Gradients on the Stanford Dish

A "Feet-On" Activity

Math 51, Winter 2005

Today, we are going to walk the trails through the hills by the Dish, and compute a few gradients and directional derivatives. As a reminder, for a function $f(x, y)$, the *gradient vector* consists of the partial derivatives of f:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right].$$

If we want to compute the directional derivative along some vector v, we take its dot product with the gradient:

$$\frac{\partial f}{\partial v} = \nabla f \cdot v\,.$$

Well, at the moment, we are standing on a very large graph of a function $z = f(x, y)$, where the z coordinate is elevation. We also have a contour map that we can use to estimate the partial derivatives of f. **The map has a scale of 1:10,000, so every millimeter on the map represents 10 meters in real life. Meanwhile, the contours of the map represent a change in elevation of 20 feet, or 6 meters.**

1. Let's estimate the partial derivatives at the point P. As we move horizontally, in the x direction, the contour lines are about 14 mm apart, and the elevation goes down as x goes up. So a run of 140 m produces a rise of -6 m. Moving in the y direction, the contour lines are about 4 mm apart, and again elevation goes down as y goes up. So a run of 40 m produces a rise of -6 m. This gives us

$$\frac{\partial f}{\partial x} = \frac{-6}{140} \text{ and } \frac{\partial f}{\partial y} = \frac{-6}{40}\,, \text{ so } \nabla f(P) = \left[\frac{-3}{70}, \frac{-3}{20}\right] \approx [-0.043, -0.15]\,.$$

Go ahead draw the gradient vector $\nabla f(P)$ at point P on the map.

2. Use this method to estimate the gradient vector $\nabla f(Q)$, and draw it in at point Q.

3. Let $u = [\frac{-4}{5}, \frac{-3}{5}]$ be a unit vector tangent to the road at point Q. Compute the directional derivative $\frac{\partial f}{\partial u}(Q)$. If you stand at point Q, does this directional derivative agree with what you see as the incline of the road?

4. Estimate the gradient vector $\nabla f(R)$, and draw it in at point R. Compute also the directional derivative $\frac{\partial f}{\partial v}(R)$, where $v = [-1, 0]$. Does the directional derivative agree with the incline of the road?

5. What is the relationship between the gradient vectors and elevation change?

6. What is the relationship between the gradient vectors and the level curves on the contour map?

7. Find three critical points of the function $f(x, y)$ on the map. Are they minima, maxima, or saddle points? (A number of critical points should be visible as you walk along the trail.)

8. Estimate the gradient vector $\nabla f(S)$, and draw it on the map. Now, suppose that a stream was to run down from point S. (There is actually a small stream that runs down from that point, even though the map doesn't show it.) Sketch the path that the stream would take. How does the path of the stream relate to the gradient vectors?