

NOTES FOR MATH 481 OVERVIEW AND SET THEORY

VIVEK DHAND

1. OVERVIEW

Discrete math is primarily about answering three types of questions: existence of a solution, counting the number of solutions, and optimization (finding the best solution). This course naturally splits into two parts. The first is combinatorics, which for us means mostly counting. The second part is graph theory, which is mostly existence questions with some counting. Below are some examples from each part of the course.

Examples from combinatorics.

- (1) There are 2,598,960 possible 5-card hands in poker. We will calculate the number of hands of each type: one pair, two pair, three of a kind, etc.
- (2) There are $n!$ ways to rearrange n distinct letters. For example, we can make $3! = 6$ words with the letters a, b and c .
- (3) More generally, we can count the number of possible words that can be made with any set of letters, like *mississippi*.
- (4) Let F_n denote the n -th Fibonacci number. We will derive a formula for F_n , without calculating all the earlier terms.
- (5) The following two sets have the same number of elements:

$$\{(x, y) \in \mathbb{Z}^2 \mid x, y \geq 0, x + y \leq 3\}, \quad \{(x, y, z) \in \mathbb{Z}^3 \mid x, y, z \geq 0, x + y + z \leq 2\}.$$

This is not a coincidence; we could replace 2 and 3 by any positive integers, and the two sets would still have the same number of elements.

- (6) Chemistry: Triphenylamine has three benzene rings bonded to a central nitrogen atom. We will count the possible number of molecules (*isomers*) if we replace some atoms by others (and we take the symmetry of the molecule into account).

Examples from graph theory.

- (1) The divisor graph: The vertices are labeled by the divisors of n , and we draw an edge between two divisors a and b if $a = pb$ for some prime number p .
- (2) Water, gas, light problem: Try to connect three houses to three utilities without any lines crossing.
- (3) Regular graphs: Try to draw graphs where every vertex has k neighbors, without any lines crossing (impossible for $k \geq 6$).
- (4) Physics: There are $(2n - 1)!!$ ways to pair $2n$ particles. You can think of this a toy model of all the things that can happen to n particles in spacetime (particles can travel, fuse together, or appear out of nothing in pairs).

2. WHAT IS A SET?

Informally speaking, a *set* is simply a collection of objects, called *elements*. There are two implicit statements in this definition:

- (1) *Every element of a set must be unique* (if two objects are *exactly* the same, then they are the same object). In other words, the elements of a set are labeled, and every element has a unique label.
- (2) *The order of the elements in a set is unimportant* (the objects in a box have no preferred arrangement). This means that we can rearrange the elements of a set and it remains the same set.

We use brackets to denote sets. For example, $X = \{a, b, c\}$ means that X is a set with elements a, b and c . We also use the symbol “ \in ” to denote that an element belongs to a set. For example, $a \in X$ means that a is an element of X .

Suppose we have two sets X and Y such that for every $a \in Y$, we have $a \in X$. Then we say that Y is a *subset* of X , and we write $Y \subset X$. Two sets are equal if and only if each one is a subset of the other, i.e. $X = Y$ if and only if $X \subset Y$ and $Y \subset X$. For example, the idea that every element of a set must be unique can be expressed by the following formula:

$$\{a, a\} = \{a\}.$$

The idea that the order of elements in a set does not matter can be expressed like this:

$$\{a, b\} = \{b, a\}$$

This definition raises a question: what are the elements of a set allowed to be? Von Neumann had the brilliant idea of proposing the following two axioms:

- (1) *There is a set with no elements.* This set is called the *empty set* and denoted \emptyset .
- (2) *The elements of a set must themselves be sets.*

This means our whole mathematical universe is built out of two symbols: \emptyset and $\{ \}$. For example, we can define the counting numbers:

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} = \{\emptyset\} \\ 2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\ 3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &\vdots \end{aligned}$$

We call this set \mathbb{N} . If we throw in the negative numbers, we get the set of integers \mathbb{Z} . Suppose we want to define a subset of \mathbb{N} with some property. For example, the set of prime numbers could be written in this way:

$$\{a \in \mathbb{N} \mid a \geq 2, \text{ and the only divisors of } a \text{ are } 1 \text{ and } a\}.$$

The vertical line is read as “such that.” By the way, Russell’s paradox shows that we have to careful not to talk about “all sets”:

$$X = \{\text{all sets } S \mid S \notin S\}.$$

Is $X \in X$? If it is, then $X \notin X$. If $X \notin X$, then $X \in X$.

3. UNIONS, INTERSECTIONS, COMPLEMENTS

The *power set* $P(X)$ is define to be the set whose elements are all the subsets of X . Note that $X \in P(X)$ and $\emptyset \in P(X)$, i.e. every set is a subset of itself, and the empty set is a subset of every set. We have two ways to combine elements of $P(X)$ together, called *union* and *intersection*. If Y and Z are subsets of X , we denote the union of Y and Z by $Y \cup Z$ and the intersection by $Y \cap Z$. They are defined as follows:

$$a \in Y \cup Z \iff a \in Y \text{ or } a \in Z \text{ (or both)}.$$

$$a \in Y \cap Z \iff a \in Y \text{ and } a \in Z.$$

It turns out that \cup and \cap behave a lot like addition and multiplication, respectively. They satisfy the following properties for any $A, B, C \in P(X)$:

$$\text{Associativity: } (A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

$$\text{Commutativity: } A \cup B = B \cup A, \quad A \cap B = B \cap A$$

$$\text{Identity: } A \cup \emptyset = A, \quad A \cap X = A$$

$$\text{Distributivity: } A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

We also have two “cross identities”: $A \cup X = X$, $A \cap \emptyset = \emptyset$, and the squaring law $A \cap A = A$.

Given $A \in P(X)$, we define the complement of A in X , denoted \overline{A} , as follows:

$$a \in \overline{A} \iff a \notin A.$$

The complement is also written $\overline{A} = X - A$ if we wish to make the ambient set explicit. We have the following formulas:

$$\overline{\overline{X}} = \emptyset, \quad \overline{\emptyset} = X$$

$$A \cup \bar{A} = X, \quad A \cap \bar{A} = \emptyset, \quad \overline{\bar{A}} = A$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}, \quad \overline{A \cap B} = \bar{A} \cup \bar{B}$$

Finally, we have the *symmetric difference* of A and B :

$$A \Delta B = A \cup B - A \cap B.$$

In other words,

$$a \in A \Delta B \iff a \in A \text{ or } a \in B \text{ (but not both)}.$$

It turns out that Δ satisfies all the properties of \cup listed above, but it has the following additional property:

$$A \Delta A = \emptyset.$$

In particular, every subset is its own inverse under Δ . Therefore, we call Δ the “addition” and \cap the “multiplication” on $P(X)$. These two operations make $P(X)$ into a *ring of sets*.

The *subset graph* of X is the graph whose vertices are labeled by subsets of X , and we draw an edge between two subsets A and B if $B \subset A$ and A has one more element than B .

For example, let $X = \{a\}$. Then $P(X) = \{\emptyset, \{a\}\}$, and the addition and multiplication tables look like:

Δ	\emptyset	a
\emptyset	\emptyset	a
a	a	\emptyset

\cap	\emptyset	a
\emptyset	\emptyset	\emptyset
a	\emptyset	a

The subset graph looks like this:



If $X = \{a, b\}$, then the power set has four elements:

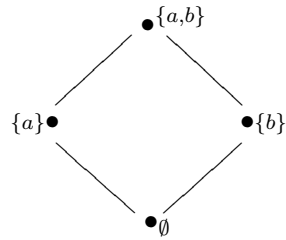
$$P(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

The addition and multiplication tables look like:

Δ	\emptyset	a	b	ab
\emptyset	\emptyset	a	b	ab
a	a	\emptyset	ab	b
b	b	ab	\emptyset	a
ab	ab	b	a	\emptyset

\cap	\emptyset	a	b	ab
\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
a	\emptyset	a	\emptyset	a
b	\emptyset	\emptyset	b	b
ab	\emptyset	a	b	ab

The subset graph is:



The subset graph of $X = \{a, b, c\}$ is:

