NOTES FOR MATH 481
LECTURE 27

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1. Trees

A *tree* is a connected graph with no cycles. A *forest* is a graph with no cycles, i.e. a graph whose connected components are trees. A vertex of degree one in a tree is called a *leaf.*

Lemma. Let G be a graph of order n. If the size of G is strictly less than $n-1$, then G is not connected.

Proof. Suppose G has connected components $G_1, \dots, G_r$. If we add an edge to G, then the number of components goes down by at most one. Therefore, if we add k edges to G, the number of components is greater than or equal to $r-k$.

Let us apply this observation to the empty graph E_n. It n vertices and no edges, so it has n components. If we draw $k \leq n-2$ edges, then number of components is greater than or equal to $n-k \geq 2$, so G is disconnected. □

Theorem. A tree T of order n has $n-1$ edges.

Proof. We use induction on n. If $n=1$, then T has no edges. For any $k < n$, assume that any tree of order k has $k-1$ edges. Let T be a tree of order n, and let e be an edge of T. Let u and v be the endpoints of e. Let $T' = T - e$ be the graph that results from removing e from T. If T' is connected, then there is a path from u to v. However, this path combined with e is cycle in T. This is a contradiction, so T' must be disconnected. Moreover, T' must have exactly two components, because $T'+e$ is connected. Therefore, T' is a forest with two trees T_1 and T_2 of orders k_1 and k_2, such that $k_1 + k_2 = n$ and $k_1, k_2 < n$. By induction, T_1 has $k_1 - 1$ edges and T_2 has $k_2 - 1$ edges. Therefore, the number of edges in T is $(k_1 - 1) + (k_2 - 1) + 1 = k_1 + k_2 - 1 = n - 1$. □

Theorem. Let G be a connected graph of order n and size $n-1$. Then G is a tree.

Proof. Suppose that G contains a cycle C. If we remove an edge e from C, then $G-e$ is still connected (any path that went through e can be rerouted through the path $C-e$). However, $G-e$ has $n-2$ edges, which contradicts the above Lemma. □

Finally, we prove a result about the number of leaves in a tree.

Proposition. Let T be a tree of order $n \geq 2$. Then T has at least one leaf.

Proof. If T has no leaves, then every vertex has degree at least 2, and the size of T is

$$\frac{1}{2}\sum_{v \in T} \deg(v) \geq \frac{1}{2}\sum_{v \in T} 2 = n$$

Since T actually has $n-1$ edges, this is a contradiction. □

In fact, we could have proved this result directly:

Proposition. If G is a graph such that $\deg(v) \geq 2$ for all v, then G has a cycle.

Proof. Let v be vertex of G. Then v has at least two neighbors, call them w and v_1. Now v_1 must have at least one more neighbor besides v, call it v_2. If $v_2 = v$ or $v_2 = w$, we are done because we have found a cycle. Otherwise, v_2 is a new vertex, which must have at least one more neighbor besides v_1, call it v_2. Continuing in this way, we can find a sequence of distinct vertices v_i such that v_i is adjacent to v_{i-1} and v_{i+1}. Since there are only finitely many vertices, there exists a number k such that v_k will have to be adjacent to one of the vertices in the set $\{v_1, \ldots, v_{k-2}, w\}$, and this will give us a cycle in G.

□