# MATH 310: EXTRA CREDIT PROBLEMS

(1) Determine whether each of the following statements is true or false. Justify your answer.

(a) For any integer $n \geq 0$, the integer $n^2 + n + 41$ is prime.

(b) There is a set (equipped with addition and multiplication) which satisfies all the axioms of a ring except commutativity of addition.

(2) (a) If $R$ is a commutative ring with identity, prove that there is a unique integer $n \geq 0$ such that $\mathbb{Z}_n \subset R$ (as a subring). This integer is called the *characteristic* of the ring $R$.

Hint: Find the kernel of the natural homomorphism $\mathbb{Z} \to R$.

(b) Prove that the characteristic of a field must be a prime number.

(3) Given $z = a + bi \in \mathbb{C}$, define $|z| = \sqrt{a^2 + b^2}$. Let $S \subset \mathbb{C}$ be the following subset:

$$S = \{z \in \mathbb{C} \mid z = 0 \text{ or } |z| = 1\}$$

Define a new addition operation $\oplus$ as follows:

$$z \oplus w = \begin{cases} \frac{z+w}{|z+w|} & \text{if } z + w \neq 0 \\ 0 & \text{if } z + w = 0 \end{cases}$$

Equip $S$ with the usual multiplication of complex numbers. Determine which of the field axioms $S$ does or does not satisfy.

(4) (Sec. 7.2, 38) Let $G$ be a nonempty set equipped with an associative operation with these properties:

(i) There is an element $e \in G$ such that $ea = a$ for every $a \in G$.

(ii) For each $a \in G$, there exists $d \in G$ such that $da = e$.

Prove that $G$ is a group.