

## MATH 310: HOMEWORK 6

- (1) Solve  $13x = 11$  in  $\mathbb{Z}_{83}$ . Can you solve  $99x = 1$  in  $\mathbb{Z}_{110000}$  ?
- (2) Let  $m, n \in \mathbb{Z}$  such that  $\gcd(m, n) = 1$ . Use Theorem 1.5 to prove that

$$(m) \cap (n) = (mn)$$

Is this formula still true if  $\gcd(m, n) \neq 1$ ?

- (3) Use the Chinese Remainder Theorem to find  $x \in \mathbb{Z}_{105}$  such that

$$x \equiv 1 \pmod{3}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 0 \pmod{7}$$

- (4) Calculate the last three digits of  $7^{800003}$ .
- (5) Consider the following subset of  $M(\mathbb{R})$ :

$$I = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$$

Prove that  $I$  is a subring of  $M(\mathbb{R})$ . Is  $I$  an ideal of  $M(\mathbb{R})$ ?

- (6) Let  $R$  be a ring with identity and let  $e$  be an idempotent element of  $R$ . Prove that  $1 - e$  is also idempotent. If  $R$  is an integral domain, prove that 0 and 1 are the only idempotents in  $R$ .
- (7) Find the order of every element of  $\mathbb{Z}_{12}$  and  $\mathbb{Z}_{12}^\times$ . Is  $\mathbb{Z}_{12}^\times$  cyclic?
- (8) Let  $G$  be a group and define  $f : G \rightarrow G$  by  $f(g) = g^{-1}$ . Prove that  $f$  is a group homomorphism if and only if  $G$  is abelian. Also, prove that  $f$  is a group isomorphism if  $G$  is abelian.
- (9) Show that  $2x^2 + x + 1$  is irreducible in  $\mathbb{Z}_3[x]$ . Find  $x^{-1}$  in  $\mathbb{Z}_3[x]/(2x^2 + x + 1)$ .
- (10) Write out the six elements of the group  $C_6$  using cycle notation. Recall the  $C_6$  is the group of rotations of a regular hexagon.
- (11) (Extra Credit, 10 points) 7.2 # 38.