MATH 482: HOMEWORK 1

(1) Find the rightmost digit of $2^{123456789}$. Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

(2) Write down the multiplication table for $\mathbb{Z}_{10}$. Which rows and columns have to be removed to make it a Latin square? Relate your answer to the elements of $\mathbb{Z}_{10}^{\times}$.

(3) Alice wants to send Bob a secret 2-digit number a. She sends the number $x = (50a \bmod 101)$. If $x = 30$, what is the secret number?

(4) Alice wants to send Bob a secret 4-digit number a. She sends $x = (a \bmod 73)$ and $y = (a \bmod 137)$. Eve the eavesdropper intercepts the message and finds that $x = 33$ and $y = 32$. What is the secret number? Why did Alice choose the primes 73 and 137 in order to send a 4-digit number?

(5) Alice claims to have invented a machine that turns lead into gold, and produces lead out of nothing. Bob is skeptical, but he decides to give it a try. He pushes the RUN button and sure enough, a piece of lead comes out. He puts the piece of lead back in the machine and it comes out as a piece of gold. Just for fun, he puts the piece of gold into the machine, and *two* pieces of lead come out. Bob is surprised, so he puts *both* pieces of lead back in the machine. Instead of two gold pieces, a piece of copper comes out! Now Bob is really puzzled. He puts the copper piece into the machine, and gets back a piece of lead and a piece of gold.

At this point, Alice tells Bob a rule about how the machine works. Whatever pieces are put in, it will either return one piece of a new material, or a collection of pieces of old materials. Let A = Lead, B = gold, C = copper, etc. Here is the sequence of outputs from the machine:

Nothing $\to$ A $\to$ B $\to$ AA $\to$ C $\to$ AB $\to$ D $\to$ AAA $\to$ BB $\to$ AC $\to$ ...

What comes next? Justify your answer.