Final Exam Review Problems

Math 132-06, Fall 2005

1. Consider the following function and its derivatives:

$$f(x) = \frac{x-3}{(x+2)^2}, \qquad f'(x) = \frac{8-x}{(x+2)^3}, \qquad f''(x) = \frac{2x-26}{(x+2)^4}.$$

What are its horizontal and vertical asymptotes? Where is $f(x)$ increasing/decreasing, concave up/concave down? Use this information to sketch a graph of the function.

Horizontal asymptote: $\lim_{x\to\infty} \frac{x-3}{(x+2)^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{2(x+2)} = 0.$

$\underline{y=0}$.

$\lim_{x\to-\infty} \frac{x-3}{(x+2)^2} = 0$ (same).

Vertical asymptote: $x = -2$.

$\lim_{x\to -2} f(x) = -\infty$.

$f'(x) = 0 \Rightarrow 8 - x = 0 \Rightarrow x = 8$.

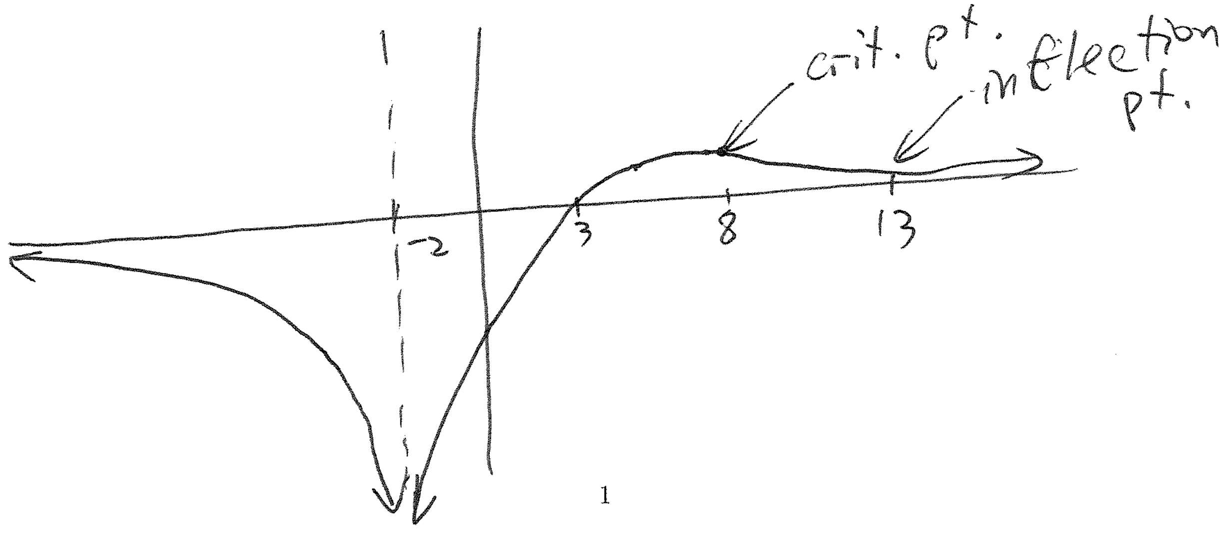
$f(x)$ is increasing on $(-2, 8)$

decreasing on $(-\infty, -2)$ and $(8, \infty)$

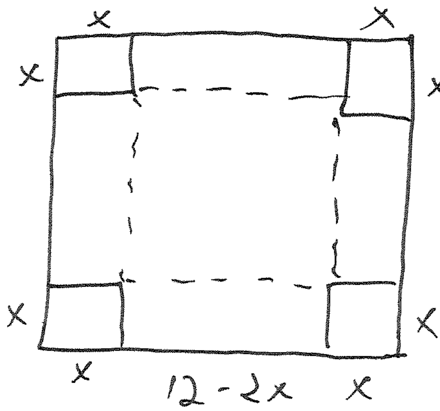
$f''(x) = 0 \Rightarrow 2x - 26 = 0 \Rightarrow x = 13$.

$f(x)$ is concave up on $(13, \infty)$

concave down on $(-\infty, 13)$.



2. You have a 12" by 12" sheet of cardboard. You want to make an open-top box out of this sheet, by cutting out four square corners and bending up the remaining sides. *What is the largest volume* of a box that you can make in this way?



$$V = x(12-2x)(12-2x)$$

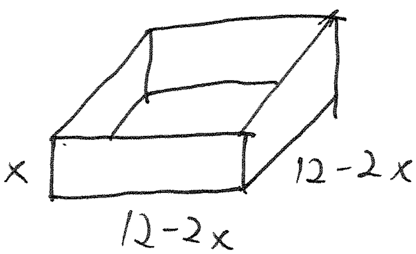
$$V = x(4x^2 - 48x + 144)$$

$$V = 4x^3 - 48x^2 + 144x$$

$$\frac{dV}{dx} = \frac{12x^2}{12} - \frac{96x}{12} + \frac{144}{12} = \frac{0}{12}$$

$$x^2 - 8x + 12 = 0$$

$$(x-2)(x-6) = 0$$



Critical points: $x=2$, $x=6$

$x=2 \Rightarrow V = 2 \cdot 8 \cdot 8 = 128 \text{ in}^3$

$x=6 \Rightarrow V = 6 \cdot 0 \cdot 0 = 0 \text{ in}^3$

(endpoints: $x=0$, $x=6$) $x=0 \Rightarrow V = 0 \cdot 12 \cdot 12 = 0$

Maximum volume: 128 in^3

3. Suppose x and y are related by the expression $x^2 + y^2 = \cos xy$. Solve for $\frac{dy}{dx}$.

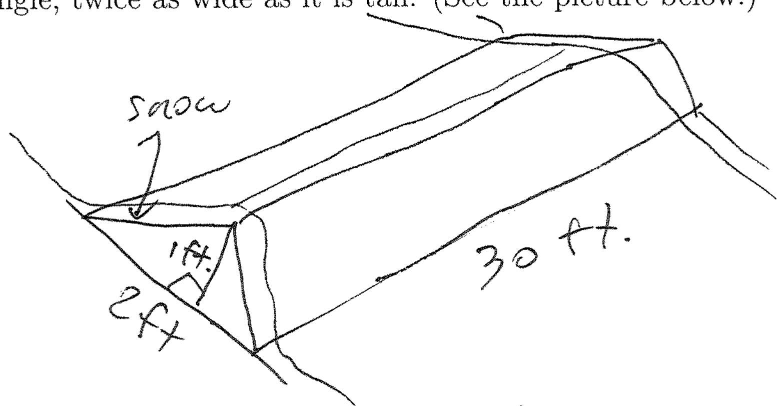
$$2x + 2y\frac{dy}{dx} = -\sin(xy)\cdot\left(y + x\frac{dy}{dx}\right)$$

$$2x + y\sin(xy) = -2y\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx}$$

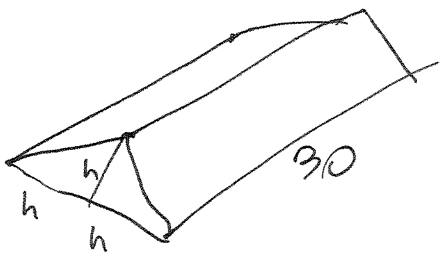
$$2x + y\sin(xy) = \frac{dy}{dx}(-2y - x\sin(xy))$$

$$\frac{dy}{dx} = -\frac{2x + y\sin(xy)}{2y + x\sin(xy)}$$

4. A warehouse has a flat roof with a ridge on it. As snow falls on top of the warehouse, the ridge of snow gets larger and larger. The ridge is 30' long, and its cross-section looks like a triangle, twice as wide as it is tall. (See the picture below.)



If the volume of snow on top of this ridge is increasing at the rate of 10 cubic feet per hour, how fast is the height of the ridge increasing when it's 1' tall?



$$V = \frac{1}{2} w \cdot h \cdot \ell$$
$$= \frac{1}{2} \cdot 2h \cdot h \cdot 30$$
$$= 30h^2$$

$$\frac{dV}{dt} = 60h \cdot \frac{dh}{dt}$$

$$\cancel{d}\, 10 = 60 \cdot 1 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{6} \frac{ft}{hr} = 2 \frac{in}{hr}.$$

5. Suppose you want to compute the area of the region enclosed between the graphs of $x = y^2$ and $y = x - 2$. Would it be easier to do the integral in terms of x or y? Sketch a graph of the region and label a typical rectangle used in the computation of the integral. Then set up an integral expression for the area, *but do not evaluate it.*

$x = y^2$
$x = y + 2$

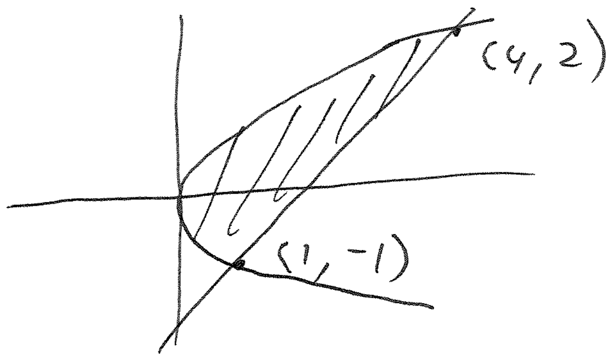
intersection points:

$y^2 = y + 2$

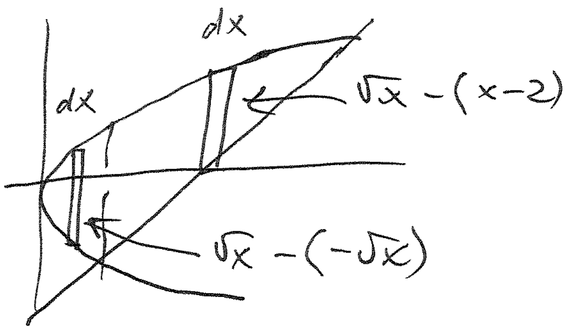
$y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y = -1$ or $y = 2$.



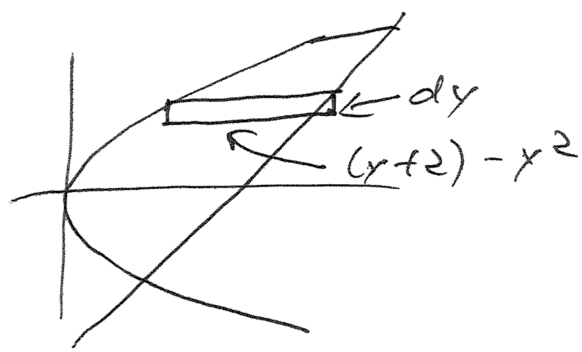
If we integrate in terms of x:



Need 2 integrals:

$\int_0^1 \sqrt{x} - (-\sqrt{x})\,dx + \int_1^4 \sqrt{x} - (x-2)\,dx$

If we integrate in terms of y:



$\int_{-1}^{2} y + 2 - y^2\,dy.$

This is easier.