

NOTES FOR MATH 482
LECTURE 38

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Theorem.

$$\Phi(x) = \sum_{k \in \mathbb{Z}} (-1)^k x^{u_k} = 1 + \sum_{k \geq 1} (-1)^k (x^{u_k} + x^{u-k})$$

Proof. Let S be the set of partitions (a_1, \dots, a_k) of n such that $a_1 > a_2 > \dots > a_k$. The coefficient of x^n in $\Phi(x)$ is:

$$\sum_{(a_1, \dots, a_k) \in S} (-1)^k = q_e(n) - q_o(n)$$

We claim that

$$q_e(n) - q_o(n) = \begin{cases} (-1)^k & \text{if } n = u_{\pm k} \\ 0 & \text{otherwise} \end{cases}$$

Let $a_1 > \dots > a_k$ be partition of n into k distinct parts. Let m be the largest integer such that the partition can be written as follows:

$$(a_1, a_1 - 1, a_1 - 2, \dots, a_1 - m + 1, a_{m+1}, \dots, a_k)$$

In other words, m is the length of the longest string of consecutive numbers at the beginning of the partition. Note that, by definition, $a_{m+1} < a_1 - m$. Let $r = a_k$ be the smallest entry of the partition. Also note that $k \geq m$.

If $m < r$ and $k \neq m$, then the following is a partition of n into $k + 1$ distinct parts:

$$(a_1 - 1, a_1 - 2, \dots, a_1 - m, a_{m+1}, \dots, a_k, m)$$

Let m' denote the length of the longest consecutive string at the beginning of this new partition, and let r' denote its new smallest entry. Note that

$$m' \geq m = r'$$

For example, $(10, 9, 8, 6, 4)$ has $m = 3, r = 4$ and $k = 5$, so it becomes $(9, 8, 7, 6, 4, 3)$, which has $m' = 4$ and $r' = 3$.

If $m \geq r$, and $k \neq m$ then the following is a partition of n into $k - 1$ distinct parts:

$$(a_1 + 1, a_1, a_1 - 1, \dots, a_1 - r + 2, a_1 - r, \dots, a_1 - m, a_{m+1}, \dots, a_{k-1})$$

We have simply added one to the first r terms and eliminated the last entry of the partition. For this new partition, we have

$$m' = r = a_k < a_{k-1} = r'$$

For example, $(7, 6, 5, 3)$ has $m = 3, r = 3$, and $k = 4$ so it becomes $(8, 7, 6)$, which has $m' = 3$ and $r' = 6$.

If $m < r - 1$ and $k = m$, then the partition looks like:

$$(a_1, a_1 - 1, \dots, a_1 - m + 1)$$

and the following partition has $k + 1$ parts:

$$(a_1 - 1, \dots, a_1 - m, m)$$

Since $r = a_1 - m + 1$, we see that $r - 1 = a_1 - m > m$, so this is a partition with distinct parts. Moreover,

$$m' \geq m = r'$$

If $m > r$ and $k = m$, then the partition again looks like:

$$(a_1, a_1 - 1, \dots, a_1 - m + 1)$$

and the following partition has $k - 1$ distinct parts:

$$(a_1 + 1, a_1, a_1 - 1, \dots, a_1 - r + 2, a_1 - r, \dots, a_1 - m + 2)$$

Again,

$$m' = r < r'$$

We can summarize these four cases as follows:

If $m < r, k \neq m$ or $m < r - 1, k = m$, then $m' \geq r'$.

If $m \geq r, k \neq m$ or $m > r, k = m$, then $m' < r'$.

If these are the only cases that occur, then these processes are exactly inverse to each other, and each partition with an odd number of distinct parts is associated to a unique partition with an even number of distinct parts, and $q_e(n) = q_0(n)$.

However, there are two exceptional cases:

If $k = m = r - 1$, then $n = s_k + t_k = u_{-k}$.

If $k = m = r$, then $n = s_k + t_{k-1} = u_k$.

If k is even, then there is an exceptional partition with an even number of distinct parts, so $q_e(n) - q_0(n) = 1$.

If k is odd, then there is an exceptional partition with an odd number of distinct parts, so $q_e(n) - q_0(n) = -1$.

□

Corollary.

$$p(n) + \sum_{k \geq 1} (-1)^k (p(n - u_k) + p(n - u_{-k})) = 0$$

Proof. Recall that

$$P(x)\Phi(x) = 1$$

If $n \geq 1$, the coefficient of x^n on the right hand side is 0. The coefficient of x^n on the left hand side is:

$$p(n) + \sum_m (-1)^k p(m)$$

where the sum runs over m such that

$$m + \frac{k(3k \pm 1)}{2} = n$$

□