can be obtained from the bubble of area w by inversion in the separating cap, with the latter bubble on the outside. Call the regions occupied by these two bubbles G_2 and H_2, respectively. Now, consider the double bubble in R^n where the three spherical caps have the same radii as the corresponding caps in R^2, and let G_n and H_n be the regions that correspond to G_2 and H_2, respectively. Then, by Corollary 4.8,

$$\frac{w}{1} < \frac{\mathrm{Vol}(H_n)}{\mathrm{Vol}(G_n)}\,.$$

Now, scaling the bubbles in R^n to make the small bubble have volume 1 will give the big bubble volume $w' = \mathrm{Vol}(H_n)/\mathrm{Vol}(G_n)$. Thus we know that $w < w'$. (It is useful to note at this point that if $w = 1$, then the two bubbles are symmetric with respect to a plane, so $w = w' = 1$ in this case. In general, for $w \geq 1$, we have $w \leq w'$.)

Recall also that the bubble G_2 of area 1 is composed of regions R_2 and R_3, symmetric under inversion in a circle, with R_2 inside the circle. Similarly, G_n is composed of regions R'_2 and R'_3. Corollary 4.8 tells us that

$$\frac{B_2}{A_2} = \frac{\mathrm{Vol}(R_3)}{\mathrm{Vol}(R_2)} < \frac{\mathrm{Vol}(R'_3)}{\mathrm{Vol}(R'_2)} = \frac{B_n}{A_n} .$$

As usual, we write $A_2 = A_2(w)$. Furthermore, because R'_2 and R'_3 make up the small bubble that has volume 1 when the big bubble has volume w', we also have $A_n = A_n(w')$. Thus

$$\begin{aligned}
1 + \frac{B_2(w)}{A_2(w)} &< 1 + \frac{B_n(w')}{A_n(w')} \\
\frac{A_2(w) + B_2(w)}{A_2(w)} &< \frac{A_n(w') + B_n(w')}{A_n(w')} \\
\frac{1}{A_2(w)} &< \frac{1}{A_n(w')} \\
A_n(w') &< A_2(w) .
\end{aligned}$$

But since A_n is an increasing function, we have

$$A_n(w) \leq A_n(w') < A_2(w) .$$

□

5.2 Case 2: $w < 1$

At the moment, very little can be said conclusively about this case. Of course, the asymptotic result of Theorem 2.3 applies in general dimension, and thus the conjecture only needs to be checked in the interval $(\epsilon_n, 1)$ for each n. Following the example of section 3, we could derive explicit formulae for $A_n(\theta)$ and $w_n(\theta)$, using integrals to express volume. Then the conjecture could be verified numerically for any desired dimension. So if the truth of Conjecture 1.1 with some specific n is needed for some particular application, this method can furnish the result. However, the formulae would get harder and harder to compute for large n, and in any case, one cannot numerically check infinitely many dimensions.

I am aware of two other approaches that could potentially be fruitful. The first is to prove an estimate of just *how much* $M_n(w)$ increases with n for small w; the increase seems to become more dramatic the smaller w gets. Then, even though $A_n(w)$ also increases with n, one can hope to prove that it increases *less.* This seems very difficult to carry out.

The second approach is to somehow use the (so far unutilized) symmetry of the double bubble. That is, region R_2 (of volume $A_n(w)$) in the $(w, 1)$ double bubble is a scaled copy of region R_1 in the $(\frac{1}{w}, 1)$ double bubble. If we rescale the problem to deal with double bubbles of volumes $(v, 1-v)$ instead of $(w, 1)$, then region R_2 in the $(v, 1-v)$ double bubble is an *exact copy* of region R_1 in the $(1-v, v)$ double bubble. Perhaps this symmetry could allow us to use the result about large volumes to prove the corresponding result about small volumes. This seems to be the most promising line of thought, although it too has not been easy so far.