NOTES FOR MATH 481
UNLABELED TREES

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1. Labeled and unlabeled trees

Recall that a labeled tree of order n is given by a Prüfer sequence $(a_1, \ldots, a_{n-2})$, where $1 \leq a_i \leq n$ for all i. This shows that there are exactly n^{n-2} labeled trees of order n.

Let $\mathcal{P}_n$ denote the set of labeled trees of order n, i.e. the set of Prüfer sequences of length $n-2$. Let $\mathcal{T}_n$ denote the set of unlabeled trees of order n. There is a map

$$\mathcal{P}_n \to \mathcal{T}_n$$

given by forgetting all the labels.

How many labeled trees have the same underlying unlabeled tree? In other words, how many ways are there to label a given unlabeled tree? There are $n!$ possible labelings of a tree of order n, but we have to take the symmetry of the tree into account. By Burnside's Lemma, the answer is

$$\frac{n!}{s(T)}$$

where $s(T)$ is the number of symmetries of the unlabeled tree T. Taking the sum over all unlabeled trees of order n, we get the following interesting formula:

$$n^{n-2} = \sum_T \frac{n!}{s(T)}$$

or, in other words:

$$\frac{n^{n-3}}{(n-1)!} = \sum_T \frac{1}{s(T)}$$