

CRYSTALS

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1. FIBERED CATEGORIES AND STACKS

Let \mathcal{C} and \mathcal{F} be categories and let $F : \mathcal{F} \rightarrow \mathcal{C}$ be a functor. Given an object $x \in \mathcal{C}$, let \mathcal{F}_x denote the category whose objects are those $a \in \mathcal{F}$ such that $Fa = x$ and whose morphisms are those $f : a \rightarrow b$ such that $F(f) = \text{id}_x$. We call \mathcal{F}_x the *category of objects over x* . Given a morphism $g : Fa \rightarrow Fb$, we define the set of *morphisms over g* as follows:

$$\text{Hom}_g(a, b) = \{f \in \text{Hom}_{\mathcal{F}}(a, b) \mid F(f) = g\}$$

By definition, for any $a, b \in \mathcal{F}_x$, we have

$$\text{Hom}_{\text{id}_x}(a, b) = \text{Hom}_{\mathcal{F}_x}(a, b).$$

A morphism $f \in \text{Hom}_{\mathcal{F}}(a, b)$ is called *cartesian* if for any $a' \in \mathcal{F}_{Fa}$ and $h \in \text{Hom}_{F(f)}(a', b)$ there exists a unique $g \in \text{Hom}_{\text{id}}(a', a)$ making the following diagram commute:

$$\begin{array}{ccc} a' & & \\ \downarrow & \searrow h & \\ g \downarrow & & \\ a & \xrightarrow{f} & b \end{array}$$

which lives over

$$Fa \xrightarrow{F(f)} Fb$$

Given a cartesian morphism $f : a \rightarrow b$, we say a is the *inverse image* of b relative to $F(f)$, and we write $F(f)^*b = a$.

We say that $F : \mathcal{F} \rightarrow \mathcal{C}$ is a *fibered category* if, for any morphism $g : x \rightarrow y$ in \mathcal{C} and $b \in \mathcal{F}_y$, the inverse image $g^*b \in \mathcal{F}_x$ exists, and the composition of cartesian morphisms is cartesian.

Let $F' : \mathcal{F}' \rightarrow \mathcal{C}$ be another fibered category. A morphism of fibered categories is a functor $\beta : \mathcal{F} \rightarrow \mathcal{F}'$ such that $F'\beta = F$. We say β is a *cartesian functor* if it sends cartesian morphisms in \mathcal{F} to cartesian morphisms in \mathcal{F}' .

1.1. Proposition. *A fibered category is the same as a 2-functor $\mathcal{C}^{op} \rightarrow \text{Cat}$.*

Proof. We can think of \mathcal{C} as a 2-category where all the 2-morphisms are identities. Therefore a 2-functor $G : \mathcal{C}^{op} \rightarrow \text{Cat}$ consists of the following data:

- (1) for any $x \in \mathcal{C}$, a category $G(x)$

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(2) for any morphism $f : x \rightarrow y$ in \mathcal{C} , a functor $G(f) : G(y) \rightarrow G(x)$

(3) for any $x \xrightarrow{f} y \xrightarrow{g} z$, a natural isomorphism $\alpha(f, g) : G(gf) \simeq G(f)G(g)$.

such that, for any $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ the following diagram commutes:

$$(1.1) \quad \begin{array}{ccc} G(hgf) & \xrightarrow{\alpha(gf, h)} & G(gf)G(h) \\ \alpha(f, hg) \downarrow & & \downarrow \alpha(f, g)_{G(h)(_)} \\ G(f)G(hg) & \xrightarrow{G(f)(\alpha(g, h))} & G(f)G(g)G(h) \end{array}$$

Given a fibered category $F : \mathcal{F} \rightarrow \mathcal{C}$, we define a 2-functor

$$G : \mathcal{C}^{op} \rightarrow \mathcal{C}at$$

$$x \mapsto \mathcal{F}_x$$

Given $f : x \rightarrow y$ in \mathcal{C} , we define $G(f) = f^* : \mathcal{F}_y \rightarrow \mathcal{F}_x$. Let us check that this is a functor. Let $m : b \rightarrow b'$ be a morphism in \mathcal{F}_y . We have the following diagram

$$\begin{array}{ccc} f^*b & \longrightarrow & b \\ f^*m \downarrow & & \downarrow m \\ f^*b' & \longrightarrow & b' \end{array}$$

which lives over

$$x \xrightarrow{f} y$$

The uniqueness of f^*m guarantees that f^* respects composition of morphisms, and that $f^* \text{id}_b = \text{id}_{f^*b}$.

Now suppose we have a diagram $x \xrightarrow{f} y \xrightarrow{g} z$ in \mathcal{C} , and let $c \in \mathcal{F}_z$. Then there exists a diagram

$$\begin{array}{ccc} (gf)^*c & & \\ \alpha(f, g)_c \downarrow & \searrow & \\ f^*g^*c & \longrightarrow & g^*c \longrightarrow c \end{array}$$

which lives over

$$x \xrightarrow{f} y \xrightarrow{g} z$$

Switching the roles of $(gf)^*c$ and f^*g^*c , we get a morphism back the other way, and uniqueness forces it to be an isomorphism.

Now for any $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ and $d \in \mathcal{F}_w$, we have the following two diagrams:

$$\begin{array}{ccccc}
 (hgf)^*d & & & & \\
 \downarrow \alpha(f,hg)_d & \searrow & & & \\
 f^*(hg)^*d & \longrightarrow & (hg)^*d & \longrightarrow & d \\
 \downarrow f^*\alpha(g,h)_d & & \downarrow & & \parallel \\
 f^*g^*h^*d & \longrightarrow & g^*h^*d & \longrightarrow & h^*d \longrightarrow d
 \end{array}$$

$$\begin{array}{ccccc}
 (hgf)^*d & & & & \\
 \downarrow \alpha(gf,h)_d & \searrow & & & \\
 (gf)^*h^*d & \longrightarrow & h^*d & \longrightarrow & d \\
 \downarrow \alpha(f,g)_{h^*d} & & \downarrow & & \parallel \\
 f^*g^*h^*d & \longrightarrow & g^*h^*d & \longrightarrow & h^*d \longrightarrow d
 \end{array}$$

By uniqueness, the composition of the dotted arrows in each diagram must be equal.

In the other direction, if $G : \mathcal{C}^{op} \rightarrow Cat$ is a 2-functor, define $\mathcal{F}_x = G(x)$ for all $x \in \mathcal{C}$. For any morphism $f : x \rightarrow y$ in \mathcal{C} , define $f^* = G(f) : \mathcal{F}_y \rightarrow \mathcal{F}_x$. We need to define a category \mathcal{F} and a functor $F : \mathcal{F} \rightarrow \mathcal{C}$. The objects of \mathcal{F} are:

$$Ob(\mathcal{F}) = \bigsqcup_{x \in \mathcal{C}} Ob(G(x))$$

Given $f : x \rightarrow y$ in \mathcal{C} and $a \in G(x), b \in G(y)$, we define

$$Hom_f(a, b) = Hom_{G(x)}(a, G(f)(b))$$

and

$$Hom_{\mathcal{F}}(a, b) = \bigsqcup_{f \in Hom_{\mathcal{C}}(x, y)} Hom_f(a, b)$$

The functor F is defined as follows: if $a \in \mathcal{F}$, then there is a unique $x \in \mathcal{C}$ such that $a \in G(x)$, so let $F(a) = x$. If $g \in Hom_{\mathcal{F}}(a, b)$, then there exists a unique morphism $f : x \rightarrow y$ in \mathcal{C} such that $F(a) = x, F(b) = y$, and $g \in Hom_{G(x)}(a, G(f)(b))$. We define $F(g) = f$.

The composition of morphisms in \mathcal{F} is defined as follows: let $m \in Hom_{\mathcal{F}}(a, b)$ and $n \in Hom_{\mathcal{F}}(b, c)$. More precisely, let $m \in Hom_{G(x)}(a, G(f)(b))$ and $n \in Hom_{G(y)}(b, G(g)(c))$, where

$$x \xrightarrow{f} y \xrightarrow{g} z$$

is the underlying diagram in \mathcal{C} . The morphism $n \circ m$ is defined by the following diagram:

$$\begin{array}{ccc} a & \xrightarrow{\quad n \circ m \quad} & G(gf)(c) \\ m \downarrow & & \downarrow \alpha(f,g)_c \\ G(f)(b) & \xrightarrow{\quad G(f)(n) \quad} & G(f)G(g)(c) \end{array}$$

The associativity of composition follows from equation 1.1. Moreover, it is clear from this definition that $F(n \circ m) = F(n)F(m)$ and $F(\text{id}_a) = \text{id}_x$.

Finally, let us show that $G(f)(b)$ satisfies the universal property of the inverse image of b . In fact, there is a natural morphism in

$$\text{Hom}_{\mathcal{F}}(G(f)(b), b) = \text{Hom}_{G(x)}(G(f)(b), G(f)(b))$$

corresponding to $\text{id}_{G(f)(b)}$. Clearly, any $h \in \text{Hom}_{\mathcal{F}}(a, b) = \text{Hom}_{G(x)}(a, G(f)(b))$ must factor uniquely through this morphism.

□

We recall the construction of the $\mathcal{2}$ -limit of a $\mathcal{2}$ -functor: an object of $\mathcal{2}\text{lim } G$ consists of the following data:

- (1) for each $x \in \mathcal{C}$, an object $a_x \in G(x)$
- (2) for each morphism $f : x \rightarrow y$ in \mathcal{C} , an isomorphism $a_x \simeq G(f)a_y$.

such that the following diagram commutes:

$$\begin{array}{ccc} G(f)G(g)a_z & \longleftarrow & G(f)a_y \\ \alpha(f,g)_{a_z} \uparrow & & \uparrow \\ G(gf)a_z & \longleftarrow & a_x \end{array}$$

The morphisms in $\mathcal{2}\text{lim } G$ are systems of morphism $a_x \rightarrow b_x$ that commute with the structure isomorphisms. We also refer to $\mathcal{2}\text{lim } G$ as the category of *cartesian sections* of G .

Now suppose \mathcal{C} is equipped with a Grothendieck topology. A *stack* on \mathcal{C} is simply a sheaf of categories for this topology. In other words, a stack is a $\mathcal{2}$ -functor $G : \mathcal{C}^{op} \rightarrow \text{Cat}$ such that the assignment

$$U \mapsto \text{Mor}(G(U))$$

is a sheaf and objects satisfy descent, i.e. given $x_i \in G(U_i)$ and isomorphisms $\phi_{i,j} : x_i|_{U_i \cap U_j} \simeq x_j|_{U_i \cap U_j}$ for all i, j , there exists an object $x \in G(\cup_i U_i)$ such that $x|_{U_i} \simeq x_i$, and this isomorphism is compatible with the $\phi_{i,j}$'s.

Let us restate this definition in the language of fibered categories. Let $F : \mathcal{F} \rightarrow \mathcal{C}$ be a fibered category. Let $f : x \rightarrow y$ be a morphism in \mathcal{C} and let $a \in \mathcal{F}_x$. A *descent datum*

on a relative to f is an isomorphism $\phi : p_1^*a \simeq p_2^*a$ such that $p_{31}^*\phi = p_{32}^*\phi \circ p_{21}^*\phi$. Here p_i and p_{ij} are the usual projections in the diagram

$$x \times_y x \times_y x \rightrightarrows x \times_y x \rightrightarrows x$$

The pairs (a, ϕ) form a category denoted $Des(f)$. There is a functor

$$\mathcal{F}_y \rightarrow Des(f)$$

$$b \mapsto (f^*b, \psi)$$

where $\psi : p_1^*f^*b \simeq p_2^*f^*b$ exists by the definition of cartesian morphism. We say $F : \mathcal{F} \rightarrow \mathcal{C}$ is a stack if

$$\mathcal{F}_y \rightarrow Des(f)$$

is an equivalence of categories for all $f : x \rightarrow y$ in \mathcal{C}

2. THE CRYSTALLINE SITE

Let X be a k -scheme. and let $Crys(X/k)$ denote the category of diagrams

$$X \xleftarrow{j} S \xrightarrow{i} T$$

where j is a quasi-finite morphism and i is a closed embedding of affine k -schemes such that the corresponding ideal $I \subset \mathcal{O}_T$ is nilpotent. A morphism in $Crys(X/k)$ from (S, T) to (S', T') is a pair of morphisms of k -schemes (f_0, f) making the following diagram commute:

$$\begin{array}{ccccc} X & \xleftarrow{j} & S & \xrightarrow{i} & T \\ \parallel & & \downarrow f_0 & & \downarrow f \\ X & \xleftarrow{j'} & S' & \xrightarrow{i'} & T' \end{array}$$

We have a functor

$$Crys(X/k) \rightarrow Sch_k$$

$$(S, T) \mapsto T$$

We can pull back various topologies via this functor.

3. DEFINITION OF A CRYSTAL

Let Sch denote the category of schemes. Let \mathcal{F} be a fibered category over Sch which is a stack for the Zariski topology. Given a scheme X , an \mathcal{F} -crystal on X is a cartesian section of $\mathcal{F} \times_{Sch} Crys(X) \rightarrow Crys(X)$.

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