

MATH 482: HOMEWORK 9

- (1) Prove that $R(p, q) \leq R(p, q - 1) + R(p - 1, q)$.

Hint: By definition, there exists a graph G of order $R(p, q) - 1$ such that G does not contain K_p and \overline{G} does not contain K_q . Let v be a vertex of G and suppose that $R(p, q) > R(p, q - 1) + R(p - 1, q)$. There are two cases: either $\deg(v) \geq R(p - 1, q)$ or $\deg(v) < R(p - 1, q)$.

- (2) Draw the following graph: the vertices are elements of \mathbb{Z}_{13} and we draw an edge from a to b if $a - b$ is a non-zero cube in \mathbb{Z}_{13} .
- (3) Use (1) and (2) to prove that $R(3, 5) = 14$.
- (4) Prove that $K_{n,n}$ is a Ramanujan graph for all $n \geq 1$.

Hint: Note that $K_{2,2} = C_4$ and make a guess what the eigenvalues are. Now think about how we showed that K_n is a Ramanujan graph.

- (5) For $n \geq 3$, let c_n be the largest possible expander constant for K_n . Let c be the largest possible expander constant for the entire family of graphs $\{K_n\}_{n \geq 3}$. Show that

$$c_n = \begin{cases} 2 & n \text{ even} \\ 2 + \frac{2}{n-1} & n \text{ odd} \end{cases}$$

and conclude that $c = 2$.