NOTES FOR MATH 481
GRAPHS

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Graph theory has applications to a staggering number of rapidly growing fields, many of them at the cutting edge of science and technology. The theory of *networks* has really come into its own in the age of the internet. This development is accompanied by a philosophical shift: we now think not only about *objects*, but about the *relationships* between them as well. We list several examples of graphs and graph-theoretic problems below.

Networks

Friendship. The vertices are people. We draw an edge between two people if they are friends. The conjecture of "*six degrees of separation*" asserts that any two people can be connected by path of length at most six.

Highways. The vertices are cities. We draw an edge between two cities if they are connected by a highway. The *traveling salesman problem* asks for the most efficient way to visit every city exactly once.

Maps. The vertices are countries on a continent (or states in a country). We draw an edge between two countries if they share a border. The *graph coloring problem* asks how many colors are needed so that no two neighboring countries get the same color.

Communications (NSA). The vertices are phone numbers. We draw an edge between two numbers if a phone call is made from one to the other. The *maximal clique problem* asks for the largest subset of people who have all called each other.

Financial markets. The vertices are stocks. We draw an edge between two stocks if their prices are correlated over a fixed period of time (there is a mathematical formula which tells if two stocks are correlated). The edge density of this graph represents the level of interdependence or *globalization* of the stock market. A set of stocks with no edges between them represents a *diversified portfolio.*

The internet. The vertices are websites. The edges are hyperlinks between websites. The *Eulerian cycle problem* asks whether a webcrawler can travel the entire web and visit every link exactly once.

The brain (memory). The vertices are neurons. The edges are physical links between neurons, which play a role in long-term memory. This graph is *sparse* (it has few edges), yet *highly connected* (lots of vertices can be removed without destroying the connectivity of the graph). This seems to be a highly efficient way to store information. See the section on *quantum computers*, below.

Chemistry and Physics

Hydrocarbons. A saturated hydrocarbon is a molecule with molecular formula C_nH_{2n+2}. It consists of a carbon backbone surrounded by hydrogens. In 1889, Cayley used his formula for counting labeled trees to predict the existence of previously unknown hydrocarbons. For example, the saturated hydrocarbons with $n \leq 4$ are methane, ethane, propane, butane, and isobutane.

Carbon allotropes. A carbon allotrope is a molecule consisting entirely of carbon atoms. Graphite consists of sheets of hexagonal lattices. Diamond consists of a three-dimensional tetrahedral lattice. Buckminsterfullerenes (or Buckyballs) are spherical arrangements of carbon atoms (e.g. the soccer ball). A carbon nano-tube is a sheet of graphite rolled up into a tube. Each of these materials has diverse applications, and even more exotic carbon allotropes are on the way, some of which were discovered using pure math!

Quantum computers (expander graphs). The quest to produce the next generation of computers revolves around the storage of information in quantum states. There are significant theoretical and experimental barriers, but the hope is that *expander graphs* will provide the right kind of memory system.

Philosophy

Category theory. Until the end of the nineteenth century, mathematics mainly involved the study of *structures*. Beginning in 1945, the focus shifted to the study of *relationships* between structures. This shift has had an immediate impact on the language of mathematics, and also on computer science, linguistics, and philosophy. Graph theory is a useful model of structures (vertices) and relationships (edges). In fact, we can also think about relationships between relationships (faces), and so on. Therefore, we can think of graph theory as a one-dimensional version of a higher-dimensional geometry of logical structures.